

TI 2012-119/VIII  
Tinbergen Institute Discussion Paper



# Hotelling Models with Price-Sensitive Demand and Asymmetric Transport Costs: An Application to Public Transport Scheduling

*Adriaan Hendrik van der Weijde*

*Erik T. Verhoef*

*Vincent A. C. van den Berg*

*Faculty of Economics and Business Administration, VU University Amsterdam, and Tinbergen Institute.*

Tinbergen Institute is the graduate school and research institute in economics of Erasmus University Rotterdam, the University of Amsterdam and VU University Amsterdam.

More TI discussion papers can be downloaded at <http://www.tinbergen.nl>

Tinbergen Institute has two locations:

Tinbergen Institute Amsterdam  
Gustav Mahlerplein 117  
1082 MS Amsterdam  
The Netherlands  
Tel.: +31(0)20 525 1600

Tinbergen Institute Rotterdam  
Burg. Oudlaan 50  
3062 PA Rotterdam  
The Netherlands  
Tel.: +31(0)10 408 8900  
Fax: +31(0)10 408 9031

Duisenberg school of finance is a collaboration of the Dutch financial sector and universities, with the ambition to support innovative research and offer top quality academic education in core areas of finance.

DSF research papers can be downloaded at: <http://www.dsf.nl/>

Duisenberg school of finance  
Gustav Mahlerplein 117  
1082 MS Amsterdam  
The Netherlands  
Tel.: +31(0)20 525 8579

# Hotelling models with price-sensitive demand and asymmetric transport costs: an application to public transport scheduling

Adriaan Hendrik van der Weijde\*, Erik T. Verhoef, Vincent A.C. van den Berg

*Department of Spatial Economics, Vrije Universiteit Amsterdam, De Boelelaan 1105,  
1081HV Amsterdam, The Netherlands*

*Tinbergen Institute, Gustav Mahlerplein 117, 1082MS Amsterdam, The Netherlands*

---

## Abstract

We formulate a horizontal differentiation model with price-sensitive demand and asymmetric transport costs, in the context of transport scheduling. Two competitors choose fares and departure times in a fixed time interval. Consumers are distributed uniformly along the interval; their location indicates their desired departure time. In a standard Hotelling model, locations are chosen before prices. In our context, the opposite order is also conceivable, but we show that it does not result in a Nash equilibrium; the same is true for a game in both variables are chosen simultaneously. We also discuss Stackelberg game structures and second-best regulation.

We conclude that the addition of price-sensitive demand results in equilibria in the traditional Hotelling model with price setting; there, services are scheduled closer together than optimal. We also show that it is possible to include asymmetric schedule delay functions. Our results show that departure times can be strategic instruments. Optimal regulatory strategies depend on the value of schedule delay, and on whether the regulator can commit.

*Keywords:* Horizontal differentiation, scheduling, transport.

*JEL Classification:* L11, L51, L91, R40.

---

## 1. Introduction

Hotelling's (1929) classic paper on horizontal differentiation shows that, when two firms compete on locations and a fixed number of consumers buy from the closest firm, the two firms locate as closely together as possible. Later

---

\*Corresponding author. Tel.: +31 20 59 86106

*Email addresses:* [h.vander.weijde@vu.nl](mailto:h.vander.weijde@vu.nl) (Adriaan Hendrik van der Weijde),  
[e.t.verhoef@vu.nl](mailto:e.t.verhoef@vu.nl) (Erik T. Verhoef), [v.a.c.vanden.berg@vu.nl](mailto:v.a.c.vanden.berg@vu.nl) (Vincent A.C. van den Berg)

work has generalized Hotelling’s model in several ways; most importantly, to also include price-setting behaviour, which results in the absence of a pure-strategy equilibrium when combined with linear transport costs, and to maximum differentiation when combined with quadratic transport costs (d’Aspremont et al., 1979).

Most analyses have kept the assumptions that demand at every location is perfectly inelastic, and that the user costs of travelling are independent of direction. Puu (2002) does formulate a Hotelling model with elastic demand, in which locations and prices are determined simultaneously, but his calculations have been shown to be flawed (Sanner, 2005). Colombo (2011) includes elastic demand and asymmetric travel costs, but his model is unidirectional: travel costs in one direction are infinite. Gu and Wenzel (2012) formulate a general Salop model with elastic demand, but their formulation is unfortunate, in that the distance between consumers and their suppliers negatively affects their utility, but their demand is a function of the price only; not of the transport costs.

The two assumptions mentioned above may yield good approximations in many applications, but in some others, they are clear oversimplifications. One obvious example is the scheduling of transport services in a certain fixed time interval. There, demand is usually price-sensitive, since people can choose not to travel or alternative modes of transport are available, amongst other reasons. Furthermore, Hotelling’s ‘travel costs’ in this setting represent the costs of schedule delay, and the cost of being late is usually higher than the cost of being early (e.g. Small, 1982). This scheduling problem has received limited attention in the literature; the only study known to us assumes perfectly inelastic demand (van Reeveen and Janssen, 2006). It concludes that an additional characteristic, such as service quality, is necessary to obtain a pure-strategy equilibrium, and even then, this equilibrium only exists if passengers are relatively insensitive to the departure time.

To generalize Hotelling’s model, we formulate a horizontal differentiation model with price-sensitive demand and asymmetric transport costs. We do so in the context of transport scheduling. Although our models can be applied in many other instances, such as telecommunications markets (e.g. as a generalization of Cancian et al., 1995), we use the scheduling problem; not only because it is a natural example, but also because of the relative lack of literature on the subject.

As in a traditional horizontal differentiation model, two competitors choose a location on a fixed interval. In our case, this is an interval in time, such that the two locations are departure times of a transport service. The two competitors also set their fares. Consumers are distributed uniformly along the interval; their location indicates their desired departure time, such that they face a schedule delay cost that increases in the deviation from their desired departure time. Hence, they minimize their generalized price, which is the sum of the fare and their schedule delay costs.

Most Hotelling models would assume that the two competitors choose their locations or departure times first, after which fares are set. In scheduling, the opposite order is also conceivable, but we show that this game does not have

a Nash equilibrium; the same is true for a game in which fares and departure times are chosen simultaneously. We first find the equilibrium fares and departure times, examine under which conditions equilibria exist, and compare them to the social optimum, assuming that schedule delay costs early and late are equal. Contrary to other horizontal differentiation models, notably that of d'Aspremont et al. (1979), the competitors schedule their services closer together than optimal. We then analyse how these equilibria change if the schedule delay cost late is higher than the schedule delay cost early, and show that the resulting equilibria can still be stable. Finally, we analyse Stackelberg game structures, which increase the parameter space in which equilibria exist, and comment on optimal second-best regulation.

## 2. Methodology

Consider a departure time choice and fare game between two duopolistic suppliers of a scheduled transport service. Consumers have different preferences over desirable departure times and, given this desired departure time, there is an elastic demand for trips. Specifically, travel demand  $d(t)$ , at any time  $t \in [-1, 1]$ , is a linear function<sup>1</sup> of the generalized price  $p(t)$ . Since demand cannot be negative

$$d(t) = \text{Max}[a - bp(t), 0] \quad (1)$$

which holds for all  $t$ . The total consumer surplus can then be calculated as

$$CS = \frac{1}{2} \int_{-1}^1 d(t) \left( \frac{a}{b} - p(t) \right) dt = \frac{1}{2b} \int_{-1}^1 (d(t))^2 dt \quad (2)$$

Operators choose a fare  $f_i$  and departure time  $t_i$ ; this assumes that each operator schedules only one service in the observed time interval. Operator profits are given by

$$\pi_i = D_i f_i - F \quad (3)$$

where  $D_i$  is the total demand for its service and  $F$  is the fixed cost associated with the operation of the service; marginal per-passenger costs are assumed to be zero. Social welfare is the sum of consumer surplus and operator profits,  $CS + \sum_i \pi_i$ .

The generalized price  $p_i(t)$  of service  $i$ , taken by a consumer with preferred departure time  $t$ , is the sum of the fare and the schedule delay cost. Schedule delay costs are assumed to be linear, but not necessarily symmetric, such that the unit cost of schedule delay when a passenger is late ( $\gamma$ ) can differ from the unit cost of schedule delay when a passenger is early ( $\beta$ ):

$$p_i(t) = f_i + \text{Max}[\gamma(t_i - t), 0] + \text{Max}[\beta(t - t_i), 0] \quad (4)$$

---

<sup>1</sup>In most Hotelling models,  $t \in [0, 1]$ , but the present specification yields more compact expressions, without affecting the conclusions.

Passengers choose the service that minimizes the generalized price they pay, so in equilibrium, their generalized price  $p(t) = \text{Min}[p_i(t), p_{i+1}(t), \dots, p_I(t)]$ . If there are only two operators, and the equilibrium demand for each operator's service is strictly positive, we can define the inner market boundary point  $t^*$ , at which passengers are indifferent between the two operators as

$$t^* = (f_2 - f_1 + \beta t_1 + \gamma t_2) / (\beta + \gamma) \quad (5)$$

where, by construction, operator 1 schedules the first service. The total demand for each operator's service,  $D_1$  and  $D_2$ , can then be calculated as

$$D_1 = \int_{-1}^{t_1} \text{Max}[a - b(f_1 + \gamma(t_1 - t)), 0] dt + \int_{t_1}^{t^*} \text{Max}[a - b(f_1 + \beta(t - t_1)), 0] dt \quad (6)$$

and

$$D_2 = \int_{t^*}^{t_2} \text{Max}[a - b(f_2 + \gamma(t_2 - t)), 0] dt + \int_{t_2}^1 \text{Max}[a - b(f_2 + \beta(t - t_2)), 0] dt \quad (7)$$

where  $t^*$  is given by Eq. 5.

### 3. Social optimum

Since there are no externalities associated with demand, and the operator's marginal costs are zero, the social optimum can be found by simply maximizing total demand. This implies that both fares should be equal to zero<sup>2</sup>. Assuming that, in the social optimum,  $d(-1), d(t^*), d(1) > 0$ , meaning that there is positive demand from all preferred arrival times, maximizing the sum of Eqs. 6 and 7 gives the social welfare-maximizing departure times:

$$\{t_1^{SW}, t_2^{SW}\} = \left\{ -\frac{\gamma}{\beta + \gamma}, \frac{\beta}{\beta + \gamma} \right\} \quad (8)$$

This implies that, if  $\beta = \gamma$ , the socially optimal departure times are at  $-1/2$  and  $1/2$ . If the cost of departing late is higher than the cost of departing early, both departures shift to an earlier time, such that more passengers depart early. Note, however, that this assumes that demand is strictly positive for any  $t \in [-1, 1]$ . Substituting the zero fares and Eq. 8 into Eq. 1, this implies that

$$\frac{a}{b} - \frac{\beta\gamma}{\beta + \gamma} > 0 \quad (9)$$

If this condition is not met, the social optimum must be a fully separated equilibrium, such that there exists a  $t$  for which demand is zero. In that case,

---

<sup>2</sup>We will disregard the possibility of setting an infinitely negative fare.

the optimal fares are still equal to zero, so any  $\{t_1, t_2\}$  that satisfies  $d(-1) = d(t^*) = d(1) = 0$  is an equilibrium, as social welfare does not depend on the exact departure times, and therewith the exact desired arrival times served. Shifting to an earlier time will allow more passengers with an earlier desired departure time to travel, but keeps exactly the same number of passengers with later desired departure times from travelling, such that the net effect on welfare is zero.

#### 4. Full market separation

It may be optimal for two operators to set their fares and departure times such that  $d(-1) = d(t^*) = d(1) = 0$ , so that their markets are fully separated and each operator acts as a monopolist on its own segment. In this case, there exists a desired departure time that is so far away from both services that nobody with this desired departure time travels. To examine when this would happen, consider a single monopolistic operator who can set any departure time and fare, and faces the linear demand function in Eq. 1 for all  $t \in \mathbb{R}$ , such that it is not constrained by a fixed time period.

Solving Eq. 1 to obtain the passengers with the earliest and latest desired departure times that are travelling in this situation gives

$$\{\underline{t}, \bar{t}\} = \left\{ t_1 - \frac{a - bf_1}{\gamma b}, t_1 + \frac{a - bf_1}{\beta b} \right\} \quad (10)$$

This operators profits are then

$$\pi_1 = \int_{\underline{t}}^{\bar{t}} d(t) dt - F = f_1 \frac{(\beta + \gamma)(a - bf_1)^2}{2\beta\gamma b} - F \quad (11)$$

Naturally, these profits do not depend on the operator's departure time choice, since there is now no unique fixed time period. The operator's profit in Eq. 11 is maximized when

$$f_1 = a/(3b) \quad (12)$$

Substituting this back in Eq. 10 gives the passengers with the earliest and latest desired departure times, as a function of the parameters:

$$\{\underline{t}, \bar{t}\} = \left\{ t_1 - \frac{2a}{3\gamma b}, t_1 + \frac{2a}{3\beta b} \right\} \quad (13)$$

Only if  $\bar{t} - \underline{t} \leq 1$  can two fully separated monopolists with fares as in Eq. 13 operate between  $t = -1$  and  $t = 1$ . This implies that

$$\frac{2(\beta + \gamma)a}{3\beta\gamma b} \leq 1 \quad (14)$$

Hence, a fully separated equilibrium is more likely to occur when the maximum number of passengers for a given  $t$  ( $a$ ) is smaller, when the demand sensitivity ( $b$ )

is higher<sup>3</sup>, for higher costs of schedule delay, and for a larger difference between the cost of schedule delay late and the cost of schedule delay earlier. However, operators can only recover their costs if  $\pi_i \geq 0$ . Using Eq. 10, this implies that

$$\frac{2(\beta + \gamma)a^3}{27\beta\gamma b^2} \geq F \quad (15)$$

This, conversely, is less likely to occur when the maximum number of passengers is smaller, the demand sensitivity is higher, and for higher costs of schedule delay. In the absence of any subsidies, Eqs. 14 and 15 can only hold simultaneously if  $F \leq a^2/(9b)$ .

## 5. Equilibria with covered markets

Having derived when a separated equilibrium occurs, we can now examine the various possibilities for an equilibrium in which the market is entirely covered, such that the two operators compete for the marginal customer. If Eq. 14 does not hold,  $D(-1), D(t^*), D(1) > 0$ , since it would be suboptimal to stay in a situation where  $D_1(-1)$  or  $D_2(1)$  equal zero while  $D(t^*) > 0$ , and vice versa. The resulting equilibrium is considerably more complicated than the fully separated one, which is why we will start by assuming that  $\beta = \gamma$ . In order to derive tractable results, before we consider asymmetric schedule delay costs.

In this case, the equilibrium fares and departure times depend on the order in which they are chosen if a sequential game structure is allowed. We will examine three possibilities: either fares and departure times are chosen simultaneously, or departure times are chosen first, while fares are chosen only after the departure times have been fixed, or vice versa. In all cases, we initially assume Nash behaviour, moving to Stackelberg games in section 6.

### 5.1. Simultaneous departure time and fare choice

Puu (2002) analyses a Hotelling game in which locations and prices are chosen simultaneously, and derives an equilibrium where both suppliers charge equal prices, and  $t_1 = -t_2$ . However, as Sanner (2005) shows, this equilibrium only appears to be stable because of a calculation error. In reality, each operator could obtain a higher profit by choosing the same location as its competitor, while undercutting its competitor's price with an arbitrarily small amount. It would then serve the entire market of its competitor, plus at least part of its own original market. Hence, in this situation, no stable equilibrium exists.

This result continues to hold if there are more than two competitors or if demand functions are nonlinear, for the same reason; it is always possible for one competitor to take its direct neighbour's place and undercut its price by an arbitrarily small amount, and then obtain the full market. As long as

---

<sup>3</sup>Note that the effects of  $b$  runs via its impact on demand, given  $a$ . In particular, for a given price and with equal  $a$ , the demand elasticity is independent of  $b$



the two operators were competing for the marginal customer, the profit of the undercutting competitor will then increase. Only if one of the operators sets its fares and locations before the other can an equilibrium exist; we will briefly examine this game in section 6 below.

### 5.2. Fares chosen before departure times

We can find the equilibrium by backward induction, first solve  $\frac{\partial \pi_i}{\partial t_i} = 0$  for  $i = \{1, 2\}$  to obtain the optimal departure times, substituting these in the operators' profit functions, and then solving for the optimal fares. The optimal timing response function for both operators are

$$\{t_1, t_2\} = \left\{ -\frac{4}{5} + \frac{1}{5\beta} \left( 2\frac{a}{b} - 3f_1 + f_2 \right) + \frac{t_2}{5}, \frac{4}{5} - \frac{1}{5\beta} \left( 2\frac{a}{b} + f_1 - 3f_2 \right) + \frac{t_1}{5} \right\} \quad (16)$$

Solving Eq. 16 to obtain the equilibrium departure times  $t_i^*(f_1, f_2)$  gives

$$\{t_1^*, t_2^*\} = \left\{ -\frac{2}{3} + \frac{1}{3\beta} \left( \frac{a}{b} - 2f_1 + f_2 \right), \frac{2}{3} - \frac{1}{3\beta} \left( \frac{a}{b} + f_1 - 2f_2 \right) \right\} \quad (17)$$

The equilibrium fares can then be obtained by solving  $\frac{\partial \pi_i | \{t_1=t_1^*, t_2=t_2^*\}}{\partial f_i} = 0$ , which gives optimal response functions<sup>4</sup>

$$f_i = \frac{16a}{39b} + \frac{10}{39}(f_j + \beta) - \frac{1}{39} \sqrt{334 \left( \frac{a}{b} \right)^2 - 460(f_j + \beta) \frac{a}{b} + 295(f_j + \beta)^2} \quad (18)$$

These derivations are tedious, and the resulting expressions have no intuitive interpretation.

The optimal response functions can be solved to obtain the equilibrium fares. However, this equilibrium is not stable. Substituting the equilibrium fares into the cross-partial derivatives of Eq. 18,  $\partial f_i / \partial f_j$ , results in a tedious expression which, however, is smaller than one for any positive  $\beta$ . This means that undercutting strategies are profitable as long as the operators are making positive profits. Instead of setting a fare equal to Eq. 18, any of the two competitors could set its fare an arbitrarily small amount lower. The other would then also adjust its fare, but by a smaller amount, since  $\partial f_i / \partial f_j < 1$ . In the timing subgame, this competitor could then simply choose the other's departure time; with its lower fare, it would get the entire market. Since both competitors can use this undercutting strategy profitably as long as positive profits are made, the equilibrium is never stable.

This result continues to hold if the value of schedule delay early is higher than the value of schedule delay late. In that case, equilibrium profits are likely to be asymmetric, so undercutting may only be a profitable strategy for one of the operators, but this still results in instability. The same is true if demand or schedule delay function are non-linear; as long as  $\partial f_i / \partial f_j < 1$  for at least one of the operators, and as long as the operators compete for the marginal traveller, any equilibrium where positive profits are made is unstable.

---

<sup>4</sup>As well as one other root, which corresponds to the minimum profit.

### 5.3. Departure times chosen before fares

#### Symmetric schedule delay costs ( $\beta = \gamma$ )

Again, we find the equilibrium by backward induction. Solving  $\frac{\partial \pi_i}{\partial f_i} = 0$  for  $i = \{1, 2\}$  gives the optimal fare response functions for both operators. Solving these gives the fare equilibrium:

$$f_1^* = \frac{a}{2\beta} + \beta \left( 2 + \frac{5}{4}t_1 + \frac{3}{4}t_2 \right) - \frac{1}{4b}$$

$$\sqrt{4 \left( \frac{a}{b} \right)^2 + \frac{4\beta a}{b} (t_1 - t_2) + \beta^2 (80 + 112t_1 + 45t_1^2 + 48t_2 + 22t_1t_2 + 13t_2^2)} \quad (19)$$

$$f_2^* = \frac{a}{2\beta} + \beta \left( 2 - \frac{3}{4}t_1 - \frac{5}{4}t_2 \right) - \frac{1}{4b}$$

$$\sqrt{4 \left( \frac{a}{b} \right)^2 + \frac{4\beta a}{b} (t_1 - t_2) + \beta^2 (80 - 112t_2 + 45t_2^2 - 48t_1 + 22t_1t_2 + 13t_1^2)} \quad (20)$$

Again, these derivations are tedious, and the resulting equations have no straightforward intuitive interpretation. Substituting them back in the original profit functions and maximizing each operator's profit with respect to its departure times gives the equilibrium departure times. These do not have a closed form, and can only be evaluated numerically, which we will do below. What is important to note here is that in this game undercutting is not a profitable strategy. Rather than an arbitrarily small deviation from the first-stage subgame equilibrium, a successful undercutting strategy now requires an operator to take its competitor's place; a major deviation. The other operator will respond with an equally large deviation, and take the place previously occupied by the undercutting operator; none of them will gain.

We can also establish the interval in which an equilibrium exists. By construction,  $-1 \leq t_1 \leq t_2 \leq 1$ . In this game, these conditions are met only when  $\beta \geq \frac{6a}{31b}$ . For a smaller  $\beta$ , there is no equilibrium. For  $\beta \geq \frac{4a}{3b}$ , the equilibrium is separated. Using these bounds, it is also possible to calculate the range of  $\{t_1^*, t_2^*\}$ :

$$\lim_{\beta \rightarrow \frac{4a}{3b}} t_1^* = -\frac{1}{2}, \quad \lim_{\beta \rightarrow \frac{6a}{31b}} t_1^* = \frac{1}{2} \quad (21)$$

So, the two services are closer together than socially optimal. This is an important difference from many other horizontal differentiation models, notably that of d'Aspremont et al. (1979), in which competitors locate as far apart from each other as possible, such that they can exert local market power. The reason that this does not happen here is that, in our model, demand is price-sensitive. If one operator schedules its service further from the other, this does indeed decrease competition at the inner market boundary, allowing it to increase its fares in the second stage, as Eqs. 19–20 show. However, by doing so, it will also lose customers with a desired departure time between the two services since, for

these travellers, both schedule delay costs and fares have increased. Of course, this will also shift the inner market boundary.

Hence, each operator's departure time must be closer to the inner market boundary than to the closest outer market boundary, precisely because the latter is fixed, while the former moves in the same direction as a change in one operator's departure time. How close it must be exactly depends on the optimal fare, and hence, on the value of schedule delay. When  $\beta$  approaches  $6a/(31b)$ , the optimal two departure times approach 0, and when  $\beta$  is even smaller, the two operators will continuously swap places; no stable equilibrium emerges. When  $\beta$  is large, however, market areas are small, so the incentive to try and steal a competitor's customers is smaller, and hence, the departure times are set further apart.

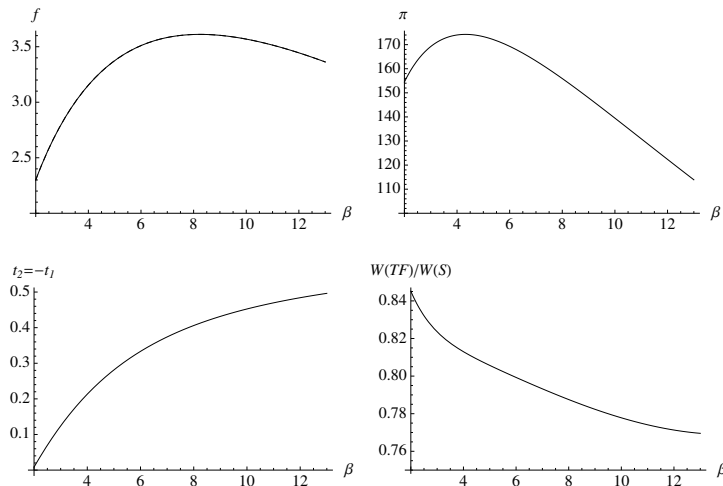


Figure 1: Setting departure times before fares ( $a/b = 10$ ,  $\gamma = \beta$ )

Fig. 1 shows the equilibrium fares, profits, departure times and social welfare relative to the optimum, for the range of values of schedule delay where an equilibrium exists. As already indicated by Eq. 21, departure times move further apart if the value of schedule delay increases. Fares and profits, however, are non-monotonic in  $\beta$ . This is because an increase in the value of schedule delay has two effects. Firstly, an increase in  $\beta$  directly decreases the number of travellers, for any set of fares and departure times, as travel costs for all commuters increase. This will reduce optimal fares and profits. However, if travel costs for all commuters increase, they also increase for the marginal commuters, who have a desired departure time  $t = t^*$ . Hence, there will be fewer marginal commuters, which will reduce competition. This allows operators to increase their fares and profits. As Fig. 1 shows, this competitive effect dominates for smaller values of schedule delay. For larger values, the optimal departure times are already so far apart that the demand effect is stronger.

Social welfare, relative to the optimum, always decreases in the value of

schedule delay. For low values of schedule delay, this is because fares are increasing in  $\beta$  and thus moving away from the optimum; although the departure times are moving closer to the optimum, this is less important, given that the values of schedule delay are relatively low. For higher values of schedule delay fares start decreasing slightly, but deviations from the optimal departure time are now so costly that although the departure times are moving towards the optimum, they are not moving too slowly to offset the negative effect of an increase in  $\beta$  on welfare.

*Asymmetric schedule delay functions ( $\gamma > \beta$ )*

If schedule delay functions are not symmetric in each commuter's desired departure time, operators in the resulting equilibrium will charge different fares, and their departure times will not be at equal distances from zero. This complicates the analysis and, hence, this situation can only be evaluated numerically. Fig. 2 shows, for  $\beta = 5$  and  $a/b = 10$ , the fares, departure times, operator profits and social welfare for a range of  $\gamma$ . Although, naturally, the exact functions are specific to these particular parameters, other parameters result in very similar figures. Moreover, all variables only depend on the ratio between  $\beta$  and  $a/b$ ; not on the individual levels of these parameters.

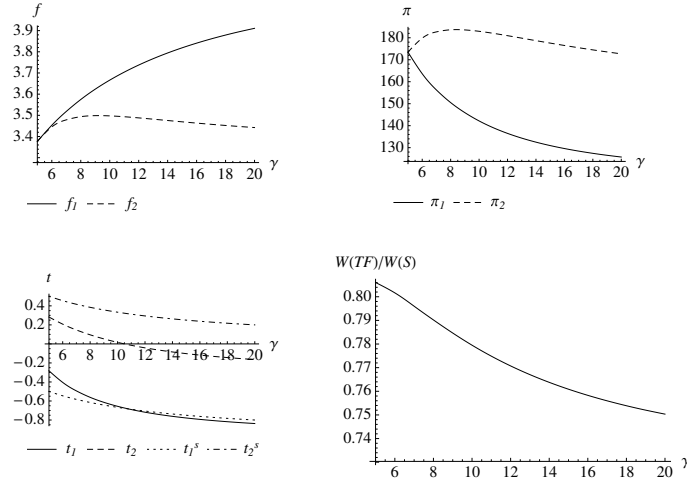


Figure 2: Effects of  $\gamma > \beta$  ( $a/b = 10$ ,  $\beta = 5$ )

Naturally, if the value of schedule delay late increases relative to the value of schedule delay early, both departures will move to an earlier time, such that fewer commuters are late. In an effort to gain the largest market share, both do so at a faster rate than the socially optimal departure times  $t_i^S$ . Hence, the first operator's departure time initially moves closer to the optimum, while the second operator's departure time, which is already earlier than optimal, continuously moves away from the optimum.

For moderate deviations of  $\gamma$  from  $\beta$ , this allows both operators to increase their fares. However, the first operator's market size decreases as it is squeezed towards its outer market boundary, and this reduces its profits. For large increases in  $\gamma$ , even the second operator loses, as demand for its service decreases too fast to be offset by its favourable position. Social welfare decreases in  $\gamma$ , relative to the first-best, as a result of higher prices, a less optimal departure time of the second operator and of course, in the same way as an increase in  $\beta$ , simply because suboptimal departure times become more costly.

## 6. Other games

As we have seen, games with simultaneous fare and timing choices never have pure strategy Nash equilibria and in games where either fares or departure times are chosen first equilibria only exist for limited ranges of parameters. It is therefore worth investigating which other game structures could result in equilibria where the above games fail. For the sake of brevity and simplicity, will limit our attention to situations with symmetric schedule delay cost functions, although, like before, it is possible to include asymmetries.

### 6.1. Stackelberg games

Stackelberg games, in which one operator sets its fare, departure time, or both before the other operator, may be a realistic representation of some real-world transport markets. A large operator, which is active not just in one market but operates many routes may, for example, have to decide on its fares and departure times much earlier than a small, flexible operator that only participates in one market. In this case, the operator that publishes its decisions first can choose them in such a way that it cannot be profitably undercut by the second operator. We will examine the Stackelberg equivalents of the three Nash games above: one situation in which the first operator sets its fare and departure time before the other, one in which the first operator sets its departure time before the other, followed by a separate second stage in which the operators set their fares in the same order, and the reverse, one in which the first operator sets its fare before the other, followed by a sequential departure time choice.

#### *Fares and departure times set simultaneously per firm, and sequentially between firms*

If the first operator decides on both its fare and its departure time before the other, and can commit to these decisions, it will choose them such that undercutting is not a profitable strategy for the second operator. This does mean that it has to accept a lower profit than it would get in some of the other games. Starting with the second stage, the second operator's optimal fares and departure times  $\{t_2^*(t_1, f_1), f_2^*(t_1, f_1)\}$  can be found by simply setting  $\frac{\partial \pi_2}{\partial t_2} = \frac{\partial \pi_2}{\partial f_2} = 0$ . The first operator then maximizes its own profits subject to not only  $\{t_2^*, f_2^*\}$ , but also another constraint, which specifies that the second operator's profit must be greater or equal to the profit it would get if it took

the first operator's departure time, and set its fare an arbitrarily small amount lower:

$$\pi_2^* \geq \int_{-1}^{\bar{t}} a - b(f_1 + \text{Max}[\gamma(t_1 - t), 0] + \text{Max}[\beta(t - t_1), 0])dt - F \quad (22)$$

where  $\pi_2^*$  is the second operator's equilibrium profit, and  $\bar{t} = \text{Min} \left[ t_1 + \frac{a - bf_1}{\beta b}, 1 \right]$ . Since this constraint will be binding for any set of parameters, Eq. 22 can be solved as a strict equality and used to substitute out one of the first operator's decision variables. Since the constraint is nonlinear, and the resulting equilibrium will have asymmetric departure times and unequal fares, this game can only be solved numerically. It does have a unique pure strategy equilibrium for a large parameter space. Fig. 3 shows the equilibrium fares, profits, departure times and relative welfare for a range of  $\beta$ .

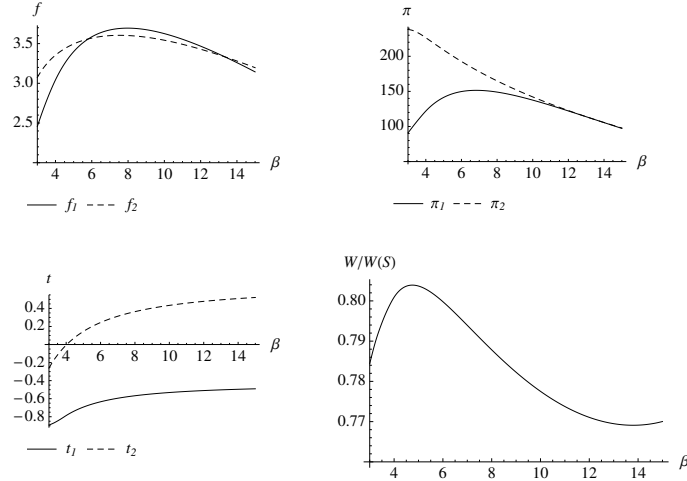


Figure 3: Varying  $\beta$  in a Stackelberg game ( $a/b = 10$ ,  $\gamma = \beta$ )

As expected, the second mover in this game has an advantage, since the first mover has to choose a position that can not be undercut profitably. Hence, the first operator's departure time is close to the outer market boundary, and its price is lower than in the corresponding Nash game. Naturally, it is therefore optimal for the second operator to set an earlier departure time and higher fare than it would do in a Nash game. As the value of schedule delay increases, resulting in a lower travel demand, the first operator can choose a more favorable position, as undercutting becomes less profitable. For very high values of schedule delay, the second-mover advantage all but disappears, and the equilibrium locations approach those of the Nash game.

#### *Fares chosen before departure times*

This game is similar to the previous, but has four separate stages;  $f_1$ ,  $f_2$ ,  $t_1$  and  $t_2$  are chosen in that order. Again, the resulting equilibrium is asymmetric;

the second operator has a second-mover advantage, so  $\beta_1^* < \beta_2^*$ . Undercutting is still possible, and can still be profitable, especially for small values of schedule delay, but the first operator can again avoid this. It can be shown numerically that this game has an equilibrium for at least some values of  $\beta$ .

#### *Departure times chosen before fares*

The equilibrium of this game, in which  $t_1$ ,  $t_2$ ,  $f_1$  and  $f_2$  are chosen in that order, is the most tedious to compute, as only the last subgame has a closed form. However, numerical simulations show that, for all  $\beta \geq 0.32a/b$ , a pure-strategy equilibrium exists; this is a considerably smaller space than in the Nash game in which both operators set their departure times at the same time, followed by a simultaneous fare decision.

#### *6.2. Regulation*

As we have seen, operators can use their departure times as strategic instruments, and hence, regulation may be desired. Even if the first-best solution, in which fares are zero and departure times given by Eq. 8, is not feasible, it may be possible for the regulator to set either fares or departure times. The operators would then be free to set the remaining variables afterwards. This requires that regulators can commit to the choices they announce; operators must be convinced that the announced fares or departure times will not be changed after they have announced their own decisions. If, for any reason, regulator cannot commit, the game collapses from a two-stage game to a simultaneous game, in which the regulator and the operators effectively set fares and departure times simultaneously.

Hence, at least four different games should be considered; two sequential games, in which regulators set either fares ( $F_S T$ ) or departure times ( $T_S F$ ), and two simultaneous games. Fig. 4 shows the performance of each game, relative to the first-best social optimum, for the range of  $\beta$  in which each game has an equilibrium.

Interestingly, all four games have stable equilibria over a large range of values of schedule delay, while only the first, in which the regulator sets the departure times, followed by a competitive pricing stage, has a stable equilibrium in an unregulated market. In reality, completely unregulated public transport markets are a rare occurrence, so this may explain why we usually observe stable equilibria. Indeed, there is some evidence that deregulation can lead to service instability (e.g. Douglas, 1987). However, even the regulatory games do not have equilibria for all possible values of schedule delay. In particular, the games in which the regulator sets fares are unstable for small values of  $\beta$ .

As Fig. 4 shows, regulators have a first-mover advantage, regardless of which variable they are controlling; welfare in each sequential game is always higher than welfare in the corresponding simultaneous game. However, this advantage is relatively small for most parameters. A game in which departure times are chosen by the regulator is preferred, from a social perspective, if values of time are low. Conversely, when values of schedule delay, a game in which the regulator chooses fares performs better. This may sound counterintuitive since,

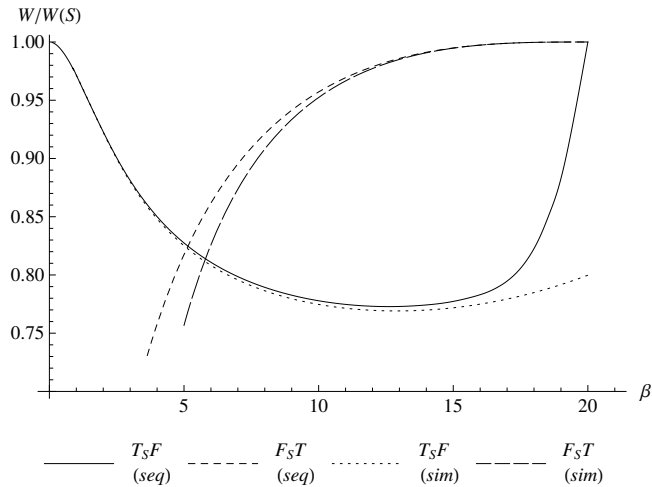


Figure 4: Relative welfare of regulatory games

when values of schedule delay are low, suboptimal departure times are relatively unimportant compared to suboptimal fares, so one would expect a more efficient outcome if the regulator set the latter. This does not happen because, for low values of schedule delay, the private operators set low fares anyway; it is better for the regulator to set the departure times, even if they are relatively unimportant. If values of schedule delay are high, on the other hand, the operators would exercise their increased market power and raise fares far above the optimum; it would then be better for the regulator to set the fares instead, even though suboptimal departure times are also relatively costly.

Of the four games in Fig. 4, only the first, in which the regulator sets the departure times, followed by a competitive pricing stage, has a stable equilibrium in an unregulated market. A comparison of the relative welfare in this game to the bottom right-hand panel in Fig. 1 shows that the increase in social welfare that results from regulatory intervention is relatively small, especially for small values of schedule delay. Only when  $\beta$  is very high, such that small deviations from the optimal departure times have a large effect on social welfare, is the regulatory game much more efficient.

## 7. Conclusions

We have shown that the inclusion of price-sensitive demand in a traditional Hotelling model with price setting can yield equilibria even if transport costs are linear. For many applications, this produces more attractive results than a similar model with fixed demand and quadratic travel costs, as equilibria in our model can be interior and do not necessarily result in minimum differentiation, and never in maximum differentiation. Indeed, the two competitors normally



locate closer together than optimal. This happens because, when demand is price-sensitive, operators have incentives to schedule their services closer to the inner market boundary than to the outer edges of the market, since the inner market boundary can be pushed in the direction of the competitor, while the outer edges of the market are fixed. Games where prices are chosen before or simultaneously with locations, on the other hand, have no stable Nash equilibria.

We have also shown that it is possible to include asymmetric travel cost functions, which can be interpreted as asymmetric schedule delay functions in our model. Asymmetric schedule delay functions generally lower the relative welfare of the game. Since asymmetric schedule delay functions result in asymmetric equilibria, they do make calculation of the equilibria much more difficult, and for some parameters result in unstable equilibria if undercutting is profitable.

A Stackelberg structure, in which one operator sets its decision variables before the other often helps to establish equilibria in games where they are not normally present. In these games, the first mover can deliberately choose a position such that its competitor has no incentives to undercut; hence, there is a second-mover advantage. Similarly, regulation can create equilibria in situations where an unregulated market fails to do so.

The models proposed above are one way of modeling scheduling decisions of transport operators; contrary to some of the previous literature, it succeeds in finding equilibria. Even though these models are simple, they do show that departure times can be strategic instruments, and should therefore be of interest to regulators. If the socially optimal locations and prices are not attainable, regulating one of these two variables can result in a small efficiency improvement; the value of schedule delay determines which of the two results in the greatest gain.

## Acknowledgements

Financial support from ERC Advanced Grant #246969 (OPTION) is gratefully acknowledged. The authors thank the participants of the 2012 Kuhmo Nectar conference for helpful comments on an earlier version of this paper.

## References

- Cancian, M., Bills, M. A., Bergstrom, T., 1995. Hotelling location problems with directional constraints: an application to television news scheduling. *Journal of Industrial Economics* 43 (1), 121–124.
- Colombo, S., 2011. Spatial price discrimination in the unidirectional hotelling model with elastic demand. *Journal of Economics* 102 (2), 157–169.
- d’Aspremont, C., Gabszewicz, J. J., Thisse, J. F., 1979. On hotelling’s ‘stability in competition’. *Econometrica* 47 (5), 1145–1150.
- Douglas, N. J., 1987. A welfare assessment of transport deregulation: The case of the express coach market in 1980. Gower, Aldershot.

- Gu, Y., Wenzel, T., 2012. Price-dependent demand in spatial models. DICE Discussion Paper 43, Heinrich Heine University, Düsseldorf.
- Hotelling, H., 1929. Stability in competition. *The Economic Journal* 39 (153), 41–57.
- Puu, T., 2002. Hotelling's "ice cream dealers" with elastic demand. *Annals of Regional Science* 36 (1), 1–17.
- Sanner, H., 2005. Instability in competition: Hotelling reconsidered. *Diskussionsbeitrag* 79, University of Potsdam.
- Small, K. A., 1982. The scheduling of consumer activities: work trips. *American Economic Review* 72 (3), 467–479.
- van Reeveen, P., Janssen, M. C. W., 2006. Stable service patterns in scheduled transport competition. *Journal of Transport Economics and Policy* 49 (1), 135–160.