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S_4 as a natural flavor symmetry for lepton mixing

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Group theoretical arguments seem to indicate the discrete symmetry S_4 as the minimal flavor symmetry compatible with tribimaximal neutrino mixing. We prove in a model-independent way that indeed S_4 can realize exact tribimaximal mixing through different symmetry breaking patterns. We present two models in which lepton tribimaximal mixing is realized in different ways and for each one we discuss the superpotential that leads to the correct breaking of the flavor symmetry.

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I. INTRODUCTION

Harrison, Perkins, and Scott (HPS) [1] proposed the so-called tribimaximal mixing matrix

$$U_{\rm TB} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \tag{1}$$

This matrix keeps in surprising agreement with experimental data [2]. Lots of theoretical models have been done to explain the mixing matrix of Eq. (1) by means of non-Abelian flavor symmetry, such as S_3 [3–13], A_4 [14–29], T' [30–34], S_4 [35–39], and Δ (27) [40–43]. The non-Abelian discrete groups have irreducible representations of dimension bigger than one [44]. The most interesting case arises when the group contains a triplet as irreducible representation, allowing one to embed the observed three generations of fermions.

Let us consider a group G and one of its subgroups G'. Then an irreducible representation r of G decomposes into the irreducible representations of G' as $r = r_1 + r_2 + \cdots$. We define $U_{G'}^{r_i}$ the projector of r into r_i and we denote with $U_{G'}$ the collection of projectors that decomposes the representations of G according to G'.

When a non-Abelian discrete group G is broken to one of its subgroup G' the projector $U_{G'}$ that decomposes the representations of G according to G' can be fixed and are completely model independent. This is the case, for example, of A_4 broken to Z_3: the triplet representation of A_4 is sent to the one-dimensional representations of Z_3, 1, $1'$, $1''$. In fact, if we choose for A_4 the basis in which the generator T of its Z_3 subgroup is given by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2}$$

the A_4 triplet representation decomposes into the singlet representations of Z_3 through the matrix U_ω defined as

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \tag{3}$$

On the contrary, the one-dimensional representations of A_4 coincide with the corresponding ones of Z_3. A good candidate to give a tribimaximal (TBM) is a discrete group G that has a triplet representation, at least two subgroups, G' that decomposes according to $U_{G'}$, and G'' that decomposes according to $U_{G''}$. It is necessary to have at least two different subgroups of G to obtain a lepton mixing matrix different from the identity: if G were broken to the same subgroup G' both in the charged lepton and in the neutrino sector the lepton mixing matrix would be given by $U_{\rm lep} = U_{G'}^\dagger U_{G'} = I$. On the other hand when G is broken in two different ways in the charged and neutral lepton sectors, such misalignment gives large angles in the lepton mixing matrix.

A priori A_4 seems to be a good candidate because it is the smallest discrete group that contains a triplet as irreducible representation. Furthermore it has two different subgroups, Z_3 and Z_2. However, while the transformation associated to Z_3 is fixed and model independent, the one associated to Z_2 is model dependent [45]. A similar analysis done with the discrete symmetry T' leads to the same conclusion [30]. This means that A_4 and T' yield exact or approximate TBM only assuming a fine-tuning in the parameters of the Yukawa Lagrangian or a particular model realization. We mention that by assuming further constraints, also models based on S_3 can yield an approximate TBM, although its largest irreducible representation is a doublet and not a triplet.

It has been recently claimed [46] that the minimal flavor symmetry naturally related to the tribimaximal mixing is S_4, the permutation symmetry of four objects. The author of [46] proved this through group theoretical arguments without entering into the details of a concrete model realization. In this paper we reconsidered S_4 and its subgroups. We have found that S_4 is able to reproduce TBM following two different symmetry breaking patterns. We have built two different models that realize TBM through the two patterns dictated by the group analysis considerations. Finally we discuss the possible superpotential that can break S_4 in the correct way.

II. THE DISCRETE SYMMETRY GROUP S_4 AS THE ORIGIN OF TBM

A. The group S_4

The discrete group S_4 is given by the permutations of four objects and it is composed of 24 elements. It can be defined by two generators S and T that satisfy

$$S^4 = T^3 = 1, \qquad ST^2S = T. \tag{4}$$

The 24 elements of S_4 belong to five classes

$\mathcal{C}_1$: I;

$\mathcal{C}_2$: $S^2, TS^2T^2, S^2TS^2T^2$;

$\mathcal{C}_3$: $T, T^2, S^2T, S^2T^2, STST^2, STS, S^2TS^2, S^3TS$;

$\mathcal{C}_4$: $ST^2, T^2S, TST, TSTS^2, STS^2, S^2TS$;

$\mathcal{C}_5$: $S, TST^2, ST, TS, S^3, S^3T^2$. (5)

The elements of $\mathcal{C}_{2,4}$ define two different sets of Z_2 subgroups of S_4, the ones of the class $\mathcal{C}_3$, a set of Z_3 Abelian discrete symmetries, and those belonging to $\mathcal{C}_5$, a set of Z_4 Abelian discrete symmetries. The S_4 irreducible representations are two singlets, 1_1, 1_2, one doublet, 2, and two triplets, 3_1 and 3_2. We adopt the following basis:

$$S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad T = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \tag{6}$$

for the doublet representation and

$$S_{+,-} = \pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{7}$$

for the triplet representations. Clearly the generators (S_+, T) and (S_-, T) define the two triplet representations 3_1, 3_2 respectively. All the product rules can be straightforwardly derived. We refer the reader to the product rules reported in [36].

B. S_4 symmetry breaking patterns: generic case

We have seen in the introduction that given a discrete non-Abelian group G a predictive lepton mixing matrix may be obtained if G is broken to one of its subgroups, with the subgroup preserved in the charged lepton sector different from the subgroup preserved in the neutrino sector.

We disregard therefore the case when S_4 is completely broken in one of the two sectors. At the same time, if the left-handed leptons transform nontrivially under S_4, the case of S_4 unbroken in one sector is ruled out. Indeed in this case we could choose for lepton families the S_4 triplet representation or the singlet plus doublet representations. With these choices the requirement of unbroken S_4 would lead to a diagonal mass matrix with at least two degenerate states—namely the doublet.

As consequence if S_4 is broken to its subgroups G' in the charged lepton sector, in the neutrino sector it has to be broken to another subgroup $G'' \neq G'$. The couple (G', G'') identifies a possible symmetry breaking pattern. In this notation the lepton mixing matrix is given by

$$U_{\text{lep}} = U_l^\dagger U_\nu = U_{G'}^\dagger U_{G''}, \tag{8}$$

with $U_{G'}$, $U_{G''}$ being the projectors that decompose the representations of S_4 into the representations of G', G'' respectively.

S_4 contains a non-Abelian subgroup S_3, the permutation group of three objects composed by six elements. The elements of S_4 that belong to S_3 correspond to C_1, T, and T^2 of C_3 and $TSTS^2$, STS^2, S^2TS of C_4. Furthermore S_4 contains the Abelian subgroups Z_2, Z_3, Z_4 corresponding to the elements of the classes $\mathcal{C}_{2,4}$, $\mathcal{C}_3$, and $\mathcal{C}_5$ respectively. The only representation that breaks S_4 to S_3 is the triplet 3_1. The reason is the following. The six elements that define S_3 belonging to S_4 are I, T, T^2, $TSTS^2$, STS^2, S^2TS, where S and T are defined in Eq. (7). When a triplet $\phi_1 \sim 3_1$ develops vacuum expectation value (VEV) as (1, 1, 1), all the S_3 elements above are preserved. On the contrary, when a triplet $\phi_2 \sim 3_2$ develops VEV as (1, 1, 1), only the three elements that define Z_3 are preserved—I, T, T^2—while $TSTS^2$, STS^2, S^2TS built according to Eq. (7) are broken.

The representations of S_3 are two singlet, 1_1 and 1_2, and a doublet, 2. In general if S_4 is broken to S_3 the representations of S_4 would transform under S_3 according to

$$\begin{gathered} 3_1 \to 1_1 + 2, \qquad 3_2 \to 1_2 + 2, \qquad 2 \to 2, \\ 1_1 \to 1_1, \qquad 1_2 \to 1_2. \end{gathered} \tag{9}$$

When S_4 is broken to S_3, a triplet of S_4, $F \sim (F_1, F_2, F_3) \sim 3_1$, will decompose under S_3 as a singlet plus a doublet $F(3_1) \to \psi_0(1_1) + \psi(2)$.

The eigenvector that identifies the singlet is given by

$$\frac{1}{\sqrt{3}}(1, 1, 1) \to \psi_0 = \frac{1}{\sqrt{3}}(F_1 + F_2 + F_3).$$

Since S_3 is not broken the doublet components are degenerate and the corresponding eigenvectors are identified up to an arbitrary rotation. This arbitrariness reflects the arbitrary freedom we have in fixing the doublet basis in an independent way with respect to the triplet basis. Indeed in the basis we have chosen the doublet reads as

$$\psi = \begin{pmatrix} (F_3 - F_2)/\sqrt{2} \\ (2F_1 - F_2 - F_3)/\sqrt{6} \end{pmatrix}, \tag{10}$$

and therefore we can rewrite

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \times \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_2 \end{pmatrix}, \qquad (11)$$

where θ is the arbitrary rotation in the doublet component space $\psi \sim (\psi_1, \psi_2)$ and where we have assumed that the second eigenvector corresponds to the singlet. The reason for the choice of this particular basis is very simple: in the limit in which θ goes to zero we recover the TBM matrix. However θ is undetermined as long as S_3 is unbroken. In the basis given by Eq. (11) the generic mass matrix for $\Psi \sim (\psi_1, \psi_0, \psi_2)$ is given by

$$M_\Psi = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & x \end{pmatrix}, \qquad (12)$$

with ψ_1 and ψ_2 degenerate as expected. If we now assume that S_3 is broken only in the doublet subspace M_Ψ becomes

$$M_\Psi = \begin{pmatrix} x_1 & 0 & x_3 \\ 0 & y & 0 \\ x_3 & 0 & x_2 \end{pmatrix}. \qquad (13)$$

It is clear that if $x_3 = 0$, then $\theta = 0$ and we are left with three nondegenerate eigenstates and the relation between the original S_4 triplet and the mass eigenstates is given by

$$F = U_{\mathrm{TB}} \cdot \Psi. \qquad (14)$$

$x_3 = 0$ is realized by requiring that S_3 in the doublet subspace is broken to Z_2 identified in the specific basis we have chosen by the S generator.[1]

C. S_4 symmetry breaking patterns: realizing exact TBM

We now assume that $F \sim L$ with L being the left-handed lepton doublets and for the moment we leave undetermined the transformation properties under S_4 of the electroweak $SU(2)$ singlets.

The first case we consider is the symmetry breaking pattern (S_3, G''), that means that we start by breaking S_4 into S_3 in the charged lepton sector while we still do not know which is its corresponding S_4 subgroup in the neutrino sector. Applying the general results obtained in Sec. II B we conclude that $M_l M_l^\dagger$ is diagonalized by U_{S_3} given by

$$U_{S_3} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \times \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \qquad (15)$$

that would lead to the wrong relation $m_e = m_\tau$. If we now break this degeneracy as sketched in Sec. II B, from Eq. (14) we would obtain $U_l = U_{\mathrm{TBM}}$ and $U_{\mathrm{lep}} = U_{\mathrm{TBM}}^T U_\nu$.

To cure this problem we could require that the neutrino mass matrix were diagonalized by $U_{\mathrm{TBM}} U_{\mathrm{TBM}}$ in order to reproduce the TBM through $U_{\mathrm{lep}} = U_{\mathrm{TBM}}^T U_{\mathrm{TBM}} U_{\mathrm{TBM}} = U_{\mathrm{TBM}}$. However there is no G'' subgroup of S_4 that yields $U_{G''} = U_{\mathrm{TBM}} U_{\mathrm{TBM}}$ and therefore exact TBM cannot be obtained according to Eq. (8). In fact the most general neutrino Majorana mass matrix diagonalized by $U_{\mathrm{TBM}} U_{\mathrm{TBM}}$ should take the following form:

$$\begin{aligned} &(U_{\mathrm{TBM}} U_{\mathrm{TBM}})^T m_\nu U_{\mathrm{TBM}} U_{\mathrm{TBM}} \\ &= m_\nu^{\mathrm{diag}} \to m_\nu \\ &\sim \begin{pmatrix} a & b & c \\ b & a + \beta_1 b + \gamma_1 c & a + \beta_2 b + \gamma_2 c \\ c & a + \beta_2 b + \gamma_2 c & a + \beta_3 b + \gamma_3 c \end{pmatrix}, \end{aligned} \qquad (16)$$

where $\beta_{1,2,3}$ and $\gamma_{1,2,3}$ are fixed coefficients. By applying all the elements of S_4, excluding the identity, according to $G^T m_\nu G$ we discover that for all of them it holds $G^T m_\nu G \neq m_\nu$. This means there is no subgroup of S_4 that leads to Eq. (16) in the basis we have chosen.

On the other hand we could require one to break the degeneracy $m_e = m_\tau$ breaking S_3 into Z_2 not only in the doublet subspace but also in the singlet one. This would mean that $M_l M_l^\dagger$ after the U_{S_3} rotation and the breaking $S_3 \to Z_2$ would read as

$$\begin{pmatrix} x_1 & 0 & 0 \\ 0 & y & x_3 \\ 0 & x_3 & x_2 \end{pmatrix}. \qquad (17)$$

It is clear that in this case the final U_l would depend on the mass parameters, and therefore the correct lepton mixing could be obtained only through a fit. We conclude that the symmetry breaking pattern with S_4 broken into S_3 in the charged lepton sector is ruled out.

We now analyze what happens considering the breaking pattern (Z_3, G''). As in the previous case the subgroup G'', corresponding to the neutrino sector, is undetermined. As already said in S_4 the breaking into Z_3 is realized when a triplet 3_2 develops a VEV in the direction (1, 1, 1) We expect that if we break S_4 into Z_3 in the charged lepton sector the charged lepton mixing matrix will send the S_4 triplet (L_1, L_2, L_3) in the Z_3 eigenstates, 1, 1′, 1″. Indeed the mixing matrix responsible for this rotation is the U_ω

[1]From the point of view of model realization the assumption that S_3 is broken only in the doublet component space is not different from assuming that S_4 is broken to different subgroups in the charged lepton sector and in the neutrino one. Indeed we will see in Sec. III B how singlet and doublet sectors can be easily separated.

defined in Eq. (3). Given U_ω the correct TBM can be reproduced if the $U_{G''}$ of Eq. (8) is given by

$$U_\nu = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \end{pmatrix}, \tag{18}$$

or in other words if the neutrino mass matrix m^ν is diagonalized by U_ν and it has the following form:

$$m^\nu = \begin{pmatrix} a & 0 & 0 \\ 0 & c & b \\ 0 & b & c \end{pmatrix}. \tag{19}$$

The matrix form of Eq. (19) is recovered by requiring the invariance of m^ν under the $G'' = Z_2 \times Z_2$ subgroup of S_4 associated to the element TST of the class $\mathcal{C}_4$ and to the element S^2 of the class C_2 respectively. This breaking pattern corresponds to the usual one used in models based on A_4—where the breaking is given by $A_4 \to Z_2$. However we stress that in the context of S_4 we have obtained TBM only according to group theory considerations.

If we consider now the case (Z_2, G'') we discover that S_4 behaves exactly as A_4 and exact TBM cannot be recovered. For a detailed analysis we refer the reader to the appendix of [45].

In the case (Z_4, G'') we found that the charged lepton mass matrix $M_l M_l^\dagger$ has a maximal angle. Taking, for example, the Z_4 associated to the S generators of S_4 we have that

$$S^T(M_l M_l^\dagger)S = (M_l M_l^\dagger) \to M_l M_l^\dagger = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & ix_3 \\ 0 & -ix_3 & x_2 \end{pmatrix}, \tag{20}$$

that gives rise to a maximal θ_{23}^l and three distinct eigenvalues $(x_1,\ x_2 - x_3,\ x_2 + x_3)$. In this case exact TBM would be recovered if U_ν would be given by a rotation in the plane θ_{12} characterized by $\tan\theta_{12}^2 = 1/2$ or in other words if m_ν would present the following form:

$$m_\nu = \begin{pmatrix} a & b & 0 \\ b & a + b/\sqrt{2} & 0 \\ 0 & 0 & c \end{pmatrix}. \tag{21}$$

However it is easy to check that in the basis we are considering there is no G' subgroup of S_4 that is left invariant by a m_ν of the form given in Eq. (21). Therefore we conclude that even the breaking of S_4 into Z_4 in the charged lepton sector does not lead to exact TBM.

So far we have considered all the possible cases in which the subgroup fixed in the charged lepton sector gives rise to a nondiagonal structure to the charged lepton mass matrix M_l. We could ask if there is any way to realize a diagonal M_l with three different mass eigenvalues. Indeed this is easily realized breaking S_4 to $Z_2 \times Z_2$ corresponding to the elements S^2 and T^2S^2T of the class $\mathcal{C}_2$. It is obvious that if the charged lepton mass matrix is diagonal all the mixing structure arises by the neutrino sector. Therefore the last symmetry breaking pattern we are going to consider is $(Z_2 \times Z_2, S_3)$ that means that we break S_4 into S_3 in the neutrino sector and then S_3 into Z_2 to have three nondegenerate mass eigenstates, as we have seen in the generic discussion in sec. II B that leaded to Eq. (14).

Contrary to what happened in the charged lepton sector, the breaking pattern $S_4 \to S_3 \to Z_2$ in the neutrino sector leads to exact TBM. Indeed from Eqs. (11)–(14) we have that $U_{\rm lep} = U_{\rm TBM}$ being the charged lepton mass matrix diagonal.

In conclusion of this first model-independent part, we have seen that on the basis of theoretical considerations based on the subgroups of S_4, the flavor symmetry S_4 has two symmetry breaking patterns giving exact TBM in the lepton sector. In the next section we will present a model realization for each breaking pattern. In the last section we build the corresponding superpotential responsible for the correct S_4 symmetry breaking patterns.

III. MODEL REALIZATION

In the standard model (SM), the most general way to introduce Majorana neutrino mass terms is by means of five-dimension Weinberg operators. These operators could arise from type-I as well as type-II or type-III seesaw mechanism. The first mechanism is based on the exchange of a heavy fermion $SU(2)$ singlet. In the second mechanism the neutrino Majorana mass term arises from the exchange of $SU(2)$ Higgs triplet while in the third mechanism the heavy particle integrated out is an isotriplet fermion. In principle using an effective operator approach we should consider all the mechanisms together. In fact ultraviolet realizations, like minimal $SO(10)$ grand unified theory (GUT) models, give rise for examples both to type-I and type-II contributions. Other nonminimal GUT scenarios could give rise to the type-III contribution as well. On the other hand we lack experimental motivations to extend the SM both with right-handed neutrinos, $SU(2)$ scalar triplets, and $SU(2)$ fermion triplets and therefore models with just one seesaw type contribution are allowed. Below we will study two models with, respectively, type-II and type-I plus type-II seesaw. As we will see in Sec. III B in the second model we present, the only type-I contribution arising by a S_4 doublet of right-handed neutrinos cannot fit the correct $\Delta m_{\rm sol}^2$. Therefore a further contribution is needed to the neutrino masses and in the model studied this is obtained from type-II seesaw. From a point of view of model realization, type-III seesaw is very similar to the type-I, so our model can be equivalently described making use of type-III plus type-II seesaw. For simplicity we have chosen the case with type-I plus type-II seesaw without entering into the details of the type-III version. It is worth emphasizing that type-I and type-III realizations present different phenomenological implications, whose study

TABLE I. Matter and scalar content of model I. The lepton mixing matrix is TBM.

	$\hat{L}$	$\hat{E}^c$	$\hat{H}^d$	$\hat{\Phi}$	$\hat{\sigma}$	$\hat{\phi}_1$	$\hat{\phi}_2$	$\hat{\Delta}$
$SU(2)$	2	1	2	3	1	1	1	1
S_4	3_1	3_1	1	1	1	3_1	3_2	3_1
Z_5	1	ω_5^4	1	1	ω_5	ω_5	ω_5	1

goes beyond the scope of this paper and could be studied elsewhere.

A. Model I: $S_4 \to Z_3$ & $S_4 \to Z_2$

The first model we consider reproduces TBM through the breaking of S_4 into Z_3 and Z_2 in the charged lepton and neutrino sector, respectively. We assume our model to be supersymmetric. Matter and scalar supermultiplets are reported in Table I. The scalar supermultiplets charged under S_4, that in the following we will identify as flavons, are electroweak $SU(2) \times U(1)$ singlets. Therefore the Yukawa superpotential $\mathcal{W}_Y$ of Eq. (22) includes effective operators of dimension 5. Λ is the cutoff of the model and an extra Z_5 symmetry has been introduced to separate the charged lepton sector from the neutrino one.

In Table I we have omitted the supermultiplets $\hat{H}^u$ and $\hat{\hat{\Phi}}$, doublet and triplet of $SU(2)$, respectively, necessary to give mass to the up-quarks and to cancel anomalies in a realistic model.

The full leading order $S_4 \times Z_5$ Yukawa superpotential $\mathcal{W}_Y$ is given by

$$\mathcal{W}_Y = \frac{1}{\Lambda} y_0 (\hat{L}\hat{E}^c)_1 \hat{\sigma} \hat{H}^d + \frac{1}{\Lambda} y_s (\hat{L}\hat{E}^c)_{3_1} \hat{\phi}_1 \hat{H}^d + \frac{1}{\Lambda} y_a (\hat{L}\hat{E}^c)_{3_2} \hat{\phi}_2 \hat{H}^d + y_1^\nu (\hat{L}\,\hat{L})_1 \hat{\Phi} + \frac{1}{\Lambda} y_2^\nu (\hat{L}\,\hat{L})_{3_1} \hat{\Delta}\, \hat{\Phi}. \tag{22}$$

When the S_4 triplet and doublet flavons align as

$$\langle \phi_1 \rangle \sim \langle \phi_2 \rangle \sim (1, 1, 1) \qquad \langle \Delta \rangle \sim (1, 0, 0), \tag{23}$$

the charged lepton and neutrino mass matrices present the usual forms

$$M_l = \begin{pmatrix} h_0 & h_1 & h_2 \\ h_2 & h_0 & h_1 \\ h_1 & h_2 & h_0 \end{pmatrix} \qquad m^\nu = \begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix} \tag{24}$$

that satisfy

$$U_\omega M_l U_\omega^\dagger = M_l^{\text{diag}}, \qquad U_\nu^T m_\nu U_\nu = m_\nu^{\text{diag}}, \tag{25}$$

with U_ω and U_ν given in Eq. (3) and (18) respectively. TBM is obtained as usual by $U_{\text{TB}} = U_\omega U_\nu$. The mass eigenvalues for the charged lepton are given by

$$m_e = h_0 + h_1 + h_2, \qquad m_\mu = h_0 + h_1\omega^2 + h_2\omega, \qquad m_\tau = h_0 + h_1\omega + h_2\omega^2, \tag{26}$$

and for the neutrino by $(a + b,\ a,\ b - a)$). By assuming that the flavon VEVs are of order $\sim\lambda^2\Lambda$ with λ the Cabibbo angle, the deviations from TBM induced by the next-to-leading order corrections to the Yukawa superpotential slightly modify lepton mixing keeping it still in agreement with neutrino data. Notice that the VEV alignments

$$\langle \phi_1 \rangle \sim \langle \phi_2 \rangle \sim (1, 1, 1) \tag{27}$$

preserve the Z_3 subgroup of S_4 associated to the element T because $\phi_2 \sim 3_2$—as we have already said $\phi_1 \sim 3_1$ alone with VEV alignment (1, 1, 1) preserves the S_3 subgroup of S_4, while the 3_2 triplet representation does not. On the contrary the VEV alignments

$$\langle \varphi \rangle \sim (0, 1) \qquad \langle \Delta \rangle \sim (1, 0, 0), \tag{28}$$

preserve the $Z_2 \times Z_2$, where the first Z_2 is associated to the element TST while the second Z_2 to the element S^2 that in the doublet and triplet representation reads, respectively, as

$$TST = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad TST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{29}$$

$$S^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{30}$$

B. Model II: $S_4 \to Z_2 \times Z_2$ and $S_4 \to S_3$

The second model we describe realizes TBM through the sequential breaking of S_4 into S_3 and then into Z_2 in the neutrino sector and the breaking of S_4 into two different $Z_2 \times Z_2$ in the charged lepton sector. The step through S_3 is crucial: if we broke S_4 directly into Z_2 in the neutrino sector we would find a generic neutrino mass matrix $\mu - \tau$ invariant not diagonalized by TBM. On the contrary, in the model that we present, the step through S_3 leads to a neutrino mass matrix m^ν which is $\mu - \tau$ invariant and satisfies the relation $m_{11}^\nu = m_{22}^\nu + m_{23}^\nu - m_{13}^\nu$ that ensures TBM diagonalization. We will see that the key ingredient in building the correct m^ν is the introduction of the right-handed neutrinos transforming as a doublet of S_4. The reason is very simple. Recovering Eqs. (12) and (13) in Sec. II Bwe have that the neutrino mass matrix obtained when S_4 is broken to S_3 has to be given by

$$m_{S_3}^\nu \sim \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & x \end{pmatrix}, \tag{31}$$

and then when S_3 is broken to Z_2 only in the doublet subspace $m_{S_3}^\nu$ becomes

$$m^\nu_{\rm diag} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & x+z \end{pmatrix}. \quad (32)$$

If we now write $m^\nu \sim U_{\rm TBM} m^\nu_{\rm diag} U^T_{\rm TBM}$ we see that the z contribution has to have the following structure:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & z/2 & -z/2 \\ 0 & -z/2 & z/2 \end{pmatrix} \sim z/2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot (0, 1, -1). \quad (33)$$

In the S_4 basis we are working, the vector $(0, 1, -1)^T$ can be obtained through the coupling $(3_1 \cdot 2_H)_{3_1}$ with $\langle 2_H \rangle \sim (1, 0)$. This means that the contribution of Eq. (33) may be obtained by an effective operator

$$\frac{1}{\Lambda}(L\phi)_{3_1}(L\phi)_{3_1} \quad (34)$$

with L, ϕ transforming under S_4 as $L \sim 3_1$ and $\phi \sim 2$. This effective operator is obtained by integrating out a S_4 doublet of right-handed neutrinos. Therefore their introduction in our model realization is crucial.

As in the case of the model presented in Sec. III A we assume our model to be supersymmetric and the flavon supermultiplets electroweak singlets. Matter and scalar supermultiplets are reported in Table II. As done in Sec. III A we have omitted the supermultiplet $\hat{\bar{\Phi}}$, triplet of $SU(2)$, necessary to cancel anomalies. Two extra discrete Abelian symmetries, Z_3 and Z_5, have been introduced in order to avoid interferences between the sectors.

The full leading order $S_4 \times Z_3 \times Z_5$ invariant Yukawa superpotential is given by

$$\mathcal{W}_Y = \frac{1}{\Lambda} y_s (\hat{L}\hat{l}^c)_1 \hat{\sigma} \hat{H}^d + \frac{1}{\Lambda} y_d (\hat{L}\hat{l}^c)_2 \hat{\phi} \hat{H}^d + y_1 (\hat{L}\,\hat{L})_1 \hat{\Phi} + \frac{1}{\Lambda} y_2 (\hat{L}\,\hat{\Delta})_2 \hat{N}^c \hat{H}^u + M_d \hat{N}^c \hat{N}^c + \tilde{y}_N \hat{\varphi} \hat{N}^c \hat{N}^c, \quad (35)$$

TABLE II. Matter and scalar content of model II. The lepton mixing matrix is TBM.

	$\hat{L}$	$\hat{l}^c$	$\hat{N}^c$	$\hat{H}^u$	$\hat{H}^d$	$\hat{\Phi}$	$\hat{\Delta}$	$\hat{\sigma}$	$\hat{\phi}$	$\hat{\varphi}$
$SU(2)$	2	1	1	2	2	3	1	1	1	1
S_4	3_1	3_1	2	1	1	1	3_1	1	2	2
Z_3	ω^2	1	1	1	ω^2	ω	ω	1	1	1
Z_5	1	ω_5^3	1	1	1	1	1	ω_5^2	ω_5^2	1

where as usual Λ is the cutoff of the model and all the Yukawa terms are of order 4 with the exception of the ones involving right-handed neutrinos. We assume that the flavons Δ and φ, triplet and doublet under S_4 respectively, align as

$$\langle \Delta \rangle \sim (1, 1, 1) \qquad \langle \varphi \rangle \sim (0, 1). \quad (36)$$

The VEV $\langle \Delta \rangle$ preserves S_3 as has been already discussed in Sec. II B. The VEV $\langle \varphi \rangle$ preserves the S generators of S_3 that coincides with the S generator of S_4 of the doublet representation—Eq. (6).

The doublet ϕ does not align and develops VEV as $\langle \phi \rangle \sim (v_1, v_2)$—this means that S_4 is broken to $Z_2 \times Z_2$ corresponding to the elements S^2 and TS^2T^2 of $\mathcal{C}_2$ that in the 3_1 triplet representation read as

$$S^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad TS^2T^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (37)$$

For the charged lepton sector we have

$$M_l = \frac{1}{\Lambda} v^d \begin{pmatrix} y'_s v_\sigma - 2y''_d v_2^\phi & 0 & 0 \\ 0 & y'_s v_\sigma + y'_d v_1^\phi + y''_d v_2^\phi & 0 \\ 0 & 0 & y'_s v_\sigma - y'_d v_1^\phi + y''_d v_2^\phi \end{pmatrix}, \quad (38)$$

with $v_\sigma = \langle \sigma \rangle$, $v^\phi_{1,2} = \langle \phi_{1,2} \rangle$, $v^d = \langle H_0^d \rangle$, and the product factors absorbed in y'_s and y'_d, y''_d. The neutrino mass matrix gets contributions both from type-I and type-II seesaw

$$m^\nu = m_{LL} - m_D \cdot \frac{1}{M_N} \cdot m_D^T, \quad (39)$$

where $m_{LL} = y_1 v_\Phi \cdot I$ with $v_\Phi = \langle \Phi \rangle$ and

$$m_D = y_2 \frac{v^\Delta}{\Lambda} v^u \begin{pmatrix} 0 & -2\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}, \quad M_N = \begin{pmatrix} M_d + V_\varphi & 0 \\ 0 & M_d - V_\varphi \end{pmatrix}, \quad (40)$$

with $v^u = \langle H_0^u \rangle$, $v^\Delta_{1,2,3} = v^\Delta$, and $V_\varphi = \tilde{y}_N \langle \varphi_2 \rangle / \sqrt{2}$. After the usual seesaw mechanism the Majorana neutrino mass matrix is given by

$$m^\nu = \begin{pmatrix} a+\frac{2}{3}b & -\frac{1}{3}b & -\frac{1}{3}b \\ -\frac{1}{3}b & a+\frac{1}{6}b+\frac{1}{2}c & \frac{1}{6}b-\frac{1}{2}c \\ -\frac{1}{6}b & \frac{1}{6}b-\frac{1}{2}c & a+\frac{1}{6}b+\frac{1}{2}c \end{pmatrix}, \quad (41)$$

with

$$a = y_1 v_\Phi, \qquad b = -y_2^2\left(\frac{v^\Delta}{\Lambda}\right)^2 \frac{(v^u)^2}{M_d - V_\varphi}, \qquad c = -y_2^2\left(\frac{v^\Delta}{\Lambda}\right)^2 \frac{(v^u)^2}{M_d + V_\varphi}. \quad (42)$$

The neutrino mass matrix m^ν is diagonalized by TBM and its eigenvalues are $(a+b, a, a+c)$ that can accommodate experimental neutrino mass splitting data being expressed in terms of three independent combinations of the parameters of the model. As in the model discussed in Sec. III A by assuming the flavon VEVs of order $\sim\lambda^2\Lambda$ next to leading order corrections to the Yukawa superpotential produce small deviations from TBM that are still compatible with neutrino data.

IV. REALIZING THE CORRECT VACUUM CONFIGURATIONS IN S_4

In the context of a flavor model based on non-Abelian discrete symmetry the lepton TBM is obtained thanks to specific alignments of the flavons. The so-called alignment problem in A_4 and T' has been extensively discussed in [18,21,25]. Different strategies have been used: the introduction of a soft breaking term of the flavor symmetry [25], the use of a continuous $U(1)_R$ symmetry [21] preserved by the scalar potential, and the promotion of the model to a fifth dimension [18]. In the context of S_4 in [39] the flavon superpotential was softly broken to guarantee the desired vacuum configuration.

In S_4 as well as in A_4 and T' it is impossible to build a flavon superpotential that guarantees the alignments needed. In the next sections we will show that the extra discrete Abelian symmetries introduced in Sec. III to separate the two lepton sectors are sufficient to give the correct vacuum configurations at leading order. In general next-to-leading (NLO) order contributions will shift the vacuum alignments of an amount of order v_i/Λ with v_i the generic flavon VEV. It has been shown that in order not to destroy lepton mixing, $v_i/\Lambda \sim \lambda^2$ with λ being the Cabibbo angle [18]. However a complete analysis of the NLO corrections is above the purposes of this work. Our motivation was to show two different S_4 based model realizations that at leading order (LO) provide exact TBM and to furnish two examples of the scalar potentials that could give the correct LO vacuum alignments.

TABLE III. Scalar content of model I including the flavons that contribute to the mass matrix structures and the ones the drive the correct vacuum alignments, the driving fields.

	$\hat\sigma$	$\hat\phi_1$	$\hat\phi_2$	$\hat\Delta$	$\hat\varphi$	$\hat\xi$	$\hat\eta$
$SU(2)$	1	1	1	1	1	1	1
S_4	1	3_1	3_2	3_1	2	2	2
Z_5	ω_5	ω_5	ω_5	1	1	ω_5^3	ω_5^2

A. Model I: Minimization of the potential

The complete flavor scalar content of the model is given in Table III. Thus the flavon potential is obtained by the following part of the full $S_4 \times Z_5$ superpotential

$$\mathcal{W}_Y = M_{\xi\eta}\hat\xi\,\hat\eta + \lambda_{\xi\eta}\hat\xi\,\hat\eta\,\hat\varphi + \lambda_{\sigma\eta}\hat\sigma\,\hat\eta\,\hat\eta + \lambda_{\xi\phi1}\hat\xi\hat\phi_1\hat\phi_1 + \lambda_{\xi\phi2}\hat\xi\hat\phi_2\hat\phi_2 + \lambda_{\xi\phi12}\hat\xi\hat\phi_1\hat\phi_2 + M_\Delta\hat\Delta\,\hat\Delta + M_\varphi\hat\varphi\,\hat\varphi + \lambda_{\varphi\Delta}\hat\Delta\,\hat\Delta\,\hat\varphi + \lambda_\varphi\hat\varphi\,\hat\varphi\,\hat\varphi + \lambda_\Delta\hat\Delta\,\hat\Delta\,\hat\Delta. \quad (43)$$

We assume that the flavor symmetry is broken in the supersymmetry (SUSY) limit and therefore the vacuum configuration is obtained solving the system $\partial\mathcal{W}_Y/\partial f_i = 0$, where f_i are the f components of the supermultiplets entering in Eq. (43) and i runs on all the supermultiplets. By assuming the general vacuum configuration

$$\langle\Delta\rangle = (v_1^\Delta, v_2^\Delta, v_3^\Delta), \qquad \langle\varphi\rangle = (v_1^\varphi, v_2^\varphi),$$
$$\langle\phi_1\rangle = (v_1^\phi, v_2^\phi, v_3^\phi), \qquad \langle\phi_2\rangle = (u_1^\phi, u_2^\phi, u_3^\phi), \quad (44)$$
$$\langle\xi\rangle = (u_1^\xi, u_2^\xi), \qquad \langle\eta\rangle = (z^\eta, z^\eta) \qquad \langle\sigma\rangle = v_\sigma,$$

the set of equations is given by

$$\partial W/\partial f_1^\Delta = \frac{2}{\sqrt{3}}M_\Delta v_1^\Delta - \frac{2}{\sqrt{3}}\lambda_{\Delta\varphi}v_1^\Delta v_2^\varphi + 2\lambda_\Delta v_2^\Delta v_3^\Delta = 0 \quad (45a)$$

$$\partial W/\partial f_2^\Delta = \frac{2}{\sqrt{3}}M_\Delta v_2^\Delta + \frac{1}{\sqrt{3}}\lambda_{\Delta\varphi}v_2^\Delta(v_2^\varphi + \sqrt{3}v_1^\varphi) + 2\lambda_\Delta v_1^\Delta v_3^\Delta = 0 \quad (45b)$$

$$\partial W/\partial f_3^\Delta = \frac{2}{\sqrt{3}}M_\Delta v_3^\Delta + \frac{1}{\sqrt{3}}\lambda_{\Delta\varphi}v_3^\Delta(v_2^\varphi - \sqrt{3}v_1^\varphi) + 2\lambda_\Delta v_1^\Delta v_2^\Delta = 0 \quad (45c)$$

$$\partial W/\partial f_1^{\varphi} = \sqrt{2}M_{\varphi}v_1^{\varphi} + \frac{\lambda_{\xi\eta}}{2}(u_2^{\xi}z_1^{\eta} + u_1^{\xi}z_2^{\eta}) + \frac{\lambda_{\Delta}}{2}[(v_2^{\Delta})^2 - (v_3^{\Delta})^2] = 0 \tag{45d}$$

$$\partial W\partial f_2^{\varphi} = \sqrt{2}M_{\varphi}v_2^{\varphi} + \frac{\lambda_{\xi\eta}}{2}(u_1^{\xi}z_1^{\eta} - u_2^{\xi}z_2^{\eta}) + \frac{\lambda_{\Delta}}{2\sqrt{3}}[-2(v_1^{\Delta})^2 + (v_2^{\Delta})^2 + (v_3^{\Delta})^2] = 0 \tag{45e}$$

$$\partial W/\partial f_1^{\eta} = \frac{M_{\xi\eta}}{\sqrt{2}}u_1^{\xi} + \frac{\lambda_{\xi\eta}}{2}(v_1^{\varphi}u_2^{\xi} + v_2^{\varphi}u_1^{\xi}) + \sqrt{2}\lambda_{\sigma\eta}v_{\sigma}z_1^{\eta} = 0 \tag{45f}$$

$$\partial W/\partial f_2^{\eta} = \frac{M_{\xi\eta}}{\sqrt{2}}u_2^{\xi} + \frac{\lambda_{\xi\eta}}{2}(v_1^{\varphi}u_1^{\xi} - v_2^{\varphi}u_2^{\xi}) + \sqrt{2}\lambda_{\sigma\eta}v_{\sigma}z_2^{\eta} = 0 \tag{45g}$$

$$\partial W/\partial f^{\sigma} = \frac{\lambda_{\sigma\eta}}{\sqrt{2}}[(z_1^{\eta})^2 + (z_2^{\eta})^2] = 0 \tag{45h}$$

$$\partial W/\partial f_1^{\xi} = \frac{1}{\sqrt{2}}M_{\xi\eta}z_1^{\eta} + \frac{1}{2}\lambda_{\xi\eta}(z_1^{\eta}v_2^{\varphi} + z_2^{\eta}v_1^{\varphi}) + \frac{1}{2}\lambda_{\xi\phi 1}[(v_2^{\phi})^2 - (v_3^{\phi})^2] + \frac{1}{2}\lambda_{\xi\phi 2}[(u_2^{\phi})^2 - (u_3^{\phi})^2] + \frac{1}{2\sqrt{3}}\lambda_{\xi\phi 12}(2v_1^{\phi}u_1^{\phi} - v_2^{\phi}u_2^{\phi} - v_3^{\phi}u_3^{\phi}) = 0 \tag{45i}$$

$$\partial W/\partial f_2^{\xi} = \frac{1}{\sqrt{2}}M_{\xi\eta}z_2^{\eta} + \frac{1}{2}\lambda_{\xi\eta}(z_1^{\eta}v_1^{\varphi} - z_2^{\eta}v_2^{\varphi}) + \frac{1}{2\sqrt{3}}\lambda_{\xi\phi 1}[-2(v_1^{\phi})^2 + (v_2^{\phi})^2 + (v_3^{\phi})^2] + \frac{1}{2\sqrt{3}}\lambda_{\xi\phi 2}[-2(u_1^{\phi})^2 + (u_2^{\phi})^2 + (u_3^{\phi})^2] + \frac{1}{2}\lambda_{\xi\phi 12}(v_2^{\phi}u_2^{\phi} - v_3^{\phi}u_3^{\phi}) = 0 \tag{45j}$$

$$\partial W/\partial f_1^{\phi_1} = \frac{1}{\sqrt{3}}(\lambda_{\xi\phi 12}u_1^{\phi}u_1^{\xi} - 2\lambda_{\xi\phi 1}u_2^{\xi}v_1^{\phi}) = 0 \tag{45k}$$

$$\partial W/\partial f_2^{\phi_1} = u_1^{\xi}\left(\lambda_{\xi\phi 1}v_2^{\phi} - \frac{1}{2\sqrt{3}}\lambda_{\xi\phi 12}u_2^{\phi}\right) + u_2^{\xi}\left(\frac{\lambda_{\xi\phi 1}}{\sqrt{3}}v_2^{\phi} + \frac{1}{2}\lambda_{\xi\phi 12}u_2^{\phi}\right) = 0 \tag{45l}$$

$$\partial W/\partial f_3^{\phi_1} = u_1^{\xi}\left(-\lambda_{\xi\phi 1}v_3^{\phi} - \frac{1}{2\sqrt{3}}\lambda_{\xi\phi 12}u_3^{\phi}\right) + u_2^{\xi}\left(\frac{\lambda_{\xi\phi 1}}{\sqrt{3}}v_2^{\phi} - \frac{1}{2}\lambda_{\xi\phi 12}u_2^{\phi}\right) = 0 \tag{45m}$$

$$\partial W/\partial f_1^{\phi_2} = \frac{1}{\sqrt{3}}(\lambda_{\xi\phi 12}v_1^{\phi}u_1^{\xi} - 2\lambda_{\xi\phi 2}u_2^{\xi}u_1^{\phi}) = 0 \tag{45n}$$

$$\partial W/\partial f_2^{\phi_2} = u_1^{\xi}\left(\lambda_{\xi\phi 2}u_2^{\phi} - \frac{1}{2\sqrt{3}}\lambda_{\xi\phi 12}v_2^{\phi}\right) + u_2^{\xi}\left(\frac{\lambda_{\xi\phi 1}}{\sqrt{3}}u_2^{\phi} + \frac{1}{2}\lambda_{\xi\phi 12}v_2^{\phi}\right) = 0 \tag{45o}$$

$$\partial W/\partial f_3^{\phi_2} = u_1^{\xi}\left(-\lambda_{\xi\phi 2}u_3^{\phi} - \frac{1}{2\sqrt{3}}\lambda_{\xi\phi 12}v_3^{\phi}\right) + u_2^{\xi}\left(\frac{\lambda_{\xi\phi 2}}{\sqrt{3}}u_2^{\phi} - \frac{1}{2}\lambda_{\xi\phi 12}v_2^{\phi}\right) = 0. \tag{45p}$$

By assuming not spontaneous breaking of *CP*, Eq. (45h) implies $z_{1,2}^{\eta} = 0$. As first consequence we have that a possible solution of Eqs. (45f) and (45g) and Eqs. (45k)–(45p) is given by

$$(u_1^{\xi}, u_2^{\xi}) = (0, 0) \quad \text{and} \quad v_{\sigma} \neq 0. \tag{46}$$

By substituting $(z_1^{\eta}, z_2^{\eta}) = (0, 0)$, $(u_1^{\xi}, u_2^{\xi}) = (0, 0)$, and $v_{\sigma} \neq 0$ in the equations not yet solved it is easy to check that a possible solution for Eqs. (45a)–(45e) is given by the vacuum configuration

$$(v_1^{\varphi}, v_2^{\varphi}) = (0, v^{\varphi}) \quad \text{with} \quad v^{\varphi} = \frac{M_{\Delta}}{\lambda_{\Delta}}$$
$$(v_1^{\Delta}, v_2^{\Delta}, v_3^{\Delta}) = (v^{\Delta}, 0, 0) \quad \text{with} \quad v^{\Delta} = 6^{1/4}\frac{\sqrt{M_{\varphi}M_{\Delta}}}{\lambda_{\Delta}}. \tag{47}$$

Finally Eqs. (45i) and (45j) are solved by the vacuum configuration

$$(v_1^{\phi}, v_2^{\phi}, v_3^{\phi}) = v^{\phi}(1, 1, 1) \quad \text{and}$$
$$(u_1^{\phi}, u_2^{\phi}, u_3^{\phi}) = u^{\phi}(1, 1, 1). \tag{48}$$

The solution found is not unique but can be stabilized once

TABLE IV. Scalar content of model II including both flavon and the driving field supermultiplets.

	$\hat{\Delta}$	$\hat{\sigma}$	$\hat{\phi}$	$\hat{\varphi}$	$\hat{\bar{\sigma}}$	$\hat{\xi}$	$\hat{\eta}$
$SU(2)$	1	1	1	1	1	1	1
S_4	3_1	1	2	2	1	1	1
Z_3	ω	1	1	1	1	ω	ω^2
Z_5	1	ω_5^2	ω_5^2	1	ω_5	1	1

we add apposite SUSY soft breaking terms. In Sec. III A we have assumed that the flavon VEVs is of order $\lambda^2\Lambda$. Therefore the next-to-leading order corrections to the Yukawa superpotential induced by the driving fields are sufficiently suppressed.

B. Model II: Minimization of the potential

The complete flavor scalar content of the model is given in Table IV. Thus the flavon potential is obtained by the following part of the full superpotential

$$\mathcal{W} = \lambda_{\Delta\xi}\hat{\xi}\,\hat{\Delta}\,\hat{\Delta} + \lambda_\Delta\hat{\Delta}\,\hat{\Delta}\,\hat{\Delta} + M_\xi\hat{\xi}\,\hat{\eta} + \lambda_\xi\hat{\xi}\,\hat{\xi}\,\hat{\xi} + \lambda_\eta\hat{\eta}\,\hat{\eta}\,\hat{\eta} + M_\varphi\hat{\varphi}\,\hat{\varphi} + \lambda_\varphi\hat{\varphi}\,\hat{\varphi}\,\hat{\varphi} + \lambda_\phi\hat{\bar{\sigma}}\,\hat{\phi}\,\hat{\phi} + \lambda_\sigma\hat{\bar{\sigma}}\,\hat{\sigma}\,\hat{\sigma}. \quad (49)$$

By assuming the general vacuum configuration

$$\langle\Delta\rangle = (v_1^\Delta, v_2^\Delta, v_3^\Delta), \qquad \langle\varphi\rangle = (v_1^\varphi, v_2^\varphi),$$
$$\langle\phi\rangle = (v_1^\phi, v_2^\phi), \qquad \langle\xi\rangle = v_\xi, \qquad \langle\eta\rangle = v_\eta \quad (50)$$
$$\langle\sigma\rangle = v_\sigma, \qquad \langle\bar{\sigma}\rangle = v_{\bar{\sigma}},$$

the minimization of the scalar potential obtained in the SUSY limit gives the following set of equations

$$\begin{aligned}
\partial\mathcal{W}_Y/\partial f_1^\Delta &= \sqrt{2}\lambda_{\Delta\xi}v_\xi v_1^\Delta + \sqrt{3}3\lambda_\Delta v_2^\Delta v_3^\Delta = 0\\
\partial\mathcal{W}_Y/\partial f_2^\Delta &= \sqrt{2}\lambda_{\Delta\xi}v_\xi v_2^\Delta + \sqrt{3}\lambda_\Delta v_1^\Delta v_3^\Delta = 0\\
\partial\mathcal{W}_Y/\partial f_3^\Delta &= \sqrt{2}\lambda_{\Delta\xi}v_\xi v_3^\Delta + \sqrt{3}\lambda_\Delta v_1^\Delta v_2^\Delta = 0\\
\partial\mathcal{W}_Y/\partial f^\xi &= \sqrt{3}\lambda_{\Delta\xi}[(v_1^\Delta)^2 + (v_2^\Delta)^2 + (v_3^\Delta)^2] + M_\xi v_\eta + 3\lambda_\xi v_\xi^2 = 0\\
\partial\mathcal{W}_Y/\partial f^\eta &= M_\xi v_\xi + 3\lambda_\eta v_\eta^2 = 0\\
\partial\mathcal{W}_Y/\partial f_1^\varphi &= \sqrt{2}M_\varphi v_1^\varphi + 3\lambda_\varphi v_1^\varphi v_2^\varphi = 0\\
\partial\mathcal{W}_Y/\partial f_2^\varphi &= \sqrt{2}M_\varphi v_2^\varphi + \frac{3}{2}\lambda_\varphi[(v_1^\varphi)^2 - (v_2^\varphi)^2] = 0\\
\partial\mathcal{W}_Y/\partial f_1^\phi &= \sqrt{2}\lambda_\phi v_1^\phi v_{\bar{\sigma}} = 0\\
\partial\mathcal{W}_Y/\partial f_2^\phi &= \sqrt{2}\lambda_\phi v_2^\phi v_{\bar{\sigma}} = 0\\
\partial\mathcal{W}_Y/\partial f^\sigma &= 2\lambda_{\bar{\sigma}}v_\sigma v_{\bar{\sigma}} = 0\\
\partial\mathcal{W}_Y/\partial f^{\bar{\sigma}} &= \frac{1}{\sqrt{2}}\lambda_\phi[(v_1^\phi)^2 + (v_2^\phi)^2] + \lambda_{\bar{\sigma}}v_\sigma^2 = 0.
\end{aligned} \quad (51)$$

Discarding for the triplet and the doublets the trivial solutions that do not break S_4, the solution of the system of Eqs. (51) is given by the following vacuum configuration:

$$\begin{aligned}
&v_1^\Delta = v_2^\Delta = v_3^\Delta = v^\Delta \quad \text{with} \quad v^\Delta = \sqrt{2}\frac{\lambda_{\Delta\xi}\lambda_\eta}{\lambda_\Delta}\frac{v_\eta^2}{M_\xi}\\
&v_\xi = -3\lambda_\eta\frac{v_\eta^2}{M_\xi} \quad \text{with} \quad v_\eta^3 = -M_\xi^3\frac{\lambda_\Delta^2}{\lambda_\eta^2(2\sqrt{3}\lambda_{\Delta\xi}^3 + 27\lambda_\xi\lambda_\Delta^2)}\\
&(v_1^\varphi, v_2^\varphi) \neq (0,0) \quad \text{with} \quad \begin{cases} (0, \frac{2\sqrt{2}}{3}\frac{M_\varphi}{\lambda_\varphi})\\ (\sqrt{\frac{2}{3}}\frac{M_\varphi}{\lambda_\varphi}, -\frac{\sqrt{2}}{3}\frac{M_\varphi}{\lambda_\varphi})\\ (-\sqrt{\frac{2}{3}}\frac{M_\varphi}{\lambda_\varphi}, -\frac{\sqrt{2}}{3}\frac{M_\varphi}{\lambda_\varphi}) \end{cases}\\
&v_\sigma^2 = -\frac{1}{\sqrt{2}}\frac{\lambda_\phi}{\lambda_\sigma}[(v_1^\phi)^2 + (v_2^\phi)^2] \neq 0 \quad \text{and} \quad v_{\bar{\sigma}} = 0.
\end{aligned} \quad (52)$$

The three solutions corresponding to $\langle\varphi\rangle$ are degenerate and corresponding to the breaking of S_3 to its 3 different Z_2 subgroups. Through appropriate choices of soft terms that break the discrete Abelian symmetry Z_3 and Z_5 and not S_4 we can stabilize as absolute minimum the vacuum configuration $\langle\varphi\rangle \sim (0, 1)$.

V. CONCLUSION

In this paper we have discussed the idea that S_4 is the minimal discrete non-Abelian group naturally related to TBM in the lepton sector. We have shown that S_4 can yield exact TBM according to a general group theory analysis and we have presented two explicit model realizations of how TBM can be obtained in S_4 once the basis of its generators is fixed. In addition we have provided a detailed study of the corresponding scalar potentials. The two models require two triplets with different VEV alignments. For each model we have built a potential that in the SUSY limit contains the minimum required. The problem of the triplet and doublet alignments is solved in a more economical way than in models based on A_4 [14–29]. To separate the charged lepton sector from the neutrino one we have introduced extra Abelian symmetries. The construction of the potentials has not required additional symmetries than such extra Abelian symmetries, but just the addition of ''driving'' fields that do not enter in the Yukawa part. We have studied neither the quark sector nor the possibility to embed such a model in a grand unified theory nor the next leading order corrections. We leave these subjects for a future publication. It is worth mentioning that in S_4 there is more freedom to generate the mixing in the quark sector than in A_4. Indeed the doublet irreducible representation could play an important role as happens in T' [30].

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