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Influence of strong magnetic fields and instantons on the phase structure of the two-flavor Nambu–Jona-Lasinio model

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Both in heavy-ion collisions and in magnetars very strong magnetic fields are produced, which has an influence on the phases of matter involved. In this paper we investigate the effect of strong magnetic fields ($B \sim 5m_\pi^2/e = 1.7 \times 10^{19}$ G) on the chiral symmetry restoring phase transition using the Nambu–Jona-Lasinio model. It is observed that the pattern of phase transitions depends on the relative magnitude of the magnetic field and the instanton interaction strength. We study two specific regimes in the phase diagram, high chemical potential and zero temperature and vice versa, which are of relevance for neutron stars and heavy-ion collisions, respectively. In order to shed light on the behavior of the phase transitions, we study the dependence of the minima of the effective potential on the occupation of Landau levels. We observe a near degeneracy of multiple minima with differing occupation numbers, of which some become the global minimum upon changing the magnetic field or the chemical potential. These minima differ considerably in the amount of chiral symmetry breaking and, in some cases, also of isospin breaking.

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I. INTRODUCTION

Recently it has been noted that very strong magnetic fields can be produced in heavy-ion collisions [1–3]. It is estimated that at the Relativistic Heavy Ion Collider magnetic fields are created of magnitude $5.3m_\pi^2/e = 1.8 \times 10^{19}$ G and at the LHC of $6m_\pi^2/e = 2 \times 10^{19}$ G, and there are even higher estimates [4]. Also, certain neutron stars called magnetars exhibit strong magnetic fields, in the range 10^{14}–10^{15} G [5,6]. These fields occur at the surface; probably in the much denser interior, even higher fields are present. Using the virial theorem it can be derived that the maximal strength is 10^{18}–10^{19}G [7]. If one assumes that the star is bound by the strong interaction instead of by gravitation, this limit can be even higher.

In both neutron stars and in heavy-ion collisions it is expected that quark matter plays a role. Therefore it is interesting to study how this form of matter behaves in a strong magnetic field. Two different regions in the QCD phase diagram are of relevance here. Heavy-ion collisions probe the low chemical potential and high temperature regime; for neutron stars it is the other way around. In this paper the effect of very strong magnetic fields will be investigated in both regimes.

Much work has been done on how an external magnetic field changes nuclear matter; for a review see Ref. [8]. The behavior of ordinary quark matter has been studied using the Nambu–Jona-Lasinio (NJL) model (see, for example, Refs. [9–18]) and recently also in the linear sigma model coupled to quarks [19]. Most studies investigate the one- and two-flavor cases, but recently the three-flavor case has also been investigated [20,21]. At high quark chemical potential, it is believed that the ground state is a color superconducting phase. The effects of an external magnetic field on such a phase are discussed in Refs. [22–28]. Here color superconductivity will not be considered.

In this paper we study the chiral symmetry restoring phase transition, which is strongly influenced by an external magnetic field. From studies of the NJL model it is known that a magnetic field enhances the chiral symmetry breaking [9]; this is related to the phenomenon of magnetic catalysis of chiral symmetry breaking, introduced in Refs. [29–31] and further studied for the NJL model in Refs. [10–16] and for QED in e.g. Refs. [32–36], where also the generation of an anomalous magnetic moment was pointed out [35,36]. This enhancement of chiral symmetry breaking can be understood as follows: the B field anti-aligns the helicities of the quarks and antiquarks, which are then more strongly bound by the NJL interaction [9]. The phenomenon of magnetic catalysis of chiral symmetry breaking leads to interesting behavior, since it allows for phases with broken chiral symmetry and nonzero nuclear density for a range of chemical potentials and magnetic fields [14,15,17]. In such a phase nonperiodic magnetic oscillations occur, which means that the constituent quark masses are strongly dependent on the magnetic field, and consequently also on other thermodynamic parameters.

In all studies of the influence of magnetic fields on chiral symmetry breaking up to now, the effects of instantons have not been studied explicitly, i.e. as a function of instanton interaction strength. Magnetic fields and instantons can lead to combined effects. In Ref. [3] it is shown that variations in topological charge, which induce variations of net chirality, in a strong magnetic field give rise to an electrical current. This effect is known as the chiral magnetic effect and could perhaps be observed in heavy-ion collisions. Variations of topological charge can, for instance, be created by instantons.

Here a related study will be performed. We will investigate the combined effect of instantons and a strong magnetic field on quark matter using the NJL model. In this model instantons induce an extra interaction, the 't Hooft determinant interaction, which leads to a mixing between the different quark flavors. Following the analysis of Refs. [37,38], the strength of the instanton interaction is set by the dimensionless parameter c. For $c = 0$ there is no contribution and the quarks are fully independent. Because of the difference in charge of the quarks, the phase transitions are decoupled. The other extreme case is $c = 1/2$, which is actually the most studied case. The quarks are then fully mixed, the constituent quark masses are equal, and the phase transitions will always coincide. In Ref. [14] this case is studied in the chiral limit. It is observed that for a range of typical value for the coupling constant, phases with broken chiral symmetry and nonzero nuclear density arise.

In general, there is a competition between the magnetic field, which tends to differentiate the constituent quark masses for different flavors, and the instanton interaction, which favors equal constituent quark masses. In this work this competition is studied. Apart from studying the ground state as a function of the magnetic field and chemical potential for various characteristic values of c, we also look at the local minima of the effective potential and the corresponding occupation of Landau levels. It is found that in the neighborhood of the chiral phase transition the phase diagram develops metastable phases, differing in the number of filled Landau levels. Some of these local minima become the global one upon increasing the magnetic field or chemical potential, but not all of them do. These phases can have rather different values for the constituent quark masses; in other words, they can display significantly different amounts of chiral symmetry breaking. Unlike in the case of $c = 1/2$ which is isospin symmetric, in these phases the values of the two constituent quark masses can be very distinct, which corresponds to large isospin violation. Furthermore, we find that for a realistic choice of parameters, the appearance of phases of broken chiral symmetry and nonzero nuclear density requires a not too large instanton interaction strength, i.e. $c \lesssim 0.1$.

As mentioned, we also investigate the role of nonzero temperature at zero chemical potential, which is of relevance for heavy-ion collisions. Without a magnetic field the chiral symmetry restoring phase transition at finite temperature is a crossover. For the linear sigma model coupled to quarks, it has been observed that the magnetic field turns it into a first order transition [19]. We will see that this is not the case for the NJL model.

This paper is organized as follows. First we derive the effective potential of the NJL model in the mean-field approximation in a magnetic background. Then we discuss the phase diagram as a function of chemical potential, concentrating on the phase with nonzero nuclear density and chiral symmetry breaking. We continue by discussing the temperature dependence and end with conclusions.

II. EFFECTIVE POTENTIAL OF THE NJL MODEL WITH A MAGNETIC FIELD

In this section the effective potential of the NJL model in a strong magnetic background is derived; this analysis is based on Refs. [18,19]. The effective potential derived in this way is equal to the one in Refs. [14–16] using the Fock-Schwinger proper time method.

First, to set our notation, we briefly review the NJL model introduced in Refs. [39,40]. It is a quark model with four-point interactions, the gluons are "integrated out." In this paper the following form of the NJL model is used, in the notation of Ref. [38]:

$$\mathcal{L}_{\mathrm{NJL}} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m - \mu\gamma_0)\psi + \mathcal{L}_{\bar{q}q} + \mathcal{L}_{\mathrm{det}}, \quad (1)$$

where $m = m_u = m_d$ is the current quark mass and $\mu = \mu_u = \mu_d$ is the quark chemical potential. Note that in contrast to Ref. [38], here the current quark masses and both chemical potentials are taken equal. Furthermore,

$$\mathcal{L}_{\bar{q}q} = G_1[(\bar{\psi}\tau_a\psi)^2 + (\bar{\psi}\tau_a i\gamma_5\psi)^2] \quad (2)$$

is the attractive part of the $\bar{q}q$ channel of the Fierz-transformed color current-current interaction [41], and

$$\mathcal{L}_{\mathrm{det}} = 8G_2 \det(\bar{\psi}_R\psi_L) + \mathrm{H.c.} \quad (3)$$

is the 't Hooft determinant interaction which describes the effects of instantons [42,43]. Note that this interaction is flavor mixing. In the literature G_1 and G_2 are often taken equal, which means that the low energy spectrum consists of σ and $\boldsymbol{\pi}$ fields only, but here we will allow them to be different. We will restrict ourselves to the two-flavor case, using τ_a with $a = 0, \ldots, 3$ as generators of U(2).

The symmetry structure of the NJL model is very similar to that of QCD. In the absence of quark masses and the instanton interaction, there is a global $\mathrm{SU}(3)c \times \mathrm{U}(2)_L \times \mathrm{U}(2)_R$ symmetry. The instanton interaction breaks it to $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B$. For nonzero but equal quark masses this symmetry is reduced to $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_V \times \mathrm{U}(1)_B$. If a magnetic field is turned on, the symmetry is reduced to $\mathrm{SU}(3)_c \times \mathrm{U}(1)^2$ due to the differences in charge.

We choose the parameters as in Refs. [37,38]. This means we write

$$G_1 = (1 - c)G_0, \qquad G_2 = cG_0, \quad (4)$$

where the parameter c controls the instanton interaction strength, while the value for the quark condensate (which is determined by the combination $G_1 + G_2$) is kept fixed. For our numerical studies we will use the following values for the parameters: $m = 6$ MeV, a three-dimensional momentum UV cutoff $\Lambda = 590\ \mathrm{MeV}/c$, and $G_0\Lambda^2 = 2.435$.

These values lead to a pion mass of 140.2 MeV, a pion decay constant of 92.6 MeV, and finally, a quark condensate $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle = (-241.5\text{ MeV})^3$ [37], all in reasonable agreement with experimental determinations.

To calculate the ground state of the theory, the effective potential has to be minimized. In this section the effective potential is calculated in the mean-field approximation. We will assume that only the charge neutral condensates $\langle\bar{\psi}\tau_0\psi\rangle$ and $\langle\bar{\psi}\tau_3\psi\rangle$ can become nonzero.

To obtain the effective potential in the mean-field approximation, first the interaction terms are "linearized" in the presence of the $\langle\bar{\psi}\tau_0\psi\rangle$ and $\langle\bar{\psi}\tau_3\psi\rangle$ condensates (this is equivalent to the procedure with a Hubbard-Stratonovich transformation used in Ref. [38]),

$$(\bar{\psi}\tau_a\psi)^2 \simeq 2\langle\bar{\psi}\tau_a\psi\rangle\bar{\psi}\tau_a\psi - \langle\bar{\psi}\tau_a\psi^2\rangle, \tag{5}$$

leading to

$$\mathcal{L}_{\rm NJL} = \bar{\psi}(i\gamma^\mu\partial_\mu - \mathcal{M} - \mu\gamma_0)\psi - \frac{(M_0-m)^2}{4G_0} - \frac{M_3^2}{4(1-2c)G_0}, \tag{6}$$

where $\mathcal{M} = M_0\tau_0 + M_3\tau_3$ and

$$\begin{aligned} M_0 &= m - 2G_0\langle\bar{\psi}\tau_0\psi\rangle, \\ M_3 &= -2(1-2c)G_0\langle\bar{\psi}\tau_3\psi\rangle. \end{aligned} \tag{7}$$

Now the Lagrangian is quadratic in the quark fields, so we can integrate over them. After going to imaginary time, this results in the following effective potential in the mean-field approximation [44]:

$$\begin{aligned} \mathcal{V} = {}& \frac{(M_0-m)^2}{4G_0} + \frac{M_3^2}{4(1-2c)G_0} - TN_c\sum_{f=u}^{d}\sum_{p_0=(2n+1)\pi T}\int\frac{d^3p}{(2\pi)^3} \\ & \times \ln\det[i\gamma_0p_0 + \gamma_ip_i - M_f - \gamma_0\mu], \end{aligned} \tag{8}$$

where we have introduced the constituent quark masses $M_u = M_0 + M_3$ and $M_d = M_0 - M_3$.

As we have mentioned earlier, often G_1 is taken equal to G_2, which is the case of $c = 1/2$. This choice implies that M_3 is then always equal to 0; i.e., the constituent quark masses are equal. Note that the reverse is not true. If the assumption is made that the constituent quark masses are equal, $c = 1/2$ or $\langle\bar{\psi}\tau_3\psi\rangle = 0$ or both. However, for $M_3 = 0$, changes in c cannot be noticed because only the combination $G_1 + G_2$ occurs. Hence, the conclusion is that $\langle\bar{\psi}\tau_3\psi\rangle = 0$ and c can be any value. Instanton effect are nevertheless present (G_2 can be nonzero after all) and η and a_0 mesons can still be present in the spectrum. The ratio G_2/G_1 simply cannot be determined if $M_3 = 0$. Since magnetic fields affect the two flavors differently because of the difference in charge, isospin breaking effects are expected and it is unnatural to choose $M_3 = 0$ from the start or to take $c = 1/2$. Using the strange quark condensate, it was argued in Ref. [37] that a realistic value of c would be around 0.2.

A. Including a magnetic field

Now we include a magnetic field, which changes the dispersion relation for the quarks in the following way:

$$p_{0n}^2 = p_z^2 + M^2 + (2n+1-\sigma)|q|B, \tag{9}$$

where n is the quantum number labeling the discrete orbits, σ the spin of the quark, and q its charge. The integral over the three-momentum is modified as

$$\int\frac{d^3p}{(2\pi)^3} \to \frac{|q|B}{2\pi}\sum_{n=0}^{\infty}\int\frac{dp_z}{2\pi}. \tag{10}$$

Performing the sum over the Matsubara frequencies gives the following effective potential [14,15,18]:

$$\begin{aligned} \mathcal{V} = {}& \frac{(M_0-m)^2}{4G_0} + \frac{M_3^2}{4(1-2c)G_0} - \frac{N_c}{2\pi}\sum_{\sigma,n,f}|q_f|B\int\frac{dp_z}{2\pi}E_{p,f}(B) \\ & - \frac{N_c}{2\pi}\sum_{\sigma,n,f}|q_f|B\int\frac{dp_z}{2\pi}\{T\ln[1+e^{-[E_{p,f}(B)+\mu]/T}] \\ & + T\ln[1+e^{-[E_{p,f}(B)-\mu]/T}]\}, \end{aligned} \tag{11}$$

where $E_{p,f}(B) = \sqrt{p_z^2 + (2n+1-\sigma)|q_f|B + M_f^2}$. Following the analysis of Ref. [18], this potential can be split into three pieces, a part that is independent of external parameters, a part that only depends on the magnetic field, and a part that depends on the magnetic field, chemical potential, and temperature:

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_1(B) + \mathcal{V}_2(B,\mu,T), \tag{12}$$

with

$$\begin{aligned} \mathcal{V}_0 &= \frac{(M_0-m)^2}{4G_0} + \frac{M_3^2}{4(1-2c)G_0} - 2N_c\sum_{f=u}^{d}\int\frac{d^3p}{(2\pi)^3}\sqrt{\mathbf{p}^2+M_f^2}, \\ \mathcal{V}_1(B) &= -\frac{N_c}{2\pi^2}\sum_{f=u}^{d}(|q_f|B)^2\left[\zeta'(-1,x_f) - \frac{1}{2}(x_f^2-x_f)\ln x_f + \frac{x_f^2}{4}\right], \\ \mathcal{V}_2(B,\mu,T) &= -\frac{N_c}{2\pi}\sum_{\sigma,n,f}|q_f|B\int\frac{dp_z}{2\pi}\{T\ln[1+e^{-[E_{p,f}(B)+\mu]/T}] + T\ln[1+e^{-[E_{p,f}(B)-\mu]/T}]\}, \end{aligned} \tag{13}$$

where we have defined $x_f = \frac{M_f^2}{2|q_f|B}$ and $\zeta'(-1, x_f) = \frac{d\zeta(z,x_f)}{dz}|_{z=-1}$ with $\zeta(z, x_f)$ the Hurwitz zeta function. We have neglected x_f-independent terms in $\mathcal{V}_1(B)$ (including a UV divergent one).

The term $\mathcal{V}_0$ is divergent and needs to be regularized. Here a conventional three-momentum UV cutoff is used, yielding the expression

$$\mathcal{V}_0 = \frac{(M_0 - m)^2}{4G_0} + \frac{M_3^2}{4(1-2c)G_0} - \frac{N_c}{8\pi^2}\sum_{f=u}^{d}|M_f|\left(M_f^3 \ln\left(\frac{\Lambda}{M_f} + \sqrt{1+\frac{\Lambda^2}{M_f^2}}\right) - \Lambda(M_f^2 + 2\Lambda^2)\sqrt{\frac{\Lambda^2}{M_f^2}+1}\right). \quad (14)$$

The expression $\zeta'(-1, x_f)$ in $\mathcal{V}_1(B)$ can be written in a more convenient form by differentiating and integrating the function with respect to x_f:

$$\zeta'(-1, x_f) = \zeta'(-1, 0) + \frac{x_f^2}{2} - \frac{x_f}{2} - \frac{x_f}{2}\log(2\pi) + \psi^{(-2)}(x_f), \quad (15)$$

where $\psi^{(m)}(x_f)$ is the mth polygamma function. The term $\zeta'(-1, 0)$ is independent of x_f and will therefore not be taken into account. The remaining expression is amenable to numerical evaluation.

The summation over σ and n in $\mathcal{V}_2(B, \mu, T)$ can be rewritten as

$$\mathcal{V}_2(B, \mu, T) = -\frac{N_c}{2\pi}\sum_{k,f}(2-\delta_{k0})|q_f|B\int\frac{dp_z}{2\pi} \times \{T\ln[1+e^{-[E_{p,k}(T)+\mu]/T}] + T\ln[1+e^{-[E_{p,k}(T)-\mu]/T}]\}, \quad (16)$$

where $E_{p,k} = \sqrt{p_z^2 + M_f^2 + 2k|q_f|B}$ and k denotes the Landau level, which has degeneracy $(2 - \delta_{k0})$.

At zero temperature, $\mathcal{V}_2$ can be simplified to

$$\begin{aligned}\mathcal{V}_2(B, \mu, 0) &= -\frac{N_c}{2\pi}\sum_{k,f}(2-\delta_{k0})\int\frac{dp_z}{2\pi}\theta(\mu - E_{p,k}) \times [\mu - E_{p,k}] \\ &= \sum_{f=u}^{d}\sum_{k=0}^{k_{f,\max}}(2-\delta_{k0})\theta(\mu - s_f(k,B))\frac{|q_f|BN_c}{4\pi^2} \times \Big\{\mu\sqrt{\mu^2 - s_f^2(k,B)} - s_f^2(k,B) \times \ln\Big[\frac{\mu + \sqrt{\mu^2 - s_f^2(k,B)}}{s_f(k,B)}\Big]\Big\},\end{aligned} \quad (17)$$

where $s_f(k, B) = \sqrt{M_f^2 + 2|q_f|Bk}$ and $k_{f,\max}$ is the upper Landau level, defined as

$$k_{f,\max} = \left\lfloor \frac{\mu^2 - M_f^2}{2|q_f|B} \right\rfloor, \quad (18)$$

where the brackets indicate the floor of the enclosed quantity.

We will now use these expressions in a numerical study of the minima of the effective potential, performed along the lines discussed in Refs. [38,44].

III. RESULTS

We start by considering the case of $\mu = 0$, $T = 0$, and $c = 0$. Figure 1 shows the results for this unmixed case. The magnetic field enhances M_u and M_d, which are proportional to $\langle\bar{u}u\rangle$ and $\langle\bar{d}d\rangle$, respectively; consequently the chiral symmetry breaking is enhanced [9]. Because of the charge difference of the quarks, the B dependence of the constituent quark masses is not equal. Nonzero c will cause mixing and will bring the masses closer to each other. As discussed, at $c = 1/2$ the constituent quark masses are exactly equal.

A. Nonzero chemical potential

In this section we turn to the phase structure near the phase transition at nonzero chemical potential and zero temperature. From Refs. [45–48] it is known that when the isospin chemical potential is nonzero, it is possible to have two phase transitions at low temperature and high baryon chemical potential. Here we study a similar case: instead of nonzero isospin chemical potential, we allow for a nonzero magnetic field; here we will also see that the possibility of separate phase transitions for the two quarks arises. We will take equal chemical potentials for the quarks, but the magnetic field acts effectively like a nonzero isospin chemical potential due to the difference in charge of the quarks. Instantons cause mixing between the quarks; if the

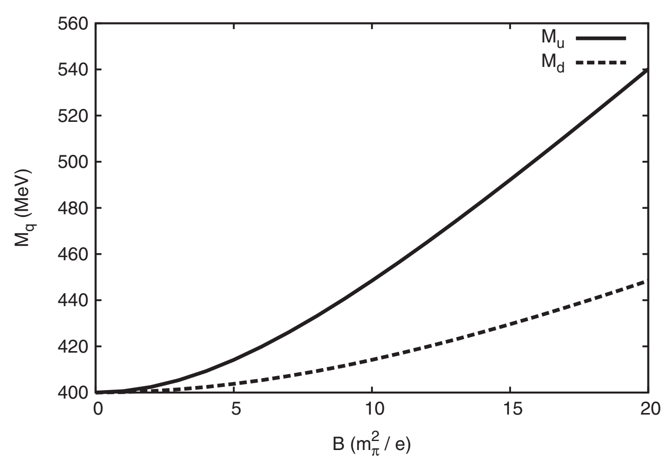

FIG. 1. The dependence of the constituent quark masses M_u and M_d on the magnetic field B.

mixing is strong enough, the two separate phase transitions merge into one. This was extensively investigated in Ref. [45] for the nonzero isospin chemical potential case.

From Refs. [14,15,17], where the NJL model in the chiral limit was studied, it is known that Landau quantization induces a more complex phase structure. Apart from the usual phase of broken chiral symmetry with zero nuclear density, there is also the possibility of such a phase with nonzero nuclear density. Here we perform a more detailed study of this case, which is a characteristic phenomenon at nonzero chemical potential and sufficiently strong magnetic fields [cf. Eq. (19) below].

1. The $c = 0$ case

When the determinant interaction is turned off, the up and down quarks are decoupled. This leads to the possibility of separate phase transitions for the quarks. In Figs. 2 and 3 we show the constituent quark mass of the up and down quarks, respectively, as a function of quark chemical potential and magnetic field. As expected, the two quarks have decoupled behavior.

Let us first discuss the behavior of the up quark. At low chemical potential we have the ''standard'' chiral symmetry breaking NJL ground state with empty Landau levels (LL). Following the nomenclature of Refs. [14,15] where the $c = 1/2$ case was studied in the chiral limit, this is called phase B. Note that this phase always has zero nuclear density. At high chemical potential chiral symmetry is restored, up to the explicit breaking. In this approximate symmetric phase, magnetic oscillations can be seen in the constituent quark masses, caused by Landau quantization. These oscillating phases are denoted by A_i, where i gives the number of filled LL. As these phases have occupied LL, they have nonzero nuclear density. The nuclear density of level k is given by [18]

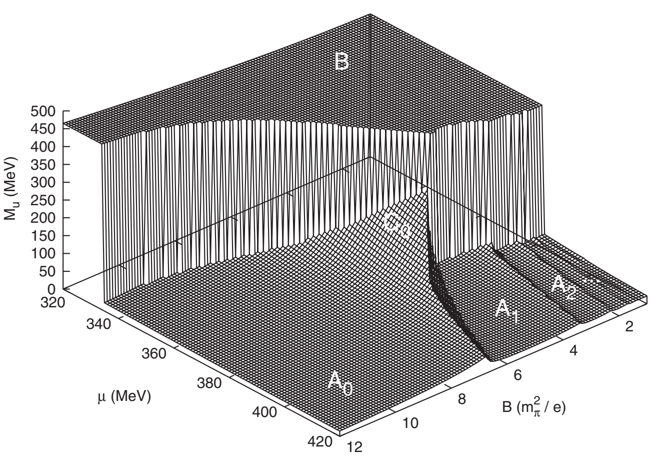

FIG. 2. The dependence of the constituent up-quark mass on B and μ. A_i, B, and C_0 denote the different phases using the scheme of Refs. [14,15]. Chiral symmetry is broken in phases B and C_0, and phases A_0 and C_0 have nonzero nuclear density.

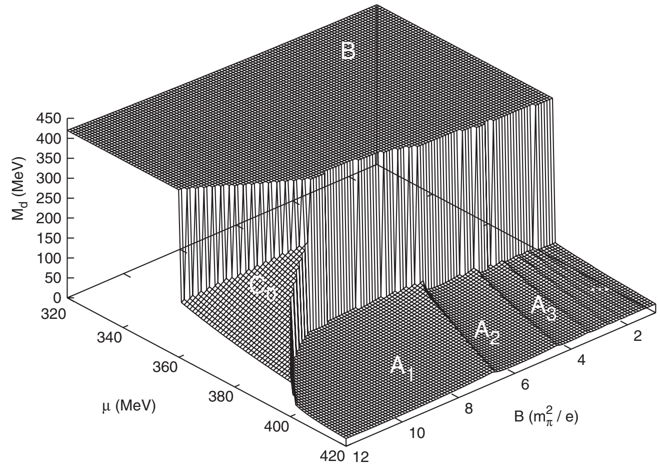

FIG. 3. Same as Fig. 2, now for the down quark.

$$\rho_{f,k}(B, \mu) = (2 - \delta_{k0})\theta(\mu - s_f(k, B))\frac{|q_f|BN_c}{6\pi^2} \times \sqrt{\mu^2 - s_f^2(k, B)}. \quad (19)$$

In the chiral limit the constituent quark masses vanish in the A_i phases.

The oscillations are due to the de Haas-van Alphen effect, which in QED and in the two-flavor NJL model for $c = 1/2$ in the chiral limit leads to second order transitions between the A_i phases [14]. However, with our choice of parameters the transitions are weakly first order. In the chiral limit they become second order, like for $c = 1/2$, as can, for instance, be seen in the nuclear density. For completeness, we mention that in the color superconducting case of Refs. [27,28], the oscillations in the gap parameter are seen to be continuous, but second order transitions can occur when neutrality conditions are imposed.

For B larger than $4.5m_\pi^2/e$ an interesting intermediate phase arises, where the up quark jumps, as a function of μ, first to a phase with a still rather large constituent mass and then to phase A_1. This intermediate phase is called C_0 in the language of Refs. [14,15] and corresponds to a phase of broken chiral symmetry having nonzero nuclear density and a filled zeroth LL. So the essential difference between this phase and A_0 is the breaking of chiral symmetry. For smaller values of the coupling constant G_0 the phases C_k with $k > 0$ (which are similar to C_0 but with more occupied LL) also occur. The transitions between the C_k are first order; furthermore, they are nonperiodic in the sense that the difference between the transitions is B dependent, as the constituent mass strongly depends on B [14]. If we are in this phase C_0 and increase the magnetic field, the constituent quark mass decreases, eventually becoming almost zero; this can be interpreted as a crossover to A_0. In the chiral limit the crossover becomes a second order transition. Finally, we would like to note that, already at

$B = 4m_\pi^2/e$, the phase C_0 exists as a metastable phase (we will discuss this in more detail later).

The qualitative behavior of the down quark is very similar, as the quarks only differ in charge. Consequently, Fig. 3 can be directly obtained from Fig. 2 by multiplying B by 2; for ease of comparison we show both figures. If one compares the two figures, one can immediately see that there are large regions where the constituent quark masses are considerably different. This is equivalent to a large nonzero $\langle\bar{\psi}\tau_3\psi\rangle$ condensate, i.e., spontaneous isospin breaking. This will influence the behavior of the mesons accordingly, for example, the masses.

Eventually, if one keeps increasing the magnetic field, the phase transitions of the quarks will take place at (almost) the same chemical potential and there will be no spontaneous isospin breaking.

2. The $c \neq 0$ case

In this section the consequences of the instanton interaction are studied; i.e., the parameter c is varied. Increasing c will cause mixing between the constituent quarks, which tends to bring the constituent quark masses together. Around the phase transition there is competition between the effect of the magnetic field and the instanton interaction.

The competition is illustrated in Fig. 4, where the constituent quark masses are plotted as a function of the quark chemical potential for three characteristic values of c, $c = 0$, 0.03, and 0.1 with $B = 5m_\pi^2/e$. The qualitative behavior for different values of the magnetic field is similar. One can see that when $c \neq 0$, the phase transitions are indeed coupled. Furthermore, one observes that the two phase transitions merge into one when c is increased and that the phase C_0 disappears. Qualitatively, the behavior is similar to the case of nonzero isospin chemical potential studied in Ref. [45], but in that case the phase C_0 does not exist.

When the coupling constant G_0 is lowered, it is possible to have C_k phases at $c = 1/2$, as in Ref. [14]. Compared to the chiral limit studied there, the region of the phase diagram with C_k phases increases for $m \neq 0$.

More insight into the phase structure and phase transitions is obtained by looking at the behavior of local minima of the effective potential. Near the phase transition at these (large) magnetic fields, metastable phases arise. These phases differ in the number of filled LL. Let us take as an example the $c = 1/2$ case, which is the easiest to discuss, as the effective potential is then only a function of $M_u = M_d = M$. In Fig. 5 we show the effective potential as a function of M with $\mu = 378$ MeV and $B = 5m_\pi^2/e$. At these values four minima can be seen; the global minimum is the phase in which the chiral symmetry breaking is largest, i.e. minimum 4. When μ is increased, minimum 1 will take over, which is A_1 for the up quark and A_2 for the down quark. The other two local minima never become the global one for our choice of G_0, but as they are almost degenerate with the other minima (also for other values of c), they are nevertheless important. These local minima correspond to C_k phases and can become the global minimum when G_0 is lowered.

Similar results hold for $c \neq 1/2$; then, metastable phases also exist with different fillings of LL. In this case some of the C_k phases can become the global minimum, as we have seen for $c = 0$. Like before, the number of such states depends on the choice of the other parameters.

As the metastable phases differ in the values of the $\langle\bar{u}u\rangle$ and $\langle\bar{d}d\rangle$ condensates at small c, they again represent rather large broken isospin and will lead to different meson masses. Whenever the system is passing through the phase transition, it could be trapped in one of those metastable phases for some time, and consequences from the changing meson masses can arise, for example, enhancing or suppressing certain decays.

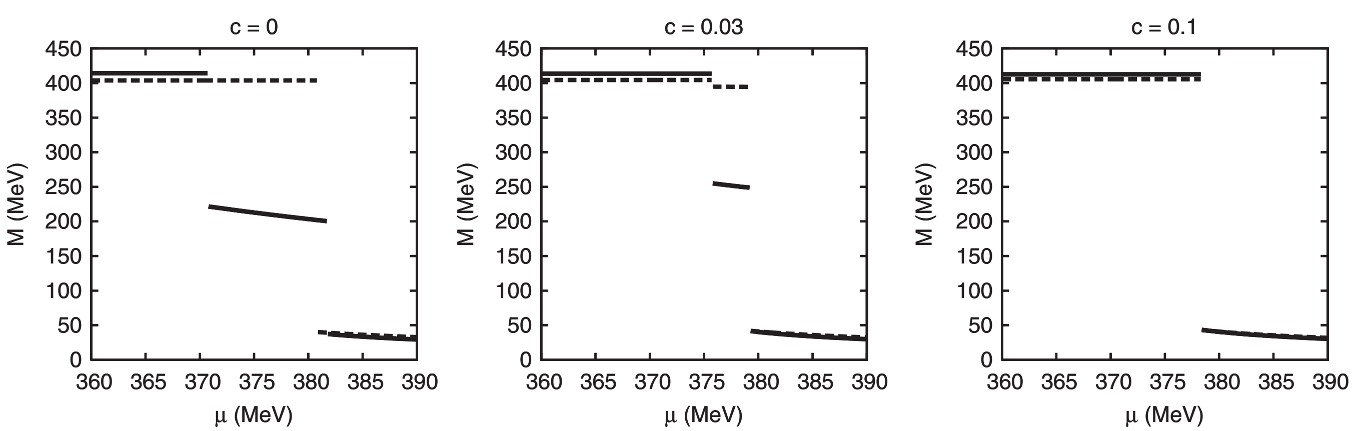

FIG. 4. The dependence of the constituent quark masses on the quark chemical potential for $B = 5m_\pi^2/e$ and various c values. Solid lines denote the up quarks, the dashed lines the down quark.

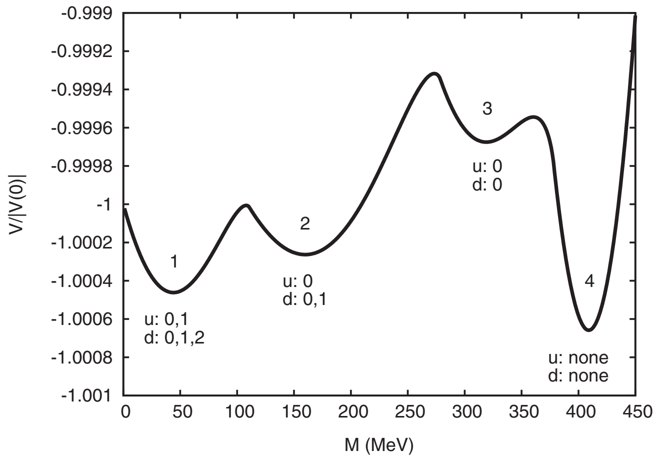

FIG. 5. The normalized effective potential at the values $B = 5m_\pi^2/e$, $\mu = 378$ MeV, and $c = 1/2$. There are four minima. The numbers below the minima denote the LL occupied for each quark. Note that the minima are almost degenerate.

B. Nonzero temperature

In this section the temperature dependence of the ground state is investigated at zero chemical potential, but with a magnetic field. As the instanton interaction does not influence the temperature dependence much, we only consider $c = 1/2$ for simplicity. Reference [19] found that, in the linear sigma model coupled to quarks, the usual crossover becomes a first order transition at very high magnetic fields. However, we find that this is not the case in the NJL model.

In Ref. [19] only the lowest Landau level was taken into account. Here more Landau levels are included, so the effect of the higher Landau levels can be investigated in the NJL model. Since the levels with large k are exponentially suppressed, the summation can be truncated in Eq. (16); we will denote the largest k by $k_{\rm trunc}$. The value of $k_{\rm trunc}$ depends on the temperature, constituent quark mass, chemical potential, and magnetic field considered. If M and T are increased or if B is decreased, $k_{\rm trunc}$ has to be increased.

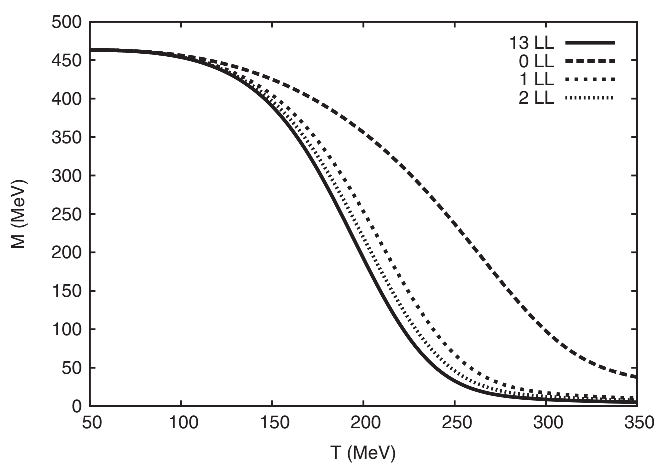

FIG. 6. The temperature dependence of the constituent quark mass for a strong magnetic field ($B = 15m_\pi^2/e$) and various $k_{\rm trunc}$ values.

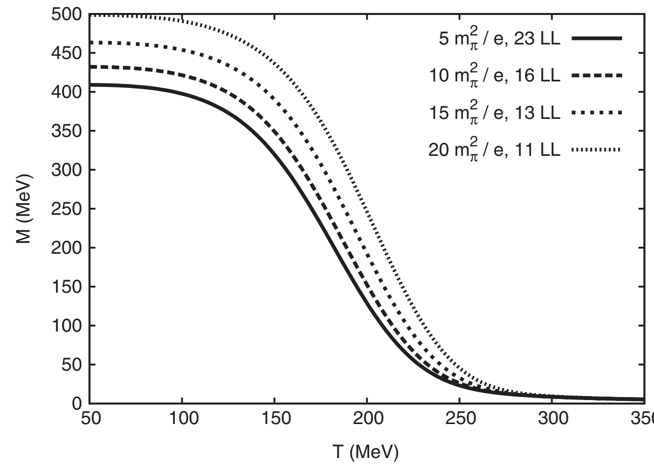

FIG. 7. The temperature dependence of the constituent quark mass for various strong magnetic fields. We also indicate $k_{\rm trunc}$.

In Fig. 6 we show the temperature dependence of the constituent quark mass at $B = 15m_\pi^2/e$ for four different values of $k_{\rm trunc}$. The 13 levels case is chosen such that the error is less than 1% at $M = 450$ MeV, $T = 450$ MeV. From the figure it can be inferred that taking more Landau levels into account makes the crossover sharper. Also, there is a significant difference between including the zeroth Landau level and including the first Landau level. It is clear that including more Landau levels influences the details of the transition. However, the qualitative aspects of the phase transition are not changed.

In Fig. 7 the temperature dependence of the constituent quark mass for different values of the external magnetic field is shown. The phase transition remains a crossover, in contrast to the results in the linear sigma model coupled to quarks. This difference is important, as a first order phase transition allows for metastable states, whereas a crossover does not.

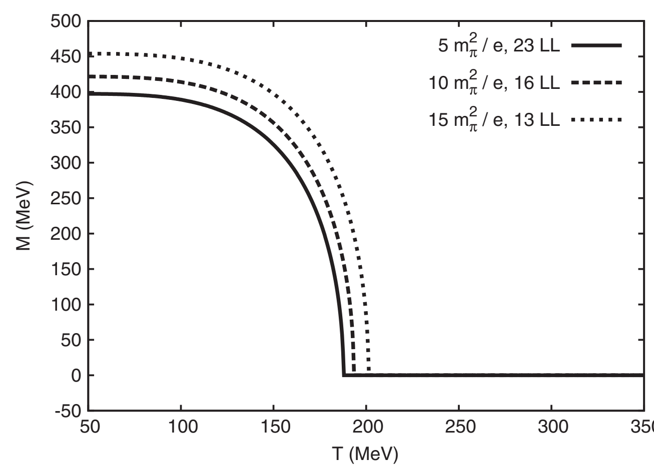

FIG. 8. Same as Fig. 7, now in the chiral limit.

In Fig. 8 the results in the chiral limit are shown, where the transition remains a second order phase transition, as is the case at zero magnetic field, and confirms the results of [17], in which the phase diagram was calculated in a strong magnetic field in the chiral limit using the Fock-Schwinger proper time method. Note that the critical temperature increases slightly with increasing magnetic field.

IV. CONCLUSIONS

The effect of a strong magnetic field on quark matter has been investigated in the NJL model in two regimes, zero temperature and finite chemical potential and vice versa. The first regime is of relevance for (the interior of) magnetars and the second for heavy-ion collisions.

At very high magnetic fields, when $M \approx 2|q|B \approx \mu$, the phase structure shows a variety of phases and phase transitions due to Landau quantization. As a function of chemical potential, more phase transitions occur, corresponding to Landau levels filling up successively. Because of the difference in charge, this pattern is different for the two quark flavors. When there is no mixing in the absence of the instanton interaction, the two patterns are uncoupled. This generally leads to rather different constituent quark masses, or equivalently, spontaneous isospin breaking $\langle \bar{\psi} \tau_3 \psi \rangle \neq 0$. This affects the mesons inside the medium, for example, their masses. It was found that for a realistic choice of parameters in the NJL model, such a phase of broken chiral and isospin symmetry arises around $B = 4.5 m_\pi^2/e$, but it is already present as a metastable phase for lower magnetic fields.

When the instanton interaction is included, a competition occurs between the strength c of this interaction and the magnetic field. This reduces the region in the phase diagram with large $\langle \bar{\psi} \tau_3 \psi \rangle$. For c sufficiently large, it disappears entirely, leaving only one phase transition. However, around this transition the phase structure is still rather complex regarding metastable phases, which are characterized by different fillings of Landau levels and which differ only slightly in energy, but much in the amount of chiral symmetry breaking. For lower values of c some of these near-degenerate minima can also differ considerably in the amount of isospin breaking.

Finally, the role of the temperature was studied at zero chemical potential. In Ref. [19] it was found that, in the linear sigma model coupled to quarks, a strong magnetic field changes the usual crossover as a function of temperature into a first order transition. In the NJL model it was found that the crossover remains a crossover. Also, including higher Landau levels in the calculation of the effective potential changes the details of the crossover; it becomes sharper, albeit the qualitative aspects of the transition are not changed. The difference between the two models is important, as the first order transition allows for metastable phases while a crossover does not.

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