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Independence Axioms for Water Allocation[1]

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Abstract

We consider the problem of sharing water among agents located along a river. Each agent has quasi-linear preferences over river water and money, where the benefit of consuming an amount of water is given by a continuous and concave benefit function. A solution to the problem efficiently distributes the river water over the agents and wastes no money. We introduce a number of (independence) axioms to characterize two new and two existing solutions. We apply the solutions to the particular case that every agent has constant marginal benefit of one up to a satiation point and marginal benefit of zero thereafter. In this case we find that two of the solutions (one existing and one new) can be implemented without monetary transfers between the agents.

Keywords: Water allocation, Harmon doctrine, concave benefit function, stability, externality, independence axiom, water claim.
JEL codes: C71, D62, Q25

1 Introduction

In this paper we consider the problem of sharing water among agents, e.g. countries, cities, firms, located along a river. As the number of agents involved in sharing (international) river water is usually small, and formal (international) water exchanges are scarce, trade in river water normally takes place by the signing of contracts between the parties involved. These contracts directly specify the amount of water to be delivered and the amount of money that has to be paid for this water, see Dinar, Rosegrant and Meinzen-Dick (1997) and its references. Cooperative game theory deals with strategic situations in which the outcome of one agent's choice depends on choices made by other agents, and the agents making the choices are able to sign binding bi- or multilateral contracts to enforce cooperation. For this reason cooperative game theory is one of the main tools that is used in modeling (international) water resource issues, see Parrachino, Dinar and Patrone (2006) for an overview.

Ambec and Sprumont (2002) introduce a model in which a group of agents is located along a single-stream river from upstream to downstream. Each agent is assumed to have quasi-linear preferences over river water and money, where the benefit of consuming an amount of water is given by a differentiable, strictly increasing and strictly concave benefit function. An allocation of the river water among the agents is efficient when it maximizes the total sum of benefits. To sustain an efficient water allocation, the agents can compensate each other by paying monetary transfers. Every water allocation and transfer schedule yields a welfare distribution, where the utility of an agent is equal to its benefit from water consumption plus its monetary transfer, which can be negative. By deriving a cooperative game from their model, Ambec and Sprumont (2002) find out how the river water should be allocated over the agents and propose what monetary transfers should be performed in order to realize a fair welfare distribution. They suggest the downstream incremental solution as a welfare distribution that satisfies both core lower bounds as well as aspiration upper bounds. This downstream incremental solution can be seen as the marginal contribution vector of their cooperative game corresponding to the ordering of agents along the river, from upstream to downstream.

Ambec and Ehlers (2008), Khmelnitskaya (2010), van den Brink, van der Laan and Moes (2010) and Wang (2011) all generalize the model of Ambec and Sprumont (2002) in a specific way. Ambec and Ehlers (2008) allow for satiable agents by assuming that the benefit function of each agent is differentiable and strictly concave, but not necessarily increasing (i.e., the benefit function can be decreasing beyond some satiation point). Khmelnitskaya (2010) considers rivers that have a so-called sink-tree or rooted-tree structure allowing multiple springs or deltas. Van den Brink, van der Laan and Moes (2010) study rivers with multiple springs (as in Khmelnitskaya (2010)) and satiable agents (as in

Ambec and Ehlers (2008)) and suggest a new class of solutions based on a water distribution principle known as Territorial Integration of all Basin States (TIBS). Finally, Wang (2011) proposes a solution to the original single-stream model in which water trading is restricted to pairs of neighboring agents.

Our paper adds to this growing literature in three ways. First, we weaken the assumption of Ambec and Ehlers (2008) (and therefore also the assumption of Ambec and Sprumont (2002)) on the benefit functions by only requiring continuity and concavity. Second, we characterize two existing solutions for the single-stream model by introducing several (independence) axioms. Third, we propose and characterize two new solutions for the single-stream model, also by using new (independence) axioms.

In contrast to the papers mentioned above, in this paper we avoid the detour of modeling the river situation as a cooperative game. Instead, we immediately impose axioms on the class of all river water sharing problems. This has as main advantage that the axioms we propose can directly be interpreted in terms of water (benefit) allocation. While most axioms used in the literature are also derived from water distribution principles, they are ultimately axioms on cooperative games and not on water allocation problems. This often leads to friction when trying to interpret the cooperative game axioms in terms of water allocation. We feel that our approach is more natural as it allows for a straightforward interpretation of the axioms.

After considering the general case, we apply the four solutions that we discuss in this paper to the particular case that every agent has constant marginal benefit of one up to a satiation point, and marginal benefit of zero thereafter, see also Ansink and Weikard (2011). This could be seen as representing a situation where the full benefit functions of the agents are unknown and each agent has only specified a single claim on water from the river. We find that in this case two of the solutions (one existing and one new) can be implemented without monetary transfers between the agents.

The paper is organized as follows. In Section 2 we recall the single-stream river sharing model of Ambec and Sprumont (2002) and weaken the assumptions on the benefit functions of that model. In Section 3 we introduce a number of axioms on river sharing problems. In Sections 4, 5, 6 and 7 we show that different sets of axioms characterize different solutions assiging fair welfare distributions to every river problem. More specifically, we characterize two existing solutions, known as the downstream incremental solution (Section 4) and the upstream incremental solution (Section 5), and propose and characterize two new solutions: the downstream solution (Section 6) and the upstream solution (Section 7). In Section 8 we apply the four solutions to the case where every agent has constant marginal benefit of one up to a satiation point, and marginal benefit of zero thereafter. We conclude with a comparison of the four solutions in Section 9.

2 River problems with concave benefit functions

In their paper 'Sharing a river', Ambec and Sprumont (2002) consider the problem of finding a 'fair' distribution of the welfare resulting from allocating the inflows of water along an international or transboundary river to the agents located along the river. Let $N = \{1, \dots, n\}$ be the set of agents, in the sequel also called countries, along the river, numbered successively from upstream to downstream, and let $e_i \geq 0$ be the inflow of water on the territory of agent i, $i = 1, \dots, n$. Every agent i is assumed to have a quasi-linear utility function assigning to every pair (x_i, t_i) with $x_i \in \mathbb{R}_+$ an amount of water allocated to i and $t_i \in \mathbb{R}$ a monetary compensation to i, the utility

$$\nu^i(x_i, t_i) = b_i(x_i) + t_i, \tag{2.1}$$

where $b_i \colon \mathbb{R}_+ \to \mathbb{R}$ is a continuous function yielding benefit $b_i(x_i)$ to agent i of the consumption x_i of water. In the following we denote such a river situation by the triple (N, e, b), where N is the set of agents, $e \in \mathbb{R}^n_+$ is the vector of nonnegative inflows and $b = (b_i)_{i \in N}$ is the collection of benefit functions.

Because of the one-directionality of the water flow from upstream to downstream, every agent can be assigned at most the water inflow at the territories of himself and his upstream agents, but the water inflow downstream of some agent cannot be allocated to this agent. Therefore, a *water allocation* $x \in \mathbb{R}^n_+$ assigns an amount of water x_i to agent i, $i = 1, \dots, n$, under the constraints

$$\sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} e_i, \quad j = 1, \dots, n,$$

i.e., $x \in \mathbb{R}^n_+$ is a water allocation if, for every agent j, the sum of the water assignments $x_1, \dots, x_j$ is at most equal to the sum of the inflows $e_1, \dots, e_j$. A water allocation x yields *total welfare* $\sum_{i=1}^{n} b_i(x_i)$. We allow for monetary transfers amongst the agents, so that agents can make monetary compensations to other agents for receiving water. A *compensation scheme* $t \in \mathbb{R}^n$ gives a monetary compensation t_i to agent i, $i = 1, \dots, n$, under the constraint

$$\sum_{i=1}^{n} t_i \leq 0.$$

As mentioned before, a pair (x, t) of a water allocation x and a compensation scheme t yields utilities $\nu^i(x_i, t_i)$ given by (2.1) for every $i = 1, \dots, n$. A pair (x, t) is *Pareto efficient* if no water and no money is wasted, i.e., (x, t) is Pareto efficient if and only if $x \in \mathbb{R}^n_+$ maximizes the welfare maximization problem

$$\max_{x_1, \dots, x_n} \sum_{i=1}^{n} b_i(x_i) \text{ s.t. } \sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} e_i, \quad j = 1, \dots, n, \text{ and } x_i \geq 0, \ i = 1, \dots, n, \tag{2.2}$$

and the compensation scheme $t \in \mathbb{R}^n_+$ is *budget balanced*: $\sum_{i=1}^n \; t_i = 0$.

In Ambec and Sprumont (2002) it is assumed that every benefit function $b_i \colon \mathbb{R}_+ \to \mathbb{R}$ is an increasing and strictly concave function, which is differentiable at every $x_i > 0$ with derivative going to infinity as x_i tends to zero. Under this assumption the maximization problem (2.2) has a unique solution x^*. We say that $z \in \mathbb{R}^n$ is a *welfare distribution* if there exists a Pareto efficient pair (x^*, t) such that

$$z_i = b_i(x_i^*) + t_i, \quad i = 1, \ldots, n.$$

Hence, a welfare distribution z distributes the maximum attainable welfare $\sum_{i=1}^n \; b(x_i^*)$ amongst the agents by allocating x_i^* to agent i, $i = 1, \ldots, n$, and implementing a budget balanced monetary compensation scheme t. Reversely, notice that for the optimal allocation x^* every budget balanced compensation scheme t induces a welfare distribution.

In Ambec and Sprumont (2002) the problem to find a 'fair' budget balanced compensation scheme, or equivalently a fair welfare distribution, is modeled by a cooperative transferable utility game. Then a solution for the cooperative game is proposed by taking into account two principles for a fair welfare distribution given in Kilgour and Dinar (1995). The principle of *Absolute Territorial Sovereignty* (ATS) states that every country has unrestricted access to use its own natural resources. For an international river this leads to the *Harmon doctrine*, stating that a country is absolutely sovereign over the inflow to the river on its own territory and thus every agent is the legal owner of its own water inflow. This principle favors upstream countries by implying that, for every $j = 1, \ldots, n$, the coalition $\{1, \ldots, j\}$ of the first j upstream countries are entitled to use the total inflow of water on their own territories without taking into account what consequences this might have for the downstream countries. In contrast to this, the principle of *Territorial Integration of all Basin States* (TIBS) favors downstream countries by stating that all the water inflows belong to all the countries together, no matter where it enters the river. It makes all countries together the legal owner of all water inflows, without regard to their own contribution to the flow. Taking the one-directionality of the water flows from upstream to downstream into account, an interpretation of the TIBS principle is the principle of *Unlimited Territorial Integrity* (UTI), stating that unrestricted use by a country of its own natural resources is only permitted in so far it does not cause damage to other sovereign countries. In its extreme form this principle implies that a country j is entitled to use all the water inflows on its own territory and on the territories of all its upstream countries. Of course, this leads to conflicting situations in the sense that the inflow e_i at the territory of country i is entitled to every country $j \geq i$.

As argued by Ambec and Sprumont (2002), the Harmon doctrine implies *stability* in the sense that for every i and every $j \geq i$ the total welfare that the collection of consecutive countries $\{i, i+1, \ldots, j\}$ receives at a Pareto efficient pair (x^*, t) should be at least equal to the sum of benefits that these countries can guarantee themselves by an optimal (welfare maximizing) allocation of their own inflows $e_i, \ldots, e_j$ amongst themselves.[1] In case $j = i$ this stability notion reduces to *individual rationality*, saying that the payoff of a country i should be at least equal to the benefit $b_i(e_i)$ of the water inflow on its own territory.[2] Taking $i = 1$ and $j \geq i$, stability implies *upstream stability,* meaning that for every upstream collection of consecutive countries $\{1, \ldots, j\}$, $j = 1, \ldots, n$, that the total welfare $\sum_{i=1}^{j} z_i$ of the first j upstream countries at a Pareto efficient pair (x^*, t) should be at least equal to the maximum that these countries can guarantee themselves by solving the welfare maximization problem

$$\max_{x_1,\ldots,x_j} \sum_{i=1}^{j} b_i(x_i) \text{ s.t. } \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} e_i, \quad k = 1, \ldots, j, \text{ and } x_i \geq 0, \ i = 1, \ldots, j. \tag{2.3}$$

Under the assumptions made on the benefit functions in Ambec and Sprumont (2002), for each j this maximization problem has a unique solution. We denote this solution by $x^j = (x_1^j, \ldots, x_j^j)$ and the corresponding total welfare by $v^j = \sum_{i=1}^{j} b_i(x_i^j)$. Notice that $x_i^n = x_i^*$, $i = 1, \ldots, n$, and $v^n = \sum_{i=1}^{n} b_i(x_i^*)$. It follows that upstream stability requires that $\sum_{i=1}^{j} z_i \geq v^j$ for every $j = 1, \ldots, n$.

On the other hand, based on the UTI principle favoring the downstream countries, Ambec and Sprumont (2002) impose the condition that for every upstream coalition $\{1, \ldots, j\}$, $j = 1, \ldots, n$, the total welfare of these countries is bounded from above by their *aspiration level*, being the maximum welfare they can obtain by distributing their own water optimally amongst themselves. Thus, the aspiration level property requires that for each j the total welfare $\sum_{i=1}^{j} z_i$ of the first j upstream countries is at most equal to the welfare obtained from solving the welfare maximization problem (2.3), i.e., $\sum_{i=1}^{j} z_i \leq v^j$ for every $j = 1, \ldots, n$. It follows that the upstream stability requirement and the aspiration level property together require that $\sum_{i=1}^{j} z_i = v^j$ for every $j = 1, \ldots, n$, and thus determine the unique welfare distribution $z_i = v^i - v^{i-1}$, $i = 1, \ldots, n$, with v^0 defined to be equal to zero. The corresponding so-called *downstream incremental solution* assigns to every river problem (N, e, b), the welfare distribution $d(N, e, b) \in \mathbb{R}^n$ given by

$$d_i(N, e, b) = v^i - v^{i-1}, \qquad i = 1, \ldots, n. \tag{2.4}$$

Although this welfare distribution is fully determined by the upstream stability requirement and the aspiration level property, it is also stable for every coalition $\{i, i+1, \ldots, j\}$, $1 \leq i \leq j \leq n$, of consecutive agents.

[1]Within the game-theoretic model of Ambec and Sprumont this equals the well-known notion of Core stability.

[2]Notice that this notion of individual rationality only holds under the assumption that a country is the legal owner of its own inflow.

Ambec and Ehlers (2008) generalized the basic river game described above by allowing for satiable agents. This means that they weaken the assumption on the benefits in Ambec and Sprumont (2002) by deleting the requirement that the benefit function is strictly increasing. They assume that every benefit function $b_i\colon \mathbb{R}_+ \to \mathbb{R}$ is a strictly concave function, differentiable at every $x_i > 0$ with derivative going to infinity as x_i tends to zero. Under this assumption it is possible that for some point $c_i > 0$, called the *satiation point* of agent i, the benefit is increasing from $x_i = 0$ to c_i, reaches its maximum value at c_i, and is decreasing for $x_i > c_i$. The existence of satiation points has serious consequences for the corresponding cooperative game. Without satiation points, only coalitions of consecutive agents are able to cooperate in order to maximize their joint welfare by allocating optimally their own water inflows amongst each other (under the Absolute Territorial Sovereignty (ATS) assumption, saying that the agents in each coalition have the rights to use their own water inflows). A non-consecutive coalition of two (consecutive) subsets of agents would never transfer water from the upstream part to the downstream part, because the increasing benefit functions would make that all water sent from the upstream agents to the downstream agents would be taken by the agents in-between the two parts. In contrast, under the weaker assumption of Ambec and Ehlers (2008), it might be profitable for a non-consecutive coalition of agents to transfer water from its upstream part to its downstream part when all agents in-between have a satiation point. Although some of this flow might be taken by the in-between agents, these agents will only take water up to their satiation points. When the flow is big enough, part of it will reach the downstream agents, possibly rendering cooperation between the two parts of the coalition profitable. This phenomenon might cause positive *externalities* on the agents in-between two parts of a non-consecutive coalition. As a result, in the corresponding cooperative game the worth that can be obtained by a coalition depends on the behavior of the other agents, leading to a more complicated model, a so-called game in partition function form. However, it is clear that for every j, the upstream coalition $\{1, \ldots, j\}$ is *externality free*, i.e., the maximum welfare that such a coalition can obtain by allocating their own water inflows optimally amongst themselves does not depend on the behavior of the agents after j, and these maximum welfare levels are still given by the values v^j, $j = 1, \ldots, n$, at the solutions of the welfare maximization problems (2.3). Therefore, the downstream incremental solution $d(N, e, b)$ is still well-defined for river problems (N, e, b) with satiation points, and in Ambec and Ehlers (2008) it is shown that also for river situations with satiable agents this solution is uniquely determined by requiring upstream stability and the aspiration level property. Although they model the river problem with satiable agents by a game in partition function form, eventually they characterize a solution which only uses the welfare levels that can be obtained by consecutive coalitions containing agent 1, and these levels are externality free. Under the assumption that $e_i \leq c_i$ for every i, the solution is also stable for every coalition $\{i, i+1, \ldots, j\}$, $1 \leq i \leq j \leq n$, of consecutive agents.

In this paper we further weaken the assumptions of Ambec and Ehlers (2008), and therefore also those of Ambec and Sprumont (2002), by allowing the benefit functions to be concave instead of strictly concave. Moreover, we weaken differentiability to continuity.

Assumption 2.1 *In a river situation (N, e, b), every benefit function $b_i\colon \mathbb{R}_+ \to \mathbb{R}$ is concave and continuous for $x_i > 0$.*

The assumption says that b_i may be nondecreasing, but also allows that there may exist an interval $[c_i, c^i]$, $c^i \geq c_i$, such that b_i is increasing on $x_i < c_i$, constant on $x_i \in [c_i, c^i]$, and decreasing when $x_i > c^i$. In the latter case the point c_i is the *satiation point* of agent i. Agent i reaches its highest benefit at c_i. All water consumption levels between c_i and c^i also yield this maximal benefit, but water consumption higher than c^i yields a lower benefit. We allow for $c_i = 0$ and $c^i = \infty$ (meaning that b_i is constant for $x_i \geq c_i \geq 0$). In particular this allows for $b_i(x_i) = b_i(0)$ for every $x_i \geq 0$.

Under Assumption 2.1, the maximization problems (2.3) do not necessarily have a unique solution, but are still well-defined. Let X^j be the set of solutions of the welfare maximization problem (2.3) for country j, $j = 1, \ldots, n$, under Assumption 2.1. Then, for every solution $x^j \in X^j$ we have that $v^j = \sum_{i=1}^{j} b_i(x_i^j)$ and for every $x^n \in X^n$, the budget balanced pair (x^n, t) yields a welfare distribution

$$z_i = b_i(x_i^n) + t_i, \quad i = 1, \ldots, n,$$

with sum of payoffs equal to the Pareto efficient total welfare $v^n = \sum_{i=1}^{n} b_i(x_i^n)$.

Under Assumption 2.1, the corresponding cooperative game is not well defined unless we make additional assumptions on the water consumption of agents that have concave, but not strictly concave, benefit functions. Consider again a non-consecutive coalition consisting of a consecutive upstream part and a consecutive downstream part. If some agent j between these two parts has a benefit function with a satiation point c_j and a point $c^j > c_j$ such that its benefit is constant between c_j and c^j and decreasing thereafter, then the cooperative game is not well-defined without an additional assumption about the water consumption of agent j in case the water flow sent by the upstream part to the downstream part becomes so big that the availability of water for agent j exceeds his satiation point c_j. Instead of making such an assumption, in the following sections we will impose axioms directly on the river situation (N, e, b) and we derive from these axioms unique solutions for the welfare distribution problem without modeling the river situation as a cooperative game. Doing so, we do not need such an additional assumption.

3 Axioms for fair welfare distribution

In this section we formulate axioms concerning the distribution of welfare in river situations (N, e, b), where the preferences of the agents over water are described by benefit functions satisfying Assumption 2.1. Let $\mathcal{W}^N$ denote the collection of all river situations (N, e, b) satisfying Assumption 2.1. Then a *solution* is a function f assigning to every $(N, e, b) \in \mathcal{W}^N$ a welfare distribution $f(N, e, b) \in \mathbb{R}^n$. In the sequel, the component $f_i(N, e, b)$ is called the *payoff* of agent i, $i = 1, \ldots, n$. Notice that the welfare levels depend on the inflow vector e and the set of benefit functions b. Therefore, in the following we denote for every j the maximum welfare level obtained by solving the welfare maximization problem (2.3) by $v^j(e, b)$ instead of just v^j.

As we have seen before, stability reflects the Harmon principle that each agent is the legal owner of its own water, and therefore stability puts a severe requirement on the welfare distribution. In particular, upstream stability may contradict the TIBS principle that each inflow belongs to all agents together. It certainly is conflicting with the UTI interpretation of the TIBS principle stating that every agent has the right to use the total inflow of himself and all his upstream agents. Therefore, one might argue that stability is too strong. In the next two axioms we weaken the stability requirement. The efficiency axiom only requires stability for the coalition of all agents and states that the total sum of payoffs equals the total welfare $v^n(e, b)$ in an optimal water allocation.

As noticed before, for individual agents stability yields individual rationality, saying that an agent gets at least a payoff equal to the highest benefit that can be obtained by consuming at most its own water inflow (this reflects that every agent is the owner of its own inflow). Under Assumption 2.1 this benefit is at least equal to the benefit of consuming a zero amount of water. The lower bound property axiom weakens individual rationality by only requiring that each agent gets at least a payoff equal to the benefit of consuming a zero amount of water.

Axiom 3.1 (Efficiency) *For every river problem (N, e, b) we have that $\sum_{i \in N} f_i(N, e, b) = v^n(e, b)$.*

Axiom 3.2 (Lower bound property) *For every river problem (N, e, b) we have that $f_i(N, e, b) \geq b_i(0)$ for all $i \in N$.*

As mentioned before, the aspiration level property reflects the UTI principle, but also utilizes this principle to put an upper bound on the total payoff to the members of all upstream coalitions $\{1, \ldots, j\}$, $j = 1, \ldots, n$, by stating that the total payoff to such a coalition is at most equal to the highest total benefit it can obtain by allocating at most their own water inflows amongst themselves. By this requirement it is a priori excluded that the total monetary compensation to an upstream coalition is more than what is needed to compensate the agents in the coalition for their loss of total benefit by allocating some of their inflow to their downstream agents. Consequently, all benefits from allocating all inflows optimally over all agents go to the downstream agents. One might wonder why a coalition of upstream agents agrees with such an optimal allocation if their total payoff is at most equal to what they can get by consuming all the water by themselves. In the next two axioms we weaken the aspiration level property in two different ways. In the first axiom, called the drought property, the aspiration level property is only required for an upstream coalition if the total water inflow of the coalition is zero. In the second axiom the aspiration level property is weakened in a way that a coalition of upstream agents can benefit from allocating some of their inflows to downstream agents. This weak aspiration level property requires that no agent earns a higher payoff as its utility when it has access to all the water inflow, that is its own inflow plus all upstream and downstream water inflows.

Axiom 3.3 (Drought property) *For every river problem* (N, e, b) *with* $e_j = 0$ *for all* $j \leq i$*, we have that* $f_i(N, e, b) \leq b_i(0)$*.*

Axiom 3.4 (Weak aspiration level property) *For every river problem* (N, e, b)*, we have that* $f_i(N, e, b) \leq \max_{x_i \leq \sum_{j \in N} e_j} b_i(x_i)$ *for all* $i \in N$*.*[3]

A more severe axiom than the drought property is the no contribution property. It states that an agent with zero inflow of water on his territory should get at most a payoff equal to its benefit of zero water consumption.

Axiom 3.5 (No contribution property) *For every river problem* (N, e, b) *and* $i \in N$ *with* $e_i = 0$ *we have that* $f_i(N, e, b) \leq b_i(0)$*.*

Next we state several independence axioms. The first one states that the payoff of an agent does not depend on the benefit functions of its downstream agents, the second one states that the payoff of an agent does not depend on the benefit functions of its upstream agents.

Axiom 3.6 (Independence of downstream benefits) *For every pair of river problems* (N, e, b) *and* (N, e, b') *such that* $b_j = (b')_j$ *for all* $j \leq i$*, we have that* $f_i(N, e, b) = f_i(N, e, b')$*.*

Axiom 3.7 (Independence of upstream benefits) *For every pair of river problems* (N, e, b) *and* (N, e, b') *such that* $b_j = (b')_j$ *for all* $j \geq i$*, we have that* $f_i(N, e, b) = f_i(N, e, b')$*.*

[3]Note that under increasing benefit functions we can write this inequality as $f_i(N, e, b) \leq b_i(\sum_{i \in N} e_i)$ for all $i \in N$.

The next two independence axioms concern the inflows. The first one states that the payoff of an agent does not depend on the inflows at the territories of its downstream agents, the second one states that the payoff of an agent does not depend on the inflows at the territories of its upstream agents.

Axiom 3.8 (Independence of downstream inflows) *For every pair of river problems* (N, e, b) *and* (N, e', b) *such that* $e_j = e'_j$ *for all* $j \leq i$*, we have that* $f_i(N, e, b) = f_i(N, e', b)$*.*

Axiom 3.9 (Independence of upstream inflows) *For every pair of river problems* (N, e, b) *and* (N, e', b) *such that* $e_j = e'_j$ *for all* $j \geq i$*, we have that* $f_i(N, e, b) = f_i(N, e', b)$*.*

As follows from the results in the following sections, not all the axioms stated in this section can be satisfied simultaneously. In the next sections we show that different sets of axioms characterize different fair welfare distributions.

4 The downstream incremental solution

The first result is that on the class $\mathcal{W}^N$ of river problems (N, e, b) the downstream incremental solution d, given by (2.4) and discussed in Section 2, is characterized by the four axioms of efficiency, lower bound property, weak aspiration level property and independence of downstream benefits.

Theorem 4.1 *A solution* f *on the class* $\mathcal{W}^N$ *of river problems is equal to the downstream incremental solution* d *if and only if* f *satisfies efficiency, the lower bound property, the weak aspiration level property and independence of downstream benefits.*

Proof. It is straightforward to show that the downstream incremental solution satisfies these four axioms. Hence, it is sufficient to show that the four axioms determine a unique solution.

Let $(N, e, b) \in \mathcal{W}^N$ be a river problem and suppose that solution f satisfies the four axioms. We prove uniqueness by induction on the labels of the agents, starting with the most upstream agent 1. We first show that $f_1(N, e, b)$ is uniquely determined by the four axioms. Consider the modified river problem (N, e, b^1) given by benefit functions $(b^1)_1 = b_1$, and $(b^1)_j(x) = 0$ for all $x \in \mathbb{R}_+$ and $j \in \{2, \ldots, n\}$. Imposing the lower bound property on (N, e, b^1) requires that $f_j(N, e, b^1) \geq (b^1)_j(0) = 0$ for all $j \in \{2, \ldots, n\}$, while the weak aspiration level property requires that $f_j(N, e, b^1) \leq \max_{x_j \leq \sum_{k \in N} e_k} (b^1)_j(x_j) = 0$ for all $j \in \{2, \ldots, n\}$. Hence, $f_j(N, e, b^1) = 0$ for all $j \in \{2, \ldots, n\}$. By efficiency we then have that $f_1(N, e, b^1) = v^n(e, b^1)$ being the welfare level at the solution of the maximization problem (2.2) for (N, e, b^1). Since $(b^1)_j(x) = 0$ for every $x \in \mathbb{R}_+$ and $j \in \{2, \ldots, n\}$, it follows that

$v^n(e, b^1) = v^1(e, b^1) = \max_{x_1 \leq e_1}(b^1)_1(x_1) = \max_{x_1 \leq e_1} b_1(x_1) = v^1(e, b)$. Independence of downstream benefits then implies that $f_1(N, e, b) = f_1(N, e, b^1) = v^1(e, b) = d_1(N, e, b)$.

Proceeding by induction, assume that $f_k(N, e, b) = d_k(N, e, b)$ for all $k < i \leq n$. Next, consider the modified river problem (N, e, b^i) given by $(b^i)_j = b_j$ for all $j \in \{1, \dots, i\}$ and $(b^i)_j(x) = 0$ for all $x \in \mathbb{R}_+$ and $j \in \{i+1, \dots, n\}$. Similar as above, the lower bound property requires that $f_j(N, e, b^i) \geq 0$ for all $j \in \{i+1, \dots, n\}$, while the weak aspiration level property requires that $f_j(N, e, b^i) \leq 0$ for all $j \in \{i+1, \dots, n\}$. Thus

$$f_j(N, e, b^i) = 0 \text{ for all } j \in \{i+1, \dots, n\}. \tag{4.5}$$

Independence of downstream benefits and the induction hypothesis imply that $f_j(N, e, b^i) = f_j(N, e, b) = d_j(N, e, b)$ for all $j \in \{1, \dots, i-1\}$. Hence,

$$\sum_{j=1}^{i-1} f_j(N, e, b^i) = \sum_{j=1}^{i-1} d_j(N, e, b) = v^{i-1}(e, b). \tag{4.6}$$

Efficiency, the induction hypothesis, (4.5) and (4.6) then determine that

$$f_i(N, e, b^i) = v^n(e, b^i) - \sum_{j=1}^{i-1} f_j(N, e, b^i) - \sum_{j=i+1}^{n} f_j(N, e, b^i) = v^n(e, b^i) - v^{i-1}(e, b). \tag{4.7}$$

Since $(b^i)_j(x) = 0$ for every $x \in \mathbb{R}_+$ and $j \in \{i+1, \dots, n\}$, similar as above it follows that $v^n(e, b^i) = v^i(e, b^i) = v^i(e, b)$. Therefore, with (4.7) we have $f_i(N, e, b^i) = v^n(e, b^i) - v^{i-1}(e, b) = v^i(e, b) - v^{i-1}(e, b) = d_i(N, e, b)$. Finally, independence of downstream benefits implies that $f_i(N, e, b) = f_i(N, e, b^i) = d_i(N, e, b)$. □

Logical independence of the axioms used in Theorem 4.1 and in the characterizations in the following sections, is shown in the appendix. Notice that the solution is fully determined by the welfare levels obtained by solving the welfare maximization problems (2.3) and that these problems are well-defined when the benefit functions satisfy Assumption 2.1. Therefore we do not need to make any additional assumption concerning agents with concave, but not strictly concave, benefit functions.

5 The upstream incremental solution

As noticed in Section 2, the aspiration level property puts an upper bound on the total payoff to the members of an upstream coalition $\{1, \dots, j\}$, $j = 1, \dots, n$. Therefore, according to the downstream incremental solution all gains in benefits that occur when some of the inflows at the territories of an upstream coalition $\{1, \dots, j\}$ are allocated to its downstream agents i, $i > j$, go to the downstream agents in the sense that the upstream coalition is only compensated for its loss of total benefit. Alternatively, van den Brink, van der Laan and Vasil'ev (2007) introduced the *upstream incremental solution*. According to this solution all gains in benefits that occur when some of the inflows at the territories of an upstream coalition $\{1, \ldots, j\}$ are allocated to its downstream agents i, $i > j$, go to the upstream agents in the sense that the total payoff to the downstream coalition $\{j+1, \ldots, n\}$ is just equal to the total benefit they can achieve by allocating their own inflows optimally amongst themselves. In van den Brink, van der Laan and Moes (2010) a class of so-called TIBS-fairness axioms is introduced that, together with efficiency, yield the downstream incremental solution and the upstream incremental solution as extreme cases.

To define the upstream incremental solution, we consider for every $j = 1, \ldots, n$, the welfare maximization problem

$$\max_{x_j,\ldots,x_n} \sum_{i=j}^{n} b_i(x_i) \text{ s.t. } \sum_{i=j}^{k} x_i \leq \sum_{i=j}^{k} e_i, \quad k = j, \ldots, n, \text{ and } x_i \geq 0, \; i = j, \ldots, n, \quad (5.8)$$

i.e., for agent j the maximization problem (5.8) optimally allocates the inflows $e_j, \ldots, e_n$ amongst the agents in the coalition $\{j, j+1, \ldots, n\}$, given the uni-directionality of the water flow. Under Assumption 2.1, these maximization problems do not have a unique solution but are well-defined. For a solution $y^j = (y^j_j, \ldots, y^j_n)$ of maximization problem (5.8) for agent j, denote $w^j(e, b) = \sum_{i=j}^{n} b_i(y^j_i)$ as the maximum welfare that the agents in $\{j, j+1, \ldots, n\}$ can obtain by distributing their own inflows. Notice that for $j = 1$ the maximization problem (5.8) is equal to problem (2.2), so that $w^1(e, b) = v^n(e, b)$ is the maximum total benefit that can be obtained when allocating all inflows optimally amongst all agents. For every solution y^1 of (5.8), the budget balanced pair (y^1, t) yields a welfare distribution

$$z_i = b_i(y^1_i) + t_i, \quad i = 1, \ldots, n,$$

with sum of payoffs equal to the Pareto efficient total welfare $w^1(e, b) = \sum_{i=1}^{n} b_i(y^1_i)$.

The upstream incremental solution assigns to every agent i its marginal contribution to the welfare when the agents enter subsequently from the most downstream agent to the most upstream agent. Hence, the upstream incremental solution assigns to every river situation (N, e, b), the welfare distribution $u(N, e, b) \in \mathbb{R}^n$ given by

$$u_i(N, e, b) = w^i(e, b) - w^{i+1}(e, b), \qquad i = 1, \ldots, n,$$

with $w^{n+1}(e, b) = 0$.

The next theorem states that on the class $\mathcal{W}^N$ of river problems (N, e, b) the upstream incremental solution u is characterized by the four axioms of efficiency, lower bound property, drought property and independence of upstream inflows.

Theorem 5.1 *A solution f on the class $\mathcal{W}^N$ of river problems is equal to the upstream incremental solution u if and only if f satisfies efficiency, the lower bound property, the drought property and independence of upstream inflows.*

Proof. It is straightforward to show that the upstream incremental solution satisfies these four axioms. Therefore, it is sufficient to prove that the four axioms determine a unique solution.

Let $(N, e, b) \in \mathcal{W}^N$ be a river problem and suppose that solution f satisfies the four axioms. We apply induction on the labels of the agents, starting with the most downstream agent n. We first consider $f_n(N, e, b)$. Consider the modified river problem (N, e^n, b) given by $(e^n)_n = e_n$, and $(e^n)_j = 0$ for all $j \in \{1, \ldots, n-1\}$. The lower bound property requires that $f_j(N, e^n, b) \geq b_j(0)$ for all $j \in \{1, \ldots, n-1\}$, while the drought property requires that $f_j(N, e^n, b) \leq b_j(0)$ for all $j \in \{1, \ldots, n-1\}$. Thus, we conclude that $f_j(N, e^n, b) = b_j(0)$ for all $j \in \{1, \ldots, n-1\}$. By efficiency we then have that

$$f_n(N, e^n, b) = w^1(e^n, b) - \sum_{j=1}^{n-1} f_j(N, e, b) = w^1(e^n, b) - \sum_{j=1}^{n-1} b_j(0). \tag{5.9}$$

Since $e^n_j = 0$ for all $j \in \{1, \ldots, n-1\}$ and $e^n_n = e_n$, it follows that $w^1(e^n, b) = \sum_{j=1}^{n-1} b_j(0) + w^n(e^n, b) = \sum_{j=1}^{n-1} b_j(0) + w^n(e, b)$, and thus with (5.9) we have $f_n(N, e^n, b) = w^n(e, b)$. Independence of upstream inflows then implies that $f_n(N, e, b) = f_n(N, e^n, b) = w^n(e, b) = u_n(N, e, b)$.

Proceeding by induction, assume that $f_k(N, e, b) = u_k(N, e, b)$ is determined for all $k > i \geq 1$. Next, consider the modified river problem (N, e^i, b) given by $(e^i)_j = e_j$ for all $j \in \{i, \ldots n\}$, and $(e^i)_j = 0$ for all $j \in \{1, \ldots, i-1\}$. Similar as above, the lower bound property requires that $f_j(N, e^i, b) \geq b_j(0)$ for all $j \in \{1, \ldots, i-1\}$, while the drought property requires that $f_j(N, e^i, b) \leq b_j(0)$ for all $j \in \{1, \ldots, i-1\}$. Thus,

$$f_j(N, e^i, b) = b_j(0) \text{ for all } j \in \{1, \ldots, i-1\}. \tag{5.10}$$

Independence of upstream inflows and the induction hypothesis imply that $f_j(N, e^i, b) = f_j(N, e, b) = u_j(N, e, b)$ for all $j \in \{i+1, \ldots, n\}$. Hence,

$$\sum_{j=i+1}^{n} f_j(N, e^i, b) = \sum_{j=i+1}^{n} u_j(N, e, b) = w^{i+1}(e, b). \tag{5.11}$$

Efficiency, the induction hypothesis, (5.10) and (5.11) then determine that

$$\begin{aligned} f_i(N, e^i, b) &= w^1(e^i, b) - \sum_{j=1}^{i-1} f_j(N, e^i, b) - \sum_{j=i+1}^{n} f_j(N, e^i, b) \\ &= w^1(e^i, b) - \sum_{j=1}^{i-1} b_j(0) - w^{i+1}(e, b). \end{aligned} \tag{5.12}$$

Since $e^i_j = 0$ for all $j \in \{1, \ldots, i-1\}$ and $e^i_j = e_j$ for all $j \in \{i, \ldots n\}$, similar as above it follows that $w^1(e^i, b) = \sum_{j=1}^{i-1} b_j(0) + w^i(e^i, b) = \sum_{j=1}^{i-1} b_j(0) + w^i(e, b)$. Thus, with (5.12) we have $f_i(N, e^i, b) = w^1(e^i, b) - \sum_{j=1}^{i-1} b_j(0) - w^{i+1}(e, b) = w^i(e, b) - w^{i+1}(e, b) = u_i(N, e, b)$. Finally, independence of downstream benefits implies that $f_i(N, e, b) = f_i(N, e^i, b) = u_i(N, e, b)$. □

Notice that the solution is fully determined by the welfare levels obtained by solving the welfare maximization problems (5.8). Hence, by definition the upstream incremental solution satisfies stability for every downstream coalition $\{i, i+1 \ldots, n\}$. Like the downstream incremental solution, it also satisfies the stability requirement for every coalition of consecutive agents, see for instance van den Brink, van der Laan and Vasil'ev (2007).

6 The downstream solution

As mentioned in the previous sections, both the downstream and the upstream incremental solution satisfy stability for every coalition of consecutive countries, and so they both meet the Harmon doctrine saying that every agent is the legal owner of its own water inflow. Under the Harmon doctrine the downstream incremental solution favors the downstream agents as much as possible. The upstream incremental solution is more in favor of the upstream agents.

As discussed in Section 2, the Harmon doctrine is conflicting with the TIBS principle which makes all countries together the legal owner of all water inflows. For example, according to the TIBS principle all agents are entitled to get a share of the water inflow e_1 at the territory of agent 1. Taking the TIBS principle in its most extreme form, one might argue that the most downstream agent is entitled to receive all the water inflows. Under this condition, the upstream agents can 'buy' water by compensating the most downstream agent for its loss of water. Taking this viewpoint on water rights, we define the *downstream solution* s, which assigns to a river situation $(N, e, b) \in \mathcal{W}^N$ the welfare distribution $s(N, e, b)$ given by

$$s_i(N, e, b) = \widehat{w}^i(e, b) - \widehat{w}^{i+1}(e, b), \quad i = 1, \ldots, n,$$

where $\widehat{w}^{n+1}(e, b) = 0$ and $\widehat{w}^j(e, b) = \sum_{i=j}^{n} b_i(y^j_i)$, $j = 1, \ldots, n$, with $y^j = (y^j_j, \ldots, y^j_n)$ a solution of the welfare maximization problem

$$\max_{x_j, \ldots, x_n} \sum_{i=j}^{n} b_i(x_i) \text{ s.t. } \sum_{i=j}^{k} x_i \leq \sum_{i=1}^{k} e_i, \quad k = j, \ldots, n, \text{ and } x_i \geq 0, \ i = j, \ldots, n. \quad (6.13)$$

Notice the difference between the maximization problems (6.13) and (5.8). In (5.8) the agents in a downstream coalition $\{j, j+1, \ldots, n\}$ can only consume their own water inflow, while in (6.13) they can use their own water inflows and also all the water inflows at the territories of all agents upstream of j. Then, for a coalition $\{j, j+1, \ldots, n\}$ the maximization problem (6.13) optimally allocates the inflows $e_1, \ldots, e_n$ amongst the agents in the coalition, given the uni-directionality of the water flow. Notice that for $j = 1$ the maximization problem (6.13) is again equal to problem (2.2), so that $\widehat{w}^1(e,b) = v^n(e,b)$ is the maximum total benefit that can be obtained when allocating all inflows optimally amongst all agents. Since $\sum_{i=1}^n s_i(N,e,b) = \widehat{w}^1(e,b) = v^n(e,b)$, also the downstream solution distributes the maximum attainable total welfare that the agents can achieve together and thus also this solution is efficient.

It turns out that the downstream solution can be characterized similarly as the downstream incremental solution in Theorem 4.1, but now the independence of downstream benefits is replaced by independence of upstream benefits.[4]

Theorem 6.1 *A solution f on the class $\mathcal{W}^N$ of river problems is equal to the downstream solution s if and only if f satisfies efficiency, the lower bound property, the weak aspiration level property and independence of upstream benefits.*

Proof. It is straightforward to prove that the downstream solution satisfies these four axioms. Therefore, it only remains to prove that the four axioms determine a unique solution.

Let $(N,e,b) \in \mathcal{W}^N$ be a river problem and suppose that solution f satisfies the four axioms. Similar as in the proof of Theorem 5.1, we apply induction on the labels of the agents, starting with the most downstream agent n. We first determine $f_n(N,e,b)$. Consider the modified river problem (N,e,b^n) given by $(b^n)_n = b_n$, and $(b^n)_j(x) = 0$ for all $x \in \mathbb{R}_+$ and $j \in \{1, \ldots, n-1\}$. The lower bound property requires that $f_j(N,e,b^n) \geq (b^n)_j(0) = 0$ for all $j \in \{1, \ldots, n-1\}$, while the weak aspiration level property requires that

$$f_j(N,e,b^n) \leq \max_{x_j \leq \sum_{k \in N} e_k} (b^n)_j(x_j) = 0, \quad \text{for all } j \in \{1, \ldots, n-1\}.$$

Thus, we conclude that $f_j(N,e,b^n) = 0$ for all $j \in \{1, \ldots, n-1\}$. By efficiency we then have that

$$f_n(N,e,b^n) = v^n(e,b^n) = \widehat{w}^1(e,b^n). \tag{6.14}$$

[4]For the auction games of Graham, Marshall and Richard (1990), it is shown in van den Brink (2004) that, together with efficiency and symmetry, independence of higher valuations characterizes the Shapley value, while independence of lower valuations characterizes the equal division solution. In bankruptcy or rationing problems independence on higher claims is used to characterize the constrained equal awards rule, while independence on lower claims is used to characterize the constrained equal losses rule, see e.g. Moulin (2003).

Since $(b^n)_j(x) = 0$ for every $x \in \mathbb{R}_+$ and $j \in \{1, \ldots, n-1\}$, and $(b^n)_n = b^n$, it follows that $\widehat{w}^1(e, b^n) = \widehat{w}^n(e, b^n) = \widehat{w}^n(e, b)$ and thus, with (6.14), $f_n(N, e, b^n) = \widehat{w}^1(e, b^n) = \widehat{w}^n(e, b) = s_n(N, e, b)$. Independence of upstream benefits then implies that $f_n(N, e, b) = f_n(N, e, b^n) = s_n(N, e, b)$.

Proceeding by induction, assume that $f_k(N, e, b) = s_k(N, e, b)$ for all $k > i \geq 1$. Next, consider the modified river problem (N, e, b^i) given by $(b^i)_j = b_j$ for all $j \in \{i, \ldots, n\}$, and $(b^i)_j(x) = 0$ for all $x \in \mathbb{R}_+$ and $j \in \{1, \ldots, i-1\}$. Similar as above, the lower bound property requires that $f_j(N, e, b^i) \geq 0$ for all $j \in \{1, \ldots, i-1\}$, while weak aspiration level fairness requires that $f_j(N, e, b^i) \leq 0$ for all $j \in \{1, \ldots, i-1\}$. Thus,

$$f_j(N, e, b^i) = 0 \text{ for all } j \in \{1, \ldots, i-1\}. \tag{6.15}$$

Independence of upstream benefits and the induction hypothesis imply that $f_j(N, e, b^i) = f_j(N, e, b) = s_j(N, e, b)$ for all $j \in \{i+1, \ldots, n\}$. Therefore,

$$\sum_{j=i+1}^{n} f_j(N, e, b^i) = \sum_{j=i+1}^{n} s_j(N, e, b) = \widehat{w}^{i+1}(e, b). \tag{6.16}$$

Efficiency, the induction hypothesis, (6.15) and (6.16) then determine that

$$f_i(N, e, b^i) = \widehat{w}^1(e, b^i) - \sum_{j=1}^{i-1} f_j(N, e, b^i) - \sum_{j=i+1}^{n} f_j(N, e, b^i) = \widehat{w}^1(e, b^i) - \widehat{w}^{i+1}(e, b). \tag{6.17}$$

Since $(b^i)_j(x) = 0$ for every $x \in \mathbb{R}_+$ and $j \in \{1, \ldots, i-1\}$, and $(b^i)_j = b_j$ for all $j \geq i$, similar as above it follows that $\widehat{w}^1(e, b^i) = \widehat{w}^i(e, b^i) = \widehat{w}^i(e, b)$. Hence, with (6.17) we have $f_i(N, e, b^i) = \widehat{w}^1(e, b^i) - \widehat{w}^{i+1}(e, b) = \widehat{w}^i(e, b) - \widehat{w}^{i+1}(e, b) = s_i(N, e, b)$. Finally, independence of upstream benefits implies that $f_i(N, e, b) = f_i(N, e, b^i) = s_i(N, e, b)$. □

Note that in the welfare maximization problems (6.13), the agents in the downstream coalition $\{j, \ldots, n\}$ are entitled to get the total water inflow $\sum_{i \in N} e_i$. When some of the water is allocated to other agents, according to the downstream solution the most downstream agent is fully compensated for his loss of benefits by monetary compensations of the other agents. This is an extreme interpretation of the TIBS principle. Consequently, the downstream solution does not satisfy upstream stability and thus violates the Harmon doctrine: clearly all water rights are given to the most downstream agent.

7 The upstream solution

As noticed in the previous section, the upstream incremental solution favors the upstream agents as much as possible under the restriction of stability for the downstream coalitions. As counterpart of the downstream solution, in this section we introduce the *upstream* *solution* r that favors the upstream agents as much as possible given the uni-directionality of the water flows. It takes the Harmon principle in its most extreme form by requiring that the agents from upstream to downstream receive the highest attainable additional benefit from their water inflows given that the inflows of their upstream agents have been distributed already.

To define the upstream solution, we first reconsider the welfare distribution according to the upstream incremental solution. This solution yields a payoff $u_n(N, e, b) = w^n(e, b)$ to the last agent n, where $w^n(e, b)$ is the highest benefit that agent n can achieve from the consumption of only his own water inflow. Then agent $n-1$ receives $u_{n-1}(N, e, b) = w^{n-1}(e, b) - w^n(e, b)$, where $w^{n-1}(e, b)$ is the total benefit that agents $n-1$ and n can jointly achieve by distributing their own water optimally. In this way agent $n-1$ receives its marginal contribution to the total benefit of his water inflow e_{n-1} to the water inflow e_n, taking all the upstream inflows equal to zero. In general, agent i receives his marginal contribution to the total benefit of his water inflow e_i to the downstream inflows e_j, $j > i$, taking all the upstream inflows e_j, $j < i$, equal to zero.

Similar to the upstream incremental solution, the upstream solution can be defined the other way around, starting with agent 1. When all inflows are zero, every agent has payoff $b_i(0)$, $i = 1, \ldots, n$. Now, let the most upstream inflow e_1 be distributed optimally amongst all agents. Then agent 1 receives in addition to $b_1(0)$ a payoff equal to the marginal contribution to the total benefit when distributing his inflow e_1 optimally amongst all other agents and assuming all other inflows to be equal to zero, i.e., the upstream solution r yields to agent 1 payoff $r_1(N, e, b,) = \widehat{v}^1(e, b)$, where

$$\widehat{v}^1(e, b) = b_1(0) + \sum_{j=1}^{n} (b_j(y_j^1) - b_j(0)) = b_1(y_1^1) + \sum_{j=2}^{n} (b_j(y_j^1) - b_j(0)),$$

with $y^1 = (y_1^1, \ldots, y_n^1)$ a solution of the welfare maximization problem

$$\max_{x_1,\ldots,x_n} \sum_{j=1}^{n} b_j(x_j) \text{ s.t. } \sum_{j=1}^{n} x_j \leq e_1,\ x_j \geq 0,\ j = 1, \ldots, n.$$

Next, the inflows e_1 and e_2 are distributed optimally over all agents assuming all other inflows to be equal to zero, and agent 2 receives his initial payoff $b_2(0)$ plus the additional total benefit that the distribution of his inflow e_2 generates to the benefits obtained already from e_1. Subsequently, for agent i all inflows e_j, $j \leq i$, are distributed optimally over all agents assuming all inflows of the downstream agents $j > i$ to be equal to zero, and agent i receives his initial payoff $b_i(0)$ plus the additional total benefit that the distribution of his inflow e_i generates to the benefit obtained already from e_1 to e_{i-1}. In general, the upstream solution r assigns to a river situation $(N, e, b) \in \mathcal{W}^N$ the welfare distribution

$r(N,e,b)$ given by

$$r_i(N,e,b) = \widehat{v}^i(e,b) - \widehat{v}^{i-1}(e,b), \quad i = 1,\dots,n,$$

where $\widehat{v}^0(e,b) = 0$ and $\widehat{v}^i(e,b) = \sum_{j=1}^{i} b_j(y_j^i) + \sum_{j=i+1}^{n} (b_j(y_j^i) - b_j(0))$, $i = 1,\dots,n$, with $y^i = (y_1^i,\dots,y_n^i)$ a solution of the welfare maximization problem

$$\max_{x_1,\dots,x_n} \sum_{j=1}^{n} b_j(x_j) \text{ s.t. } \begin{cases} \sum_{j=1}^{n} x_j \le \sum_{j=1}^{i} e_j, \\ \sum_{j=1}^{k} x_j \le \sum_{j=1}^{k} e_j, \; k = 1,\dots,i-1, \\ x_j \ge 0, \; j = 1,\dots,n. \end{cases} \tag{7.18}$$

Observe that this maximization problem optimally distributes the water inflow of the agents in $\{1,\dots,i\}$ over all agents, taking into account that for every agent $k < i$ the total consumption of the first k agents is at most equal to the sum of their own inflows. The payoffs of the welfare distribution $r(N,e,b)$ can also be written as

$$\begin{aligned} r_i(N,e,b) &= \widehat{v}^i(e,b) - \widehat{v}^{i-1}(e,b) \\ &= \sum_{j=1}^{i} b_j(y_j^i) + \sum_{j=i+1}^{n} (b_j(y_j^i) - b_j(0)) - \left(\sum_{j=1}^{i-1} b_j(y_j^{i-1}) + \sum_{j=i}^{n} (b_j(y_j^{i-1}) - b_j(0)) \right) \\ &= b_i(0) + \sum_{j=1}^{n} (b_j(y_j^i) - b_j(y_j^{i-1})), \quad i = 1,\dots,n. \end{aligned}$$

For $j = n$, the maximization problem (7.18) is again equal to problem (2.2), so that $\widehat{v}^n(e,b) = v^n(e,b)$ is the maximum total benefit that can be obtained when allocating all inflows optimally amongst all agents. Since $\sum_{i=1}^{n} r_i(N,e,b) = \widehat{v}^n(e,b) = v^n(e,b)$, also the upstream solution distributes the maximum attainable total welfare that the agents can achieve together, and thus also this solution is efficient. It turns out that the upstream solution can be characterized similarly as the upstream incremental solution by means of efficiency and the lower bound property, but now the drought property is changed for the no contribution property and the independence of upstream inflows is replaced by independence of downstream inflows.

Theorem 7.1 *A solution f on the class $\mathcal{W}^N$ of river problems is equal to the upstream solution r if and only if f satisfies efficiency, the lower bound property, the no contribution property and independence of downstream inflows.*

Proof. It is straightforward to show that the upstream solution satisfies these four axioms. Therefore, it is sufficient to prove that the four axioms determine a unique solution.

Let $(N,e,b) \in \mathcal{W}^N$ be a river problem and suppose that solution f satisfies the four axioms. Similar as in the proof of Theorem 4.1, we apply induction on the labels of the agents, starting with the most upstream agent 1. We first determine $f_1(N,e,b)$. Consider the modified river problem (N,e^1,b) given by $(e^1)_1 = e_1$, and $(e^1)_j = 0$ for all $j \in \{2,\dots,n\}$. The lower bound property requires that $f_j(N,e^1,b) \geq b_j(0)$ for all $j \in \{2,\dots,n\}$, while the no contribution property requires that $f_j(N,e^1,b) \leq b_j(0)$ for all $j \in \{2,\dots,n\}$. Thus, we conclude that $f_j(N,e^1,b) = b_j(0)$ for all $j \in \{2,\dots,n\}$. By efficiency we then have that $f_1(N,e^1,b) = v^n(e^1,b) - \sum_{j=2}^n f_j(N,e^1,b) = v^n(e^1,b) - \sum_{j=2}^n b_j(0)$. Since $e_j^1 = 0$ for all $j \in \{2,\dots,n\}$ and $e_1^1 = e_1$, it follows that $v^n(e^1,b) = \sum_{j=1}^n b_j(y_j^1) = \widehat{v}^1(e,b) + \sum_{j=2}^n b_j(0)$, and thus $f_1(N,e^1,b) = \widehat{v}^1(e,b) + \sum_{j=2}^n b_j(0) - \sum_{j=2}^n b_j(0) = \widehat{v}^1(e,b)$. Independence of downstream inflows then implies that $f_1(N,e,b) = f_1(N,e^1,b) = \widehat{v}^1(e,b) = \widehat{v}^1(e,b) - \widehat{v}^0(e,b) = r_1(N,e,b)$.

Proceeding by induction, assume that $f_k(N,e,b) = r_k(N,e,b)$ is determined for all $k < i \leq n$. Next, consider the modified river problem (N,e^i,b) given by $(e^i)_j = e_j$ for all $j \in \{1,...,i\}$ and $(e^i)_j = 0$ for all $j \in \{i+1,\dots,n\}$. Similar as above, the lower bound property requires that $f_j(N,e^i,b) \geq b_j(0)$ for all $j \in \{i+1,\dots,n\}$, while the no contribution property requires that $f_j(N,e^i,b) \leq b_j(0)$ for all $j \in \{i+1,\dots,n\}$. Thus,

$$f_j(N,e^i,b) = b_j(0) \text{ for all } j \in \{i+1,\dots,n\}. \tag{7.19}$$

Independence of downstream inflows and the induction hypothesis imply that $f_j(N,e^i,b) = f_j(N,e,b) = r_j(N,e,b)$ for all $j \in \{1,...,i-1\}$. Therefore,

$$\sum_{j=1}^{i-1} f_j(N,e^i,b) = \sum_{j=1}^{i-1} r_j(N,e,b) = \sum_{j=1}^{i-1} [\widehat{v}^j(e,b) - \widehat{v}^{j-1}(e,b)] = \widehat{v}^{i-1}(e,b). \tag{7.20}$$

Efficiency, the induction hypothesis, (7.19) and (7.20) then determine that

$$\begin{aligned} f_i(N,e^i,b) &= v^n(e^i,b) - \sum_{j=1}^{i-1} f_j(N,e,b) - \sum_{j=i+1}^{n} f_j(N,e,b) \\ &= v^n(e^i,b) - \widehat{v}^{i-1}(e,b) - \sum_{j=i+1}^{n} b_j(0). \end{aligned} \tag{7.21}$$

Since $e_j^i = 0$ for all $j \in \{i+1,\dots,n\}$ and $e_j^i = e_j$ for all $j \in \{1,\dots,i\}$, similar as above it follows that $v^n(e^i,b) = \sum_{j=1}^n b_j(y_j^i) = \widehat{v}^i(e,b) + \sum_{j=i+1}^n b_j(0)$. Thus, with (7.21) we have $f_i(N,e^i,b) = \widehat{v}^i(e,b) + \sum_{j=i+1}^n b_j(0) - \widehat{v}^{i-1}(e,b) - \sum_{j=i+1}^n b_j(0) = \widehat{v}^i(e,b) - \widehat{v}^{i-1}(e,b) = r_i(N,e,b)$. Finally, independence of downstream inflows implies that $f_i(N,e,b) = f_i(N,e^i,b) = r_i(N,e,b)$. □

According to the upstream solution, every upstream coalition $\{1,\dots,j\}$ receives the total welfare that can be attained by allocating the water inflows of such a coalition optimally over all agents. Clearly, the welfare at a solution of the corresponding welfare maximization problem (7.18) is at least as high as the welfare at a solution of the corresponding welfare maximization problem (2.3) in which the inflows of a coalition $\{1, \ldots, j\}$ are distributed optimally amongst themselves. So, the upstream solution certainly satisfies stability for the upstream coalitions and thus satisfies the Harmon principle for the upstream coalitions. However, the upstream solution does not satisfy stability in general. For example, agent n receives the marginal benefit $\widehat{v}^n(e,b) - \widehat{v}^{n-1}(e,b)$, being the difference between the total benefit of the water consumptions y^n and y^{n-1}. Nothing can be said about this difference and the benefit $b_n(e_n)$ that agent n can obtain by consuming his own water. Therefore, it might happen that $r_n(N,e,b) < b_n(e_n)$, violating individual rationality and thus stability.[5] However, we could say that for every coalition $\{i, i+1, \ldots, j\}$ of consecutive agents the upstream solution reflects a weaker form of the Harmon principle in the sense that such a coalition receives as much as possible under the restriction that the total payoff to its upstream coalition $\{1, \ldots, i-1\}$ is equal to the highest attainable total welfare of all countries that can be achieved by allocating their inflows $e_1, \ldots, e_{i-1}$ amongst all countries.

8 A special case: the claims model

In this section we consider the particular case that every country has constant marginal benefit of one up to its satiation point c_i and has constant benefit of every water consumption above its satiation point, i.e., for every i there exists a $c_i > 0$ such that

$$b_i(x_i) = \begin{cases} x_i & \text{if } x_i \leq c_i \\ c_i & \text{if } x_i > c_i. \end{cases} \tag{8.22}$$

Such benefit functions have been considered in Ansink and Weikard (2011) within river situations in which it is not allowed or possible to make monetary transfers. In such a model the satiation point c_i can be considered as the *claim* of country i for water and the problem is to find, without monetary compensations, a fair distribution of the water inflows amongst the countries given their claims and the uni-directionality of the water flows. We now consider the four solutions in case of such benefit functions within the model considered in this paper, allowing for monetary compensations.

Recall that the downstream incremental solution d is given by

$$d_j(N,e,b) = v^j(e,b) - v^{j-1}(e,b), \quad j = 1, \ldots, n, \tag{8.23}$$

[5] Note that the definition of stability is based on the game theoretical model proposed in Ambec and Sprumont (2002). This model is a thought experiment to study the allocation problem at hand that takes into account the Harmon principle. A different model could be proposed taking into account the TIBS principle and/or the UTI principle, which would lead to stability of the upstream solution.

where $v^0(e,b) = 0$ and $v^j(e,b)$ is the welfare at a solution of the welfare maximization problem (2.3) for agent j. For benefit functions of type (8.22) it follows straightforwardly that

$$v^1(e,b) = \min[c_1, e_1],$$

and successively

$$v^j(e,b) = v^{j-1}(e,b) + \min[c_j, \sum_{i=1}^{j} e_i - v^{j-1}(e,b)], \quad j = 2, \ldots, n.$$

Substituting this in the equations (8.23) we obtain

$$d_1(N,e,b) = \min[c_1, e_1]$$

and, for $j = 2, \ldots, n$, recursively,

$$d_j(N,e,b) = \min\left[c_j, e_j + \sum_{i=1}^{j-1}(e_i - d_i(N,e,b))\right],$$

using the fact that by definition $\sum_{i=1}^{j-1} d_i(N,e,b) = v^{j-1}(e,b)$, $j = 2, \ldots, n$. This downstream incremental solution can be implemented by assigning to each upstream coalition $\{1, \ldots, j\}$ as much water as possible given the uni-directionality of the water flows and under the constraint that no country gets water above its satiation point. We conclude that, when each country has a benefit function of type (8.22), monetary compensations are not needed to implement the downstream incremental solution.

Recall that the downstream solution s is given by

$$s_j(N,e,b) = \widehat{w}^j(e,b) - \widehat{w}^{j+1}(e,b), \quad j = 1, \ldots, n, \tag{8.24}$$

where $\widehat{w}^{n+1}(e,b) = 0$ and $\widehat{w}^j(e,b)$ is the welfare at a solution of the welfare maximization problem (6.13) for agent j. To avoid notational burden, in the following we assume, without loss of generality, that $\sum_{i=j}^{n} e_i < \sum_{i=j}^{n} c_i$ for every $j = 1, \ldots, n$. Suppose this does not hold for $j = n$. Then $e_n \geq c_n$ and the water inflow at agent n is at least as high as his satiation point. Then this country does not need to get water from his upstream countries. Because also his excess $e_n - c_n$ of water cannot be assigned to his upstream countries, in this case country n stands alone and does not affect the welfare of the other agents. Therefore, it is sufficient to consider the agents $1, \ldots, n-1$. Similarly, suppose that the condition holds for all agents $i > j$ and not for agent j. Then $e_j \geq c_j + \sum_{i=j+1}^{n}(c_i - e_i)$ and the water inflow at agent j is high enough to assign agent j and all its downstream agents water up to their satiation points, and so the water consumptions of these agents do not affect the upstream agents. In this case, we have to solve two independent water distribution problems, one for the upstream coalition $\{1, \ldots, j-1\}$ and the other for the downstream coalition $\{j, \ldots, n\}$. Therefore, the assumption can be made without loss of generality. Under this assumption it follows straightforwardly that for benefit functions of type (8.22)

$$\widehat{w}^n(e, b) = \min[c_n, \sum_{i=1}^{n} e_i].$$

By the above assumption we have that $\widehat{w}^n(e, b) \geq e_n$ and then

$$\widehat{w}^{n-1}(e, b) = \widehat{w}^n(e, b) + \min[c_{n-1}, \sum_{i=1}^{n} e_i - \widehat{w}^n(e, b)],$$

and again by the assumption we have that $\widehat{w}^{n-1}(e, b) \geq e_{n-1} + e_n$. Continuing, it follows successively from $j = n-2$ to $j = 1$ that

$$\widehat{w}^j(e, b) = \widehat{w}^{j+1}(e, b) + \min[c_j, \sum_{i=1}^{n} e_i - \widehat{w}^{j+1}(e, b)].$$

Substituting this in the equations (8.24) we obtain

$$s_n(N, e, b) = \min[c_n, \sum_{i=1}^{n} e_i]$$

and, recursively from $j = n-1$ to $j = 1$,

$$s_j(N, e, b) = \min[c_j, \sum_{j=1}^{n} e_j - \sum_{i=j+1}^{n} s_i(N, e, b))],$$

using the fact that by definition $\sum_{i=j+1}^{n} s_i(N, e, b) = \widehat{w}^{j+1}(e, b)$, $j = 1, \ldots, n-1$. From these expressions it follows that the downstream solution can be implemented by assigning to each downstream coalition $\{j, \ldots, n\}$ as much water as possible given the uni-directionality of the water flows and under the constraint that no country gets water above its satiation point. Again, when each country has a benefit function of type (8.22), monetary compensations are not needed to implement the downstream solution and thus the downstream solution also can be implemented when monetary compensations cannot be made.

Recall that the upstream incremental solution u is given by

$$u_j(N, e, b) = w^j(e, b) - w^{j+1}(e, b), \quad j = 1, \ldots, n, \tag{8.25}$$

where $w^{n+1}(e, b) = 0$ and $w^j(e, b)$ is the welfare at a solution of the welfare maximization problem (5.8) for agent j. Again, without loss of generality we assume that $\sum_{i=j}^{n} e_i <$

$\sum_{i=j}^{n} c_i$ for every $j = 1, \ldots, n$. For benefit functions of type (8.22) it then follows straightforwardly that

$$w^j(e, b) = \sum_{i=j}^{n} e_i, \quad j = 1, \ldots, n$$

and substituting this in the equations (8.25) we obtain

$$u_j(N, e, b) = \sum_{i=j}^{n} e_i - \sum_{i=j+1}^{n} e_i = e_j, \quad j = 1, \ldots, n.$$

Therefore, the upstream incremental solution gives precisely payoff e_j to each country j. Clearly, it might be possible that this cannot be implemented without monetary transfers. For example, take $n = 2$, $c_1 < e_1 < c_1 + c_2$ and $e_2 = 0$. The total welfare e_1 is obtained for every solution x^* of the welfare maximization problem (2.2), thus for every x^* with $\max[0, e_1 - c_2] \leq x_1^* \leq c_1$ and $x_2^* = e_1 - x_1^*$. Thus, to implement the welfare distribution $u(N, e, b) = (e_1, 0)^\top \in \mathbb{R}_+^2$ it is required that agent 2 pays a monetary compensation $t = x_2^*$ to agent 1 since by only consuming water, agent 1 cannot reach a higher payoff than $c_1 < e_1$. We conclude that in general the upstream incremental solution cannot be applied when monetary compensations are not allowed or not possible. However, notice that when $e_i \leq c_i$ for all i, then the downstream incremental solution reduces to $d_j(N, e, b) = e_j$ for all j, and thus the downstream incremental and upstream incremental solution coincide.

Finally, recall that the upstream solution r given by

$$r_j(N, e, b) = \widehat{v}^j(e, b) - \widehat{v}^{j-1}(e, b), \quad j = 1, \ldots, n, \tag{8.26}$$

where $\widehat{v}^0(e, b) = 0$ and $\widehat{v}^j(e, b)$ is obtained from a solution of the welfare maximization problem (7.18) for agent j. Again under the assumption that $\sum_{i=j}^{n} e_i < \sum_{i=j}^{n} c_i$ for every $j = 1, \ldots, n$, it follows straightforwardly that

$$\widehat{v}^j(e, b) = \sum_{i=1}^{j} e_i, \quad j = 1, \ldots, n$$

and substituting this in the equations (8.26) we obtain

$$r_j(N, e, b) = \sum_{i=1}^{j} e_i - \sum_{i=1}^{j-1} e_i = e_j, \quad j = 1, \ldots, n.$$

Therefore, the upstream solution and the upstream incremental solution coincide and thus in general the upstream solution cannot be applied when monetary compensations are not allowed or not possible.

9 Comparison of the four solutions and concluding remarks

In this paper we consider the problem of sharing water among agents located along a river. We adapted the model of Ambec and Sprumont (2002) by weakening the assumption on the benefit functions of the agents. Using nine different axioms we were able to characterize four solutions for this model. The downstream incremental solution, originally suggested by Ambec and Sprumont (2002), can be characterized by efficiency, the lower bound property, the weak aspiration level property and independence of downstream benefits. The upstream incremental solution, originally suggested by van den Brink, van der Laan and Vasil'ev (2007), can be characterized by efficiency, the lower bound property, the drought property and independence of upstream inflows. The new downstream solution can be characterized by efficiency, the lower bound property, the weak aspiration property and independence of upstream benefits, and the new upstream solution can be characterized by efficiency, the lower bound property, the no contribution property and independence of downstream inflows.

The taxonomy is shown in Table 9.1 (where, for every solution, the four boldface 'yes' give an axiomatization of the solution).

Solution:	downstream incr.	upstream incr.	downstream	upstream
Axiom:				
efficiency	**yes**	**yes**	**yes**	**yes**
lower bound	**yes**	**yes**	**yes**	**yes**
drought	yes	**yes**	yes	yes
weak aspiration	**yes**	no	**yes**	no
no contribution	no	yes	no	**yes**
downstream benefits	**yes**	no	no	no
upstream benefits	no	yes	**yes**	no
upstream inflows	no	**yes**	no	no
downstream inflows	yes	no	no	**yes**

Table 9.1 Table of axioms satisfied by the four solutions.

Notice from this table that none of the four solutions satisfies simultaneously the weak aspiration level property and independence of upstream inflows. Similarly none of the four solutions satisfies simultaneously the no contribution property and the independence of downstream benefits. Further, the independence of downstream benefits is only satisfied by the incremental downstream solution and the independence of upstream inflows is only satisfied by the incremental upstream solution. Thus, if the countries along an international river agree to impose one of these properties, then it selects a unique solution out of the four solutions presented in this paper and therefore also provides the countries with a compensation scheme.

From the selected axioms it follows that independence of downstream benefits gives lower compensations to the upstream countries than the independence of upstream inflows. The two other independence properties are satisfied by two solutions (one extreme and one incremental solution). The independence of upstream benefits is satisfied by the downstream solution and the upstream incremental solution, the independence of downstream inflows is satisfied by the upstream solution and the downstream incremental solution.

Finally, when we apply the four solutions to the particular case that every agent has constant marginal benefit of one up to a satiation point and marginal benefit of zero thereafter, only the downstream and downstream incremental solutions can be implemented without monetary transfers between the agents. This means that when countries along an international river only state a claim on the river water and are not willing to transfer money to each other, out of the four solutions presented in this paper only these two solutions are viable.

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Appendix: Logical independence

In this appendix we show logical independence of the axioms in the four axiomatizations by, for each axiomatization, giving four alternative solutions, each of these solutions only satisfying three of the four axioms.

The axioms of Theorem 4.1

1. The solution $f_i(N,e,b) = b_i(0)$ for all $i \in N$ and all river problems (N,e,b) satisfies the lower bound property, the weak aspiration level property, and independence of downstream benefits. It does not satisfy efficiency.

2. The solution $f_i(N,e,b) = \max_{x_i \leq \sum_{j\in N} e_j} b_i(x_i)$ for all $i \in N \setminus \{n\}$, and $f_n(N,e,b) = v^n(e,b) - \sum_{j=1}^{n-1} f_i(N,e,b)$ assigns to every agent except the most downstream agent its highest benefit when it would have access to all water inflows, while the benefit of the most downstream agent is obtained by subtracting all these benefits from the total benefit in an efficient allocation. It satisfies efficiency, independence of downstream benefits, and the weak aspiration level property. It does not satisfy the lower bound property.

3. The solution $f_n(N,e,b) = v^n(e,b) - \sum_{j=1}^{n-1} b_i(0)$ and $f_i(N,e,b) = b_i(0)$ for all $i \in N \backslash \{n\}$ satisfies efficiency, the lower bound property, and independence of downstream benefits. It does not satisfy the weak aspiration level property.

4. The downstream solution satisfies efficiency, the lower bound property, and the weak aspiration level property. It does not satisfy independence of downstream benefits.

The axioms of Theorem 5.1

1. The solution $f_i(N,e,b) = b_i(0)$ for all $i \in N$ satisfies the lower bound property, the drought property, and independence of upstream inflows. It does not satisfy efficiency.

2. For some $\epsilon > 0$, define the solution f as $f(N,e,b) = u(N,e,b)$ if $e_n = 0$. Otherwise, define $f_1(N,e,b) = u_1(N,e,b) - \epsilon$, $f_i(N,e,b) = u_i(N,e,b)$ for $i = 2,\ldots,n-1$ and $f_n(N,e,b) = u_n(N,e,b) + \epsilon$. It is easily seen that f satisfies efficiency, the drought property, and independence of upstream inflows since u satisfies these properties. It does not satisfy the lower bound property.

3. The solution $f_1(N,e,b) = w^1(e,b) - \sum_{j=2}^{n} b_j(0)$ and $f_i(N,e,b) = b_i(0)$ for all $i \in N \setminus \{1\}$ satisfies efficiency, the lower bound property, and independence of upstream inflows. It does not satisfy the drought property.

4. The downstream incremental solution d satisfies efficiency, the lower bound property, and the drought property. It does not satisfy independence of upstream inflows.

The axioms of Theorem 6.1

1. The solution assigning $f_i(N,e,b) = b_i(0)$ to i, $i = 1,\ldots,n$, satisfies the lower bound property, the weak aspiration level property, and independence of upstream benefits. It does not satisfy efficiency.

2. The solution $f_i(N,e,b) = \max_{x_i \leq \sum_{j\in N} e_j} \; b_i(x_i)$ for all $i \in N \setminus \{1\}$, and $f_1(N,e,b) = v^n(e,b) - \sum_{j=2}^{n} \; f_i(N,e,b)$ assigns to every agent except the most upstream agent its highest benefit when it would have access to all water inflows, while the benefit of the most upstream agent is obtained by subtracting all these benefits from the total benefit in an efficient allocation. This solution satisfies efficiency, independence of upstream benefits, and the weak aspiration level property. It does not satisfy the lower bound property.

3. The upstream incremental solution satisfies efficiency, the lower bound property, and independence of upstream benefits. It does not satisfy the weak aspiration level property.

4. From Theorem 4.1 it follows that the downstream incremental solution satisfies efficiency, the lower bound property, and the weak aspiration level property. It does not satisfy independence of upstream benefits.

The axioms of Theorem 7.1

1. The solution $f_i(N,e,b) = b_i(0)$ for all $i \in N$ satisfies the lower bound property, the no contribution property, and independence of downstream inflows. It does not satisfy efficiency.

2. For some $\epsilon > 0$, define the solution f by $f(N,e,b) = r(N,e,b)$ if $e_1 = 0$. Otherwise, define $f_1(N,e,b) = r_1(N,e,b) + \epsilon$, $f_i(N,e,b) = r_i(N,e,b)$ for $i = 2,\ldots,n-1$ and $f_n(N,e,b) = r_n(N,e,b) - \epsilon$. It is easily seen that f satisfies efficiency, the no contribution property, and independence of upstream inflows. It does not satisfy the lower bound property.

3. The solution $f_i(N, e, b) = b_i(0)$ for all $i \in N \setminus \{n\}$ and $f_n(N, e, b) = v^n(e, b) - \sum_{j=1}^{n-1} b_j(0)$ satisfies efficiency, the lower bound property, and independence of downstream inflows. It does not satisfy the no contribution property.

4. The upstream incremental solution u satisfies efficiency, the lower bound property, and the no contribution property. It does not satisfy independence of downstream inflows.