Network Reliability: A Fresh Look at Some Basic Questions

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Abstract

In this paper, the general problem of comparing the performance of two communication networks is examined. The standard approach, using stochastic ordering as a metric, is reviewed, as are the mixed results on the existence of uniformly optimal networks (UONs) which have emerged from this approach. While UONs have been shown to exist for certain classes of networks, it has also been shown that no UON network exists for other classes. Results to date beg the question: Is the problem of identifying a Universally Optimal Network (UON) of a given size dead or alive? We reframe the investigation into UONs in terms of network *signatures* and the alternative metric of *stochastic precedence*. While the endeavor has been dead, or at least dormant, for some twenty years, the findings in the present paper suggest that the question above is by no means settled. Specifically, we examine a class of networks of a particular size for which it was shown that no individual network was universally optimal relative to the standard metric (the uniform ordering of reliability polynomials), and we show, using the aforementioned alternative metric, that this class is totally ordered and that a uniformly optimal network exists after all. Optimality with respect to "performance per unit cost" type metrics is also discussed.

1 Introduction

Communication networks have become pervasive in modern society, and the study of their performance (in terms of the persistence of connectivity of a desired type) has received increasing attention in recent years. Important applications abound. For example, Rivera and Young (2009) provide both an explanation of this growing interest and motivation for further research in the area, as it relates to military applications, as evidenced in the following remark: "The recent National Research Council Report on Network Science identified the need to develop fundamental knowledge about large, complex networks that

will enable a better understanding of how to apply technology to the Army's network-centric operations. Two key topics of our research are the basic issues of the fundamental capacity of MANETs (Mobile *Ad Hoc* Networks) and the connectivity of nodes in a MANET." In an edited Proceedings volume from a conference of the Army Research Laboratory's Collaborative Alliance in Communications and Networks, Gowens et al. (2009) feature seventy research papers on a multiplicity of subjects dealing with the design and performance characteristics of communications networks, studied with a view toward their assessment relative to a variety of metrics including speed, reliability, security, survivability and performance per unit cost. Among the themes receiving special emphasis were optimality issues, the development of secure, scalable, reliable communications in dynamic environments and the performance of networks as a function of their topologies (or designs).

The general problem of comparing the performance of two communications networks has been investigated in various ways. We will review a particular approach, one involving the comparison of network signatures as described in Boland, Samaniego and Vestrup (2003). Our discussion of the concept of network signatures leads naturally to an examination of the question of primary interest here: is the problem of identifying a Universally Optimal Network (UON) dead or alive? The latter problem has an interesting, if somewhat rocky, history. We will review the early successes in finding UONs among networks of a given size, and we will revisit the stunning, infinite array of counterexamples of Myrvold et al. (1992) showing that UONs did not exist among networks of certain specific sizes. The latter paper essentially dashed the hopes of network researchers seeking to develop a general methodology for finding UONs. This type of endeavor has been dead, or at least dormant, for some twenty years. The findings in the present paper suggest that the question above is by no means settled. Specifically, we will examine a class of networks of a particular size for which it was shown that no individual network was universally optimal relative to a standard metric (the uniform ordering of reliability polynomials), and we will show, using an alternative metric, that this class is totally ordered and that a uniformly optimal network exists after all.

Before proceeding, we will set some basic definitions and briefly discuss the needed background on network signatures and on two approaches to ordering random variables that will be central to the comparisons that lie ahead. We will follow the standard practice of representing a communications network as an undirected graph. Such graphs are completely specified by a collection of vertices (or nodes) and a set of edges joining selected pairs of vertices. The well-known "Wheatstone Bridge" network is shown in Figure 1.

The family of networks with v vertices and n edges will be denoted by $G(v,n)$. Following the standard convention (see Colbourn (1987)), we assume that vertices function with certainty, while the edges in a network are subject to failure. We will restrict attention to "coherent" networks, defined as follows.

Definition 1.1. A network is *coherent* is every edge is relevant and the network's functioning cannot be diminished when a failed edge is replaced by a
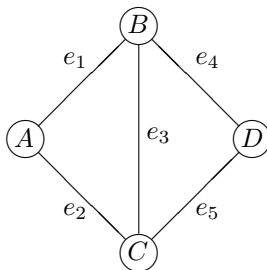
Figure 1: The Wheatstone Bridge

working edge.

The main quality of interest in a communications network is *connectivity*. Different types of connectivity may be relevant at a given time or for a particular purpose. We will distinguish among the following options: "2-terminal connectivity" (that is, there exists a working path—a series of adjacent working edges—linking two distinguished vertices), "k-terminal connectivity" (for $2 < k < v$, there exists a working path linking any pair in a set of k distinguished vertices) and "all-terminal connectivity" (there exists a working path linking any pair of vertices).

The reliability of a network is defined, simply, as the probability that the network meets its connectivity goal. If T represents the time at which the network's connectivity fails, then the reliability function of the network is represented by $\overline{F}_T(t) = P(T > t)$. The reliability of a network is of course a function of the reliability of its edges. A variety of options exist for the modeling of edge reliabilities. Consider a network in the family $G(v, n)$. A completely general model for edge reliabilities would posit that the edge failure times $X_1, \ldots, X_n$ have a continuous multivariate distribution which allows for possible dependencies and singularities. Rather little work has been done at this level of generality, partly because of the paucity of tractable multivariate models for random vectors with positive elements and partly because of the complexity of characterizing the precise dependencies that might be present in a given application. One viable option that should be mentioned is the Marshall–Olkin (1967) multivariate exponential model, which can be derived as a shock model and gives rise to edge failure times that are positively correlated. However, the fact that the marginal distributions of edge failure times are exponential limits the model's applicability. A less general but reasonably tractable alternative to an unconstrained multivariate model is a model which posits independent but not identically distributed (i.n.i.d.) edge failure times. If X_i is the failure time of the ith edge, then the X_i are independent, with $X_i \sim F_i$, $i = 1, \ldots, n$. The probability that edge i is working at time t_0 is $p_i = \overline{F}_i(t_0)$, where $\overline{F}_i = 1 - F_i$. This intermediate type of stochastic model makes it possible to represent the reliability of a network at a fixed time t_0 as a multinomial expression of the edge reliabilities

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$\{p_1, \ldots, p_n\}$ that is linear in each p_i. While work does exist under the i.n.i.d. assumption, the model does not readily lend itself to the comparison of network performance.

The most commonly encountered model in network reliability studies is the following important special case: Edge failure times are assumed to be independent and are assumed to have identical distributions, that is, $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} F$. At a fixed time t_0, the probability that any given edge is working is $p = \overline{F}(t_0)$. In this case, the reliability at time t_0 of a network in the $G(v, n)$ class may be written as an nth degree polynomial in the argument p. The monograph by Colbourn (1987), as well as much of the published work on the comparison of the reliability of various networks, including research work on UONs, restrict attention to the i.i.d. framework. The i.i.d. assumption can be defended on several levels, and we put forward a brief defense here. First, it is fair to say that the assumption reasonably approximates the stochastic behavior of edges in certain networks used in wired or even wireless communications. There are many instances in which all the edges of a network are reasonably thought to be equally vulnerable to failure and actually fail in similar but unrelated ways. Further, if the independence of edge failure times is deemed a reasonable assumption, then the study of network reliability under the additional assumption of a common edge reliability p may serve as a helpful way of bounding the reliability of the network. Specifically, if p may reasonably be assumed to be a lower bound on all $p_i = \overline{F}_i(t_0)$, or if $\overline{F}_i(t) \geq \overline{F}(t)$ for all t, then the network's reliability is bounded below by the reliability of the network based on the i.i.d. assumption with $p_i \equiv p$ or $\overline{F}_i \equiv \overline{F}$. Finally, the i.i.d. framework "levels the playing field" when comparing two networks. It is clear that a poorly-designed network with highly reliable edges will outperform a well-designed network with quite unreliable edges. Further, under the i.i.d. assumption on edge failure times, the differences between network designs can be characterized through distribution-free summaries like "network signatures." We now turn to a discussion of network signatures and some of their properties.

Consider a network in the $G(v, n)$ class, and assume that its n edges have failure times $\{X_i\}$ which can be modeled as $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} F$. Suppose that a given type of connectivity is of interest. Let T be the time at which connectivity of the network fails. The failure of connectivity necessarily coincides with a particular edge failure. The signature of the network, given its specific connectivity goal, is defined as follows.

Definition 1.2. The *signature* **s** of a $G(v, n)$ network is an n-dimensional probability vector whose ith element is $s_i = P(T = X_{i:n})$, where $X_{i:n}$ is the ith smallest X among the sample of edge failure times $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} F$, or, alternatively, the ith order statistic in the random sample $X_1, X_2, \ldots, X_n$ of edge failure times.

Consider again the Wheatstone Bridge Network shown in Figure 1. We will compute the signature for two types of connectivity.

Example 1.1. Suppose 2-terminal connectivity (between vertices A and D) is

of interest. The "minimal cut sets" for this network's connectivity are

$$\{1,2\}, \quad \{4,5\}, \quad \{1,3,5\}, \quad \{2,3,4\}$$

Under the i.i.d. assumption, the 120 permutations of edge failure times X_1, X_2, X_3, X_4 and X_5 have equal likelihood. We may verify that the signature vector $\mathbf{s} = (0, \ 1/5, \ 3/5, \ 1/5, \ 0)$ as follows:

(a) It is clear from inspection that $P(T = X_{1:5}) = 0 = P(T = X_{5:5})$.

(b) Since $T = X_{2:5}$ if and only if permutations of the forms

$$(1, 2, \text{-}, \text{-}, \text{-}), \quad (2, 1, \text{-}, \text{-}, \text{-}), \quad (4, 5, \text{-}, \text{-}, \text{-}) \quad \text{or} \quad (5, 4, \text{-}, \text{-}, \text{-})$$

occur, we obtain that $P(T = X_{2:5}) = 24/120 = 1/5$.

(c) Since $T = X_{4:5}$ if and only if permutations of the forms

$$(\text{-}, \text{-}, \text{-}, 2, 5), \quad (\text{-}, \text{-}, \text{-}, 5, 2), \quad (\text{-}, \text{-}, \text{-}, 1, 4) \quad \text{or} \quad (\text{-}, \text{-}, \text{-}, 4, 1)$$

occur, we obtain that $P(T = X_{2:5}) = 1/5$.

(d) It follows that $P(T = X_{3:5}) = 3/5$.

Example 1.2. Now, suppose that all-terminal connectivity in the Wheatstone bridge is of interest. The "minimal cut sets" for this network connectivity are

$$\{1,2\}, \quad \{4,5\}, \quad \{1,3,4\}, \quad \{1,3,5\}, \quad \{2,3,4\}, \quad \{2,3,5\}$$

Under the i.i.d. assumption, the 120 permutations of edge failure times X_1, X_2, X_3, X_4 and X_5 have equal likelihood. We may verify that $\mathbf{s} = (0, 1/5, 4/5, 0, 0)$ as follows:

• It is clear from inspection that $P(T = X_{1:5}) = 0 = P(T = X_{5:5})$.

• Since $T = X_{2:5}$ if and only if permutations of the forms

$$(1, 2, \text{-}, \text{-}, \text{-}), \quad (2, 1, \text{-}, \text{-}, \text{-}), \quad (4, 5, \text{-}, \text{-}, \text{-}) \quad \text{or} \quad (5, 4, \text{-}, \text{-}, \text{-})$$

occur, we have $P(T = X_{2:5}) = 1/5$.

• Since it is not possible to connect 4 vertices with just 2 edges, connectivity will fail at or before the 3rd edge failure. Thus, $P(T = X_{4:5}) = 0$ and $P(T = X_{3:5}) = 4/5$.

Network signatures have proven useful in the analysis of network performance and in comparisons between and among different network designs. Although the definition of a network's signature, as given above, involves the assumption of i.i.d. edge failure times, the signature vector is in fact a topological invariant which may be used as an index of the network's design. The following representation theorem shows that, under the i.i.d. assumption, the distribution of the failure times of a network is solely a function of its signature vector and the underlying common distribution F of the failure times of its edges.

Theorem 1.1 (Samaniego (1985)). *Consider a network in the $G(v,n)$ class. Assume that the failure times of its n edges are i.i.d. with common distribution F. Let $\boldsymbol{s}$ be its signature vector. Then the failure time of the network is given by*

$$\overline{F}_T(t) = \sum_{j=0}^{n-1} \left(\sum_{i=j+1}^{n} s_i \right) \binom{n}{j} (F(t))^j \left(\overline{F}(t) \right)^{n-j}. \tag{1.1}$$

For extensions of the representation theorem above to the reliability function of a system with heterogeneous components, see Navarro, Samaniego and Balakrishnan (2011).

In addition to "representation results" such as Theorem 1.1 above, reliability analysts are often also interested in "preservation theorems" which show that certain characteristics of an index of a class of systems are inherited by the systems themselves. Such results are often essential tools in studying the comparative performance of systems. The result below shows that several types of stochastic relationships enjoyed by pairs of system signatures are preserved by the lifetimes of the corresponding networks. The most commonly used criterion for comparing the relative sizes of two random variables is "stochastic ordering." This ordering is defined as follows:

Definition 1.3. Given two independent random variables X and Y, X is smaller than Y in the *stochastic ordering* (denoted by $X \leq_{\text{st}} Y$) if and only if their respective survival functions satisfy $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for all t.

For definitions of hazard-rate (hr) and likelihood-ratio (lr) ordering between random variables (or their distributions), see Shaked and Shantikumar (2007). In the following result, signatures are seen as the distributions of discrete variables taking values in $\{1, 2, \ldots, n\}$.

Theorem 1.2 (Kochar, Mukerjee and Samaniego (1999)). *Let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be the signatures of the two networks, both containing n edges whose failure times are i.i.d. with common distribution F. Let T_1 and T_2 be their corresponding network failure times. The following preservation results hold:*

  (a) if $\boldsymbol{s}_1 \leq_{\text{st}} \boldsymbol{s}_2$, then $T_1 \leq_{\text{st}} T_2$,

  (b) if $\boldsymbol{s}_1 \leq_{\text{hr}} \boldsymbol{s}_2$, then $T_1 \leq_{\text{hr}} T_2$, and

  (c) if $\boldsymbol{s}_1 \leq_{\text{lr}} \boldsymbol{s}_2$ and F is absolutely continuous with density f, then $T_1 \leq_{\text{lr}} T_2$.

The result above makes it clear that one may compare the reliability of two networks by examining properties of the corresponding signature vectors.

In making stochastic comparisons among networks in the sections that follow, we will examine in detail two specific types of orderings between random variables. The first of these, "stochastic ordering," is defined above. We note that stochastic ordering applies to both discrete and continuous variables, and it is known to be a weaker ordering than hazard-rate and likelihood-ratio ordering. When these three orderings are well-defined, it is well known that $\text{lr} \Rightarrow \text{hr}$

$\Rightarrow$ st. An alternative concept capturing the notion that X is smaller than Y is that of "stochastic precedence." Arcones, Kvam and Samaniego (JASA, 2002) studied this ordering in a reliability context. Since it will play an important role in the sequel, we include a formal definition here.

Definition 1.4. Two independent random variables X and Y are ordered in *stochastic precedence* (denoted by $X \leq_{\mathrm{sp}} Y$) if and only if $P(X < Y) \geq P(X > Y)$; the variables are *equivalent in stochastic precedence* (i.e., $X =_{\mathrm{sp}} Y$) if and only if $P(X < Y) = P(X > Y)$.

We note that the "sp" ordering applies to both discrete or continuous (X, Y). For independent random variables X and Y, the sp ordering is weaker than stochastic ordering, that is, st $\Rightarrow$ sp. The sp ordering has a natural interpretation when comparing the (continuous) time to failure of two competing networks: If network failure times T_1 and T_2 satisfy $T_1 <_{\mathrm{sp}} T_2$, then $P(T_1 < T_2) > 0.5$, that is, the chances are that network 2 will last longer than network 1.

2 Comparing two $G(v, n)$ networks

When all edges work independently of each other and have a common probability p of working, the reliability of a network with n edges can be written as an nth degree polynomial. The reliability polynomial of the network can be expressed, in standard form, as

$$h(p) = \sum_{r=1}^{n} d_r p^r \ . \tag{2.1}$$

Satyanarayana and Prabhakar (1978) provided an efficient technique for computing the "signed dominations" $\{d_r\}$ in the reliability polynomial. This polynomial provides a closed form expression for the probability that the network will retain its connectivity goal when all n edges operate independently and each works with probability p. The reliability of the network at a fixed time t is given by $h(\overline{F}(t))$.

The survival function of a network's lifetime T can also be written as a function of $\mathbf{s}$ and F. At a fixed time t, where $P(X_j > t) = p$ for all j, the representation in Theorem 1.1 reduces to the reliability polynomial in "pq-form," where $q = 1 - p$:

$$
\begin{aligned}
h(p) &= \sum_{j=1}^{n} \left(\sum_{i=n-j+1}^{n} s_i \right) \binom{n}{j} p^j q^{n-j} \\
&= \sum_{j=1}^{n} a_j \binom{n}{j} p^j q^{n-j} \ ,
\end{aligned}
\tag{2.2}
$$

where $a_j = \sum_{i=n-j+1}^{n} s_i$ for $j = 1, \ldots, n$. The vectors $\mathbf{a}$ and $\mathbf{s}$ are linearly

related. We will express this relationship as $\mathbf{a} = \mathbf{Ps}$, where

$$P_{uv} = \begin{cases} 0 & \text{if } u + v \leq n \\ 1 & \text{if } u + v > n \end{cases}.$$

Expanding $q^{n-j} = (1-p)^{n-j}$ by the binomial theorem, we may identify each domination d_i as a linear combination of the elements $a_1, \ldots, a_n$. More specifically, we may write $\mathbf{d} = \mathbf{Ma}$, where

$$\mathbf{M} = \begin{pmatrix} \binom{n}{1}\binom{n-1}{0} & 0 & 0 & \cdots & 0 \\ -\binom{n}{1}\binom{n-1}{1} & \binom{n}{2}\binom{n-2}{0} & 0 & \cdots & 0 \\ \binom{n}{1}\binom{n-1}{2} & -\binom{n}{2}\binom{n-2}{1} & \binom{n}{3}\binom{n-3}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \pm\binom{n}{1}\binom{n-1}{n-1} & \mp\binom{n}{2}\binom{n-2}{n-2} & \pm\binom{n}{3}\binom{n-3}{n-3} & \cdots & \binom{n}{n}\binom{n-n}{n-n} \end{pmatrix}$$

Since $\mathbf{d} = \mathbf{MPs}$, we may express the relationship of interest to us as

$$\mathbf{s} = \mathbf{P}^{-1}\mathbf{M}^{-1}\mathbf{d}. \tag{2.3}$$

We will use the symbol $(k)_j$ for the number of permutations of k items taken j at a time, that is, $(k)_j = k(k-1)\cdots(k-j+1)$. Then $\mathbf{M}^{-1}$ is the matrix $\mathbf{M}^*$ whose ith row is given by

$$(m_{i1}^*, \cdots, m_{ii}^*, 0, \cdots, 0) = \left(\underbrace{\frac{(i)_1}{(n)_1}, \frac{(i)_2}{(n)_2}, \cdots, \frac{(i)_i}{(n)_i}}_{i \text{ slots}}, \underbrace{0, \cdots, 0}_{n-i \text{ slots}} \right).$$

The claim above, and the following result, are proven in Boland, Samaniego and Vestrup (2003).

Theorem 2.1. *Let $\mathbf{d}$ and $\mathbf{s}$ denote the domination and signature vectors for a given network of order n. Then for $i = 1, \ldots, n$, we have*

$$s_i = \sum_{j=1}^{n-i} \frac{(n-i+1)_j - (n-i)_j}{(n)_j} d_j + \frac{(n-i+1)_{n-i+1}}{(n)_{n-i+1}} d_{n-i+1}. \tag{2.4}$$

Having the relationship $\mathbf{s} = f(\mathbf{d})$ in hand enables us to exploit both the computational advantages of dominations and the interpretive value of signatures.

Example 2.1. Consider the comparison between the two $G(9, 27)$ networks pictured in Figure 2. It is difficult to determine by a visual inspection of these two network schematics which of the two might offer better performance. The
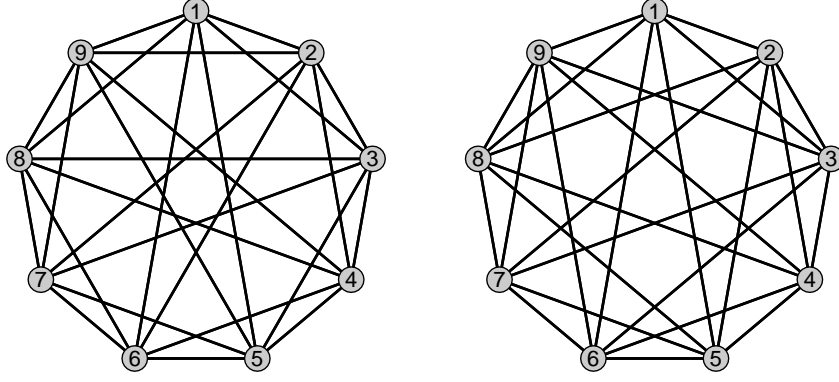
Figure 2: Networks G_1 and G_2

reliability polynomials of these two networks are displayed below:

$$\begin{aligned}
h_{G_1}(p) &= 419904p^{27} - 6021144p^{26} + 41705280p^{25} - 18489826p^{24} \\
&\quad + 586821717p^{23} - 1413876060p^{25} + 2677774329p^{21} \\
&\quad - 4074363810p^{20} + 5048856414p^{19} - 5135792742p^{18} \\
&\quad + 4303029693p^{17} - 2967712776p^{16} + 1676975886p^{15} \\
&\quad - 769265910p^{14} - 282176568p^{13} + 80853282p^{12} \\
&\quad + 17445456p^{11} - 2667060p^{10} + 257634p^9 - 11828p^8
\end{aligned}$$

$$\begin{aligned}
h_{G_2}(p) &= 414720p^{27} - 5934288p^{26} + 41015964p^{25} - 181453380p^{24} \\
&\quad + 574666025p^{23} - 1381692972p^{22} + 2611463517p^{21} \\
&\quad - 3965536554p^{20} - 4904464002p^{19} + 4979513718p^{18} \\
&\quad + 4164454729p^{17} - 2867022480p^{16} + 1617256842p^{15} \\
&\quad - 740601350p^{14} - 271201476p^{13} + 77576922p^{12} \\
&\quad + 16709916p^{11} - 2550156p^{10} + 245898p^9 - 11268p^8
\end{aligned}$$

Since the difference polynomial $h_{G_1}(p) - h_{G_2}(p)$ has alternating signs, the identification of the better performing network by this means is cumbersome. But a comparison of the signatures of these networks readily yields an answer.

From the second and third columns of Table 1, we see that $\mathbf{s}_{G_1} \geq_{\mathsf{st}} \mathbf{s}_{G_2}$, an inequality that immediately implies that $h_{G_1}(p) \geq h_{G_2}(p)$ for $p \in (0,1)$. Further, $\mathbf{s}_{G_1} \geq_{\mathsf{hr}} \mathbf{s}_{G_2}$, a conclusion that is not possible to obtain from an analysis of the polynomials $h_{G_1}(p)$ and $h_{G_2}(p)$ alone. This additional fact establishes that network G_1 is not only better than network G_2, it is actually better in quite a strong sense. More importantly, the example above illustrates well the potential utility of network signatures in the comparative analysis of network prerformance.

Table 1: Signature Tail Probabilities $S(x) = \sum_{i=x}^{27} s_i$ and Their Ratios

x	$S_{G_1}(x)$	$S_{G_2}(x)$	$S_{G_1}(x)/S_{G_2}(x)$
1	1.0	1.0	1.0
2	1.0	1.0	1.0
3	1.0	1.0	1.0
4	1.0	1.0	1.0
5	1.0	1.0	1.0
6	1.0	1.0	1.0
7	0.999970	0.999970	1.0
8	0.999787	0.999787	1.0
9	0.999149	0.999149	1.0
10	0.997367	0.997367	1.0
11	0.993612	0.993612	1.0
12	0.985922	0.985922	1.0
13	0.971744	0.971743	1.0000005
14	0.947220	0.947214	1.0000063
15	0.906907	0.906867	1.0000442
16	0.843421	0.843240	1.0002148
17	0.747317	0.746717	1.0008024
18	0.607883	0.606416	1.0024183
19	0.417560	0.415077	1.0059834
20	0.189140	0.186804	1.0125000
21	0.0	0.0	—
22	0.0	0.0	—
23	0.0	0.0	—
24	0.0	0.0	—
25	0.0	0.0	—
26	0.0	0.0	—
27	0.0	0.0	—

3 The traditional approach to identifying Uniformly Optimal Networks

Consider a class of networks of the same size, that is, with the same number of vertices (v) and the same number of edges (n). Suppose that, for any member of the $G(v, n)$ class, the failure times of the n edges are independent and have a common distribution F. The "traditional" approach to the problem of identifying a uniformly optimal network (UON) in the $G(v, n)$ class is to find, if possible, the $G(v, n)$ network (or equivalent group of networks) for which the time T^* to failure of connectivity (of a predetermined type) has a reliability function $\overline{F}_{T^*}(t)$ that is greater than or equal to the reliability function of every other network in the class. Letting $p = \overline{F}_{T^*}(t)$, where F is the common lifetime distribution of the network's edges, the UON G^* (with lifetime T^*) satisfies, for all $p \in (0, 1)$, or equivalently, for all positive values of t, the inequality

$$P_{G^*}(T^* > t) = h_{G^*}(p) \geq h_G(p) = P_G(T > t)$$

for any network G (with lifetime T) in the class of interest. This inequality is equivalent to the statement that $T \leq_{\mathrm{st}} T^*$ for all network failure times T corresponding to a network in the class $G(v, n)$. The search for Uniformly Optimal Networks (UONs) among networks $G(v, n)$ of a given size includes work by Boesch, Li and Suffel (1991), who, for example, identified the unique UON among networks in the $G(v, v - 1)$, $G(v, v)$, $G(v, v + 1)$ and $G(v, v + 2)$ classes. The UON in the $G(v, v + 3)$ class was later identified by Wang (1994). This work appeared, at the time, to be the beginning of a major surge in the study of methods and results associated with identifying UONs.

Around this same time, a group based in Victoria, British Columbia, had its doubts about the potential for success in these endeavors. In a stunning paper, they demonstrated quite dramatically that such searches for UONs might well be for naught. Specifically, Myrvold, Cheung, Page and Perry (1991) showed that for some classes of networks, e.g., the class $G\left(v, \binom{v}{2} - \frac{v}{2} - 1\right)$ for any even $v \geq 6$, a UON does not exist. They presented a similar collection of network classes with an odd number of vertices which, likewise, contained no UON. They proved the existence of a network in each such class which dominated every other network in the class for p sufficiently large, but was inferior to an alternative network if p is suitably small. The reliability polynomials of the two $G(6, 11)$ networks pictured in Figure 3 have precisely the crossing property alluded to above.

The Myrvold et al. paper all but squelched the vigorous research that had focused on the identification of UONs. Apparently, any further research in the area would need to face the fact that a given search might come up empty. Further, the problem of characterizing the classes of networks for which a UON does exist remained an open problem that appeared to be intractable.

This leads us to the main theme of the present paper. Is the search for UONs truly fraught with peril? Is this area of research dead, or are there other formulations of the UON problem that hold some real promise? The question if
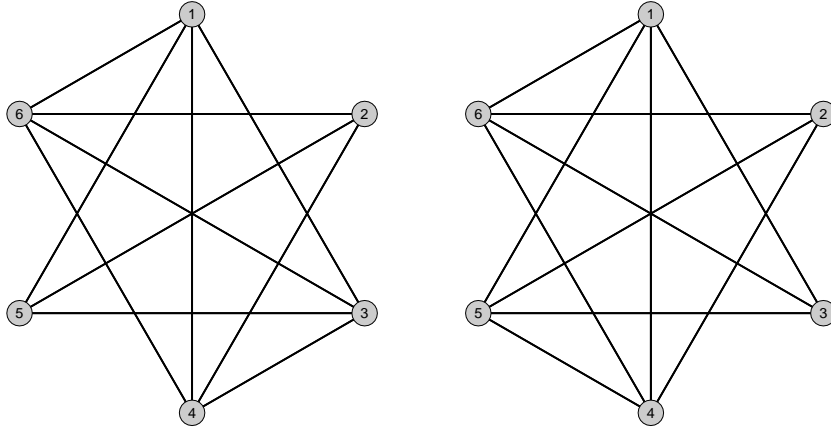
Figure 3: Networks G_8 and G_9

interest to us is: Could it be that stochastic ordering is too strong a criterion to expect uniform optimality of a single member of a class $G(v, n)$? The following empirical study of $G(6, 11)$ networks represents, to us, a "reversal of fortune" in the study of optimality questions for communications networks.

The $G(6, 11)$ class contains $\binom{15}{11} = 1365$ possible network designs. Suppose we are interested in all-terminal connectivity. When all edge probabilities p_i are equal to p, we can compute the signatures of each of these networks. As shown in Table 2, there are precisely nine distinct signatures.

Networks G_8 and G_9 pictured in Figure 3 have signatures $\mathbf{s}_8$ and $\mathbf{s}_9$, respectively. The signatures $\mathbf{s}_8$ and $\mathbf{s}_9$ are not comparable relative to stochastic ordering. Further, it is clear that there are many $G(6, 11)$ networks that are

Table 2: The Nine Possible Signatures of $G(6, 11)$ Networks.

$\mathbf{s}_1 = (0.0909, \ 0.0909, \ 0.0909, \ 0.1061, \ 0.1407, \ 0.2100, \ 0.2706, \ 0, \ 0, \ 0, \ 0)$
$\mathbf{s}_2 = (0.0000, \ 0.0182, \ 0.0485, \ 0.0939, \ 0.1662, \ 0.2835, \ 0.3896, \ 0, \ 0, \ 0, \ 0)$
$\mathbf{s}_3 = (0.0000, \ 0.0182, \ 0.0424, \ 0.0848, \ 0.1619, \ 0.2922, \ 0.4004, \ 0, \ 0, \ 0, \ 0)$
$\mathbf{s}_4 = (0.0000, \ 0.0182, \ 0.0364, \ 0.0758, \ 0.1489, \ 0.2879, \ 0.4329, \ 0, \ 0, \ 0, \ 0)$
$\mathbf{s}_5 = (0.0000, \ 0.0000, \ 0.0242, \ 0.0788, \ 0.1697, \ 0.3117, \ 0.4156, \ 0, \ 0, \ 0, \ 0)$
$\mathbf{s}_6 = (0.0000, \ 0.0000, \ 0.0182, \ 0.0636, \ 0.1541, \ 0.3117, \ 0.4524, \ 0, \ 0, \ 0, \ 0)$
$\mathbf{s}_7 = (0.0000, \ 0.0000, \ 0.0182, \ 0.0606, \ 0.1485, \ 0.3052, \ 0.4675, \ 0, \ 0, \ 0, \ 0)$
$\mathbf{s}_8 = (0.0000, \ 0.0000, \ 0.0121, \ 0.0515, \ 0.1398, \ 0.3095, \ 0.4870, \ 0, \ 0, \ 0, \ 0)$
$\mathbf{s}_9 = (0.0000, \ 0.0000, \ 0.0121, \ 0.0485, \ 0.1385, \ 0.3160, \ 0.4848, \ 0, \ 0, \ 0, \ 0)$

"equivalent" (under the assumption of i.i.d. edge failure times) to a given network with any one of the signatures above. For example, there are 180 networks in the $G(6, 11)$ class that have $\mathbf{s}_9$ as a signature vector.

The following claims about $G(6, 11)$ networks (with i.i.d. edge reliabilities) are easily confirmed.

- The signatures of the 1365 possible $G(6, 11)$ networks are totally ordered in stochastic precedence, and the nine distinct network signatures shown above are strictly sp-ordered:

$$\mathbf{s}_1 <_{\mathrm{sp}} \mathbf{s}_2 <_{\mathrm{sp}} \mathbf{s}_3 <_{\mathrm{sp}} \mathbf{s}_4 <_{\mathrm{sp}} \mathbf{s}_5 <_{\mathrm{sp}} \mathbf{s}_6 <_{\mathrm{sp}} \mathbf{s}_7 <_{\mathrm{sp}} \mathbf{s}_8 <_{\mathrm{sp}} \mathbf{s}_9$$

- Further, the following preservation result holds for all $G(6, 11)$ networks under stochastic precedence: if $\mathbf{s}_i \leq_{\mathrm{sp}} \mathbf{s}_j$, then $T_i \leq_{\mathrm{sp}} T_j$, where T_k represents the time of connectivity failure for a network of type k, with $k = 1, \ldots, 9$.

- The following comparisons show that the network G_9 is the Uniformly Optimal Network relative to the stochastic precedence ordering:

$$P(T_9 > T_8) = 0.501, \quad P(T_9 > T_7) = 0.510, \quad P(T_9 > T_6) = 0.514$$
$$P(T_9 > T_5) = 0.528, \quad P(T_9 > T_4) = 0.534, \quad P(T_9 > T_3) = 0.546$$
$$P(T_9 > T_2) = 0.553, \quad P(T_9 > T_1) = 0.659$$

So, there does exist a uniformly optimal network after all! It is clear that the criterion used in comparing networks makes a critical difference in both determining the existence of a UON and in identifying it.

4   Reliability-economics analysis of network designs.

Relative to the sp criterion, one is able to identify G_9 as the Universally Optimal Network within the $G(6, 11)$ class. This network, and those with the same signature, have the uniformly best performance among all $G(6, 11)$ networks. Now, suppose that network costs are taken into account. Consider the criterion function

$$m_r(\mathbf{s}, \mathbf{a}, \mathbf{c}) = \sum_{i=1}^{n} a_i s_i \bigg/ \left(\sum_{i=1}^{n} c_i s_i \right)^r , \tag{4.1}$$

where the vectors $\mathbf{a}$ and $\mathbf{c}$ can be chosen arbitrarily within the context of two natural constraints: $0 < a_1 < \cdots < a_n$ and $0 < c_1 < \cdots < c_n$; the constant $r > 0$ is a calibration parameter that places more or less weight on costs depending on whether $r > 1$ or $r < 1$. The function $m_r(\mathbf{s}, \mathbf{a}, \mathbf{c})$ in (4.1) represents, when $r = 1$, one reasonable way of measuring performance per unit cost; its natural variants (for $r \neq 1$) may serve as criterion functions for identifying optimal networks when either performance or cost is deemed to merit greater weight

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than the other. The criterion was utilized in Dugas and Samaniego (2007) in identifying optimal systems of a given size (see also Samaniego (2007)). Since the numerical measure m results in a total ordering of networks, the existence of an optimal network (or networks) is guaranteed, and the identification of optimal networks is reduced to a tractable minimization problem.

Before proceeding further, we interject a brief word which provides some motivation for the criterion in (4.1). Suppose that the coefficients $\{a_i, \ i = 1, \ldots, n\}$ are chosen to be $a_i = EX_{i:n}$ for $i = 1, \ldots, n$. In this case, the numerator of $m_r(\mathbf{s}, \mathbf{a}, \mathbf{c})$ is simply ET, the expected failure time of the network. On the other hand, the linear form of the denominator of $m_r(\mathbf{s}, \mathbf{a}, \mathbf{c})$ arises, for example, in the "salvage model" for a wired network which yields an expected cost of the network equal to

$$EC = \sum_{i=1}^{n}(C_f + n(A - B) + Bi)s_i,$$

where C_f is the fixed cost of manufacturing the networks of interest, A is the cost of an individual edge and B is the salvage value of an edge that is used but working when the network fails.

Example 4.1. Suppose the lifetimes of edges in a $G(6, 11)$ network are i.i.d. exponential variables with a mean life of 100 hours. Taking $a_i = EX_{i:11}$ for $i = 1, 2, \ldots, 11$, we may calculate the vector $\mathbf{a}$ as

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
9.1	19.1	30.2	42.7	57.0	73.7	93.7	118.7	152.0	202.0	302.0

Suppose we use the cost factors $c_i = 1 + 0.5 \times i$ for $i = 1, 2, \ldots, 11$. For $r = 1$, $m_1(\mathbf{s})$ is maximized at $\mathbf{s}_9$. Setting $r = 2$, the criterion function m becomes

$$m_2(\mathbf{s}, \mathbf{a}, \mathbf{c}) = \sum_{i=1}^{11} a_i s_i \left/ \left(\sum_{i=1}^{11} c_i s_i \right)^2 \right., \tag{4.2}$$

and we obtain the following results for the criterion function $m_2(\mathbf{s}) = m_2(\mathbf{s}, \mathbf{a}, \mathbf{c})$ for the 9 distinct signatures of $G(6, 11)$ networks:

$m(\mathbf{s}_1)$	$m(\mathbf{s}_2)$	$m(\mathbf{s}_3)$	$m(\mathbf{s}_4)$	$m(\mathbf{s}_5)$	$m(\mathbf{s}_6)$	$m(\mathbf{s}_7)$	$m(\mathbf{s}_8)$	$m(\mathbf{s}_9)$
5.034	4.757	4.748	4.740	4.710	4.698	4.697	4.687	4.686

From this, we see that any $G(6, 11)$ network with signature $\mathbf{s}_1$ is optimal on the basis of this performance vs. cost analysis, with $r = 2$.

We now report on a numerical search for optimal $G(6, 11)$ networks relative to $m_r(\mathbf{s}, \mathbf{a}, \mathbf{c})$ for values of r and c located in a grid. Specifically, given this criterion function with $c_i = U + Vi$ and a_i as above, we varied r from 1 to 10, and for each r we varied both U and V independently from 1 to 100. At each pair (U, V) we evaluated the criterion function for each of the nine signatures, and noted which signature produces the maximum. For each r, the relative

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Table 3: Relative frequency of optimality.

r	$\mathbf{s}_1$	$\mathbf{s}_4$	$\mathbf{s}_8$	$\mathbf{s}_9$
1	0.0000	0.0000	0.0000	1.0000
2	0.8718	0.0000	0.0353	0.0929
3	0.9450	0.0015	0.0118	0.0417
4	0.9657	0.0019	0.0069	0.0255
5	0.9761	0.0013	0.0046	0.0180
6	0.9820	0.0012	0.0033	0.0135
7	0.9859	0.0009	0.0032	0.0100
8	0.9887	0.0011	0.0019	0.0083
9	0.9906	0.0008	0.0019	0.0067
10	0.9921	0.0008	0.0021	0.0050

frequency distribution for the optimal signature over the grid of (U, V) pairs is shown in Table 3. When $r = 1$, signature $\mathbf{s}_9$ is optimal at all 10,000 (U, V) pairs. For $r > 1$, optimality is distributed among signatures 1, 4, 8 and 9 over the grid, with signature $\mathbf{s}_1$ dominating.

An intriguing feature of Table 3 is the fact that, among the 100,000 outcomes for which the optimal $G(6, 11)$ network signature relative to the metric $m_2(\mathbf{s}, \mathbf{a}, \mathbf{c})$ was recorded, the networks G_2, G_3, G_5, G_6 and G_7 never surfaced as optimal. This suggests that a certain "discontinuity" exists in the metric $m_2(\mathbf{s}, \mathbf{a}, \mathbf{c})$ as a function of the index of the network signatures ordered by their "sp ranking."

5   Discussion

Our examination of networks in the $G(6, 11)$ class is striking in a variety of different ways. First, it affirms that, relative to the stochastic precedence criterion, which is, arguably, a reasonable alternative to the stronger and more restrictive stochastic ordering criterion, the class does contain a group of equivalent networks that are uniformly optimal, that is, better than all others at every possible value of the common edge reliability p. Secondly, the sp criterion induces a special structure among $G(6, 11)$ networks; all 1365 networks in the class are totally ordered, satisfying the reflexive, anti-symmetric, transitive and trichotomy properties which characterize "order relations." It is well-known that stochastic precedence need not, in general, be transitive (see, for example, Blyth (1972)). The fact that transitivity holds here is an intriguing fact that begs the question: Is this a general phenomenon in the comparison of the performance of communications networks? In the comparisons made above, it is clear that the inequalities $\mathbf{s}_i \leq_{\mathrm{sp}} \mathbf{s}_j$ and $\mathbf{s}_j \leq_{\mathrm{sp}} \mathbf{s}_k$ imply that $\mathbf{s}_i \leq_{\mathrm{sp}} \mathbf{s}_k$. Finally, it is apparent that stochastic precedence of the signatures of coherent networks is preserved in the lifetimes of the networks themselves, that is, for any signatures $\mathbf{s}_i$ and $\mathbf{s}_j$ corresponding to networks in the $G(6, 11)$ class, $\mathbf{s}_i \leq_{\mathrm{sp}} \mathbf{s}_j$ implies that

$T_i \leq_{\mathrm{sp}} T_j$.

Many challenging questions remain to be investigated. The questions which the example above elicits may be summarized in one succinct query: To what extent do the findings in the paper regarding the use of "stochastic precedence" in the comparison of communication networks generalize?

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