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Local Definitizability of $T^{[*]}T$ and $TT^{[*]}$

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Abstract. The spectral properties of two products AB and BA of possibly unbounded operators A and B in a Banach space are considered. The results are applied in the comparison of local spectral properties of the operators $T^{[*]}T$ and $TT^{[*]}$ in a Krein space. It is shown that under the assumption that both operators $T^{[*]}T$ and $TT^{[*]}$ have non-empty resolvent sets, the operator $T^{[*]}T$ is locally definitizable if and only if $TT^{[*]}$ is. In this context the critical points of both operators are compared.

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1. Introduction

In the paper [15] three conditions on a closed and densely defined operator T in a Krein space $\mathcal{K}$ were considered:

(t1) $T^{[*]}T$ and $TT^{[*]}$ are selfadjoint operators in $\mathcal{K}$;
(t2) $T^{[*]}T$ and $TT^{[*]}$ have non-empty resolvent sets;
(t3) $T^{[*]}T$ is definitizable.

Under these conditions it was shown that also $TT^{[*]}$ is definitizable, and the spectral properties of the operators $T^{[*]}T$ and $TT^{[*]}$ were compared. In particular, it was shown, that if (t1)–(t3) are satisfied, then the non-zero spectra, as well as the non-zero singular and the non-zero regular critical points, respectively, of $T^{[*]}T$ and $TT^{[*]}$ coincide. On the other hand, there were given counterexamples, showing that for the point zero the same will not be true.

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The present contribution can be regarded as a continuation of the paper [15], although it also contains some more general results which we consider to be of independent interest. The paper consists of two sections. In the first one we consider the spectra of the products AB and BA of two arbitrary linear operators A and B acting between Banach spaces. We give a simple proof of a theorem from [5], saying that the non-zero spectra of AB and BA are equal, provided the resolvent sets of AB and BA are non-empty. As a by-product we establish some estimates on the norms of the resolvents of AB and BA. Moreover, we show that not only the non-zero spectra of AB and BA coincide, but also the most prevalent types of spectra.

The second part of the paper is devoted to the situation when $A = T$ is a closed, densely defined operator in a Krein space and $B = T^{[*]}$ is its Krein space adjoint. The first of our main objectives is to show that (t2) implies (t1) and is accomplished in Theorem 3.1. Our main tool here is a Banach-space result from [6]. Our second aim is to prove analogues of central results of [15] assuming—instead of (t3)—that the operator $T^{[*]}T$ is only definitizable over a subset Ω of $\overline{\mathbb{C}}$ (see Definition 3.8). First, we provide a natural correspondence between the sign types of the spectra of $T^{[*]}T$ and $TT^{[*]}$ (Proposition 3.7). Later on, this fact is used in the proof of Theorem 3.9. This theorem states that under condition (t2) the operator $T^{[*]}T$ is definitizable over a set Ω if and only if $TT^{[*]}$ is. This was proved already in [15] for $\Omega = \overline{\mathbb{C}}$, since definitizability over $\overline{\mathbb{C}}$ is equivalent to definitizability. However, in the present situation we cannot use the technique of definitizing polynomials as in [15]. Instead, we have to tackle the problem by comparing the local sign type properties of the spectra of $T^{[*]}T$ and $TT^{[*]}$. In this setting we also prove the equality of the sets on non-zero critical points of $T^{[*]}T$ and $TT^{[*]}$ (Theorem 3.10), which also has its analogue in [15]. The following simple example shows that all these generalizations are substantial, i.e. locally definitizable but not definitizable operators of the form $T^{[*]}T$ do exist.

Example 1. Let $(T_n)_{n=0}^{\infty}$ be a bounded sequences of linear operators in $\mathbb{C}^2$, and let the fundamental symmetry $J_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ determine the indefinite inner product on $\mathbb{C}^2$. Suppose additionally, that for each $n \in \mathbb{N}$ the operator $T_n^{[*]}T_n$ (and thus also $T_nT_n^{[*]}$) has exactly one (real) eigenvalue λ_n and that the sequence $(\lambda_n)_{n=1}^{\infty}$ is strictly decreasing to zero[1]. In the space $\ell^2(\mathbb{C}^2)$ we consider the operator T and the fundamental symmetry J, defined by

$$T = \bigoplus_{n=1}^{\infty} T_n, \qquad J = \bigoplus_{n=1}^{\infty} J_0.$$

Then the operators $T^{[*]}T$ and $TT^{[*]}$ are given by

$$T^{[*]}T = \bigoplus_{n=1}^{\infty} T_n^{[*]}T_n \quad \text{and} \quad TT^{[*]} = \bigoplus_{n=1}^{\infty} T_nT_n^{[*]}$$

[1] E.g. $T_n = \begin{pmatrix} 1/n & 0 \\ 0 & 1/n \end{pmatrix} U_n$ with any U_n satisfying $U_n^{[*]} = U_n^{-1}$.

and they satisfy (t1) and (t2) as bounded operators. Note that the algebraic eigenspace of each of the operators $T^{[*]}T$ and $TT^{[*]}$ corresponding to the eigenvalue λ_n ($n \in \mathbb{N}$) is two-dimensional and indefinite. Therefore, both operators are (locally) definitizable over $\overline{\mathbb{C}} \setminus \{0\}$, but not definitizable (over $\overline{\mathbb{C}}$).

For a history of the problem of comparing the operators $T^{[*]}T$ and $TT^{[*]}$ and its relation with indefinite polar decompositions we refer the reader to the papers [15,16]. At this point we only mention that the finite dimensional instance has found a complete solution in terms of canonical forms, see [14,16].

2. On the Pair of Operators AB and BA in Banach Spaces

We start this section by recalling some definitions and notions concerning the spectrum of a linear operator. Let $\mathcal{X}$ be a Banach space. The algebra of all bounded linear operators $T : \mathcal{X} \to \mathcal{X}$ will be denoted by $L(\mathcal{X})$. Let T be a linear operator in $\mathcal{X}$ with domain $\operatorname{dom} T \subset \mathcal{X}$. By $\rho(T)$ we denote the *resolvent set of* T which is the set of all points $\lambda \in \mathbb{C}$ for which the operator $T - \lambda : \operatorname{dom} T \to \mathcal{X}$ is bijective and $(T - \lambda)^{-1} \in L(\mathcal{X})$. Note that according to this definition of $\rho(T)$ the operator T is closed if its resolvent set is non-empty. The *spectrum of* T is defined by $\sigma(T) := \mathbb{C} \setminus \rho(T)$. We define the *point spectrum* $\sigma_{\mathrm{p}}(T)$ as the set of eigenvalues of T.

Throughout this section we assume that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, A is a closed and densely defined operator acting from $\operatorname{dom} A \subset \mathcal{X}$ to $\mathcal{Y}$, and B is a closed and densely defined operator acting from $\operatorname{dom} B \subset \mathcal{Y}$ to $\mathcal{X}$. Note that the following lemma is based on linear algebra only.

Lemma 2.1. *For $n \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ the operator A maps* $\ker((BA - \lambda)^n)$ *bijectively onto* $\ker((AB - \lambda)^n)$. *In particular, we have*

$$\sigma_{\mathrm{p}}(BA) \setminus \{0\} = \sigma_{\mathrm{p}}(AB) \setminus \{0\}.$$

Proof. We prove the statement for $n = 1$ only, the case of arbitrary n follows by induction. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and let $x \in \ker(BA - \lambda)$. Then from $BAx = \lambda x$ we conclude that $BAx \in \operatorname{dom} A$ and $ABAx = \lambda Ax$. Hence, $Ax \in \ker(AB - \lambda)$. It is now easy to check, that $\lambda^{-1}B|\ker(AB - \lambda)$ is the inverse of $A|\ker(BA - \lambda)$. □

The following lemma is a simple consequence of the closed graph theorem.

Lemma 2.2. *If the operators B and AB are closed then B is AB-bounded*[2], *i.e. there exists $c > 0$ such that*

$$\|Bx\| \le c(\|ABx\| + \|x\|), \quad x \in \operatorname{dom}(AB).$$

[2] This relation is also sometimes called *domination of B by AB*.

The first statement of the theorem below [Eq. (2.1)] has already been proved by Hardt and Mennicken in [6] with the use of an operator matrix construction. However, we present a different proof, which is necessary to obtain the estimate (2.4) playing an important role in the second part of the paper.

Theorem 2.3. *Assume that the resolvent sets $\rho(AB)$ and $\rho(BA)$ of the operators AB and BA are non-empty. Then we have*

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}. \tag{2.1}$$

Moreover, for $\lambda \in \rho(AB) \setminus \{0\}$ and $\mu \in \rho(BA)$ the following connection between the resolvents of AB and BA holds:

$$\begin{aligned}(BA-\lambda)^{-1} &= \lambda^{-1}[\overline{B(AB-\lambda)^{-1}A} - I] & (2.2)\\ &= \lambda^{-1}\left(\mu + (\lambda-\mu)B(AB-\lambda)^{-1}A\right)(BA-\mu)^{-1}. & (2.3)\end{aligned}$$

Consequently, there exists a constant $C > 0$, which depends on A and B only, such that for $\lambda, \mu \in \rho(BA), \lambda \neq 0$, the following inequality is satisfied

$$\|(BA-\lambda)^{-1}\| \leq \frac{CM_1(\lambda)M_2(\mu)}{|\lambda|}\left(|\mu| + |\lambda-\mu|\,(2+|\lambda|)(2+|\mu|)\right), \tag{2.4}$$

with $M_1(\lambda) := \max\{1, \|(AB-\lambda)^{-1}\|\}$ and $M_2(\mu) := \max\{1, \|(BA-\mu)^{-1}\|\}$.

Proof. Let $\lambda \in \rho(AB)\setminus\{0\}$. By $(BA-\lambda)^{-1}$ we denote the inverse of $BA-\lambda$ which exists due to Lemma 2.1 and maps $\operatorname{ran}(BA-\lambda)$ bijectively onto $\operatorname{dom}(BA)$. Consider the operator $R_\lambda : \operatorname{dom} A \to \mathcal{X}$, defined by

$$R_\lambda x := \lambda^{-1}[B(AB-\lambda)^{-1}Ax - x], \quad x \in \operatorname{dom} A.$$

For $x \in \operatorname{dom} A$ we obtain

$$AR_\lambda x = \lambda^{-1}[(AB-\lambda+\lambda)(AB-\lambda)^{-1}Ax - Ax] = (AB-\lambda)^{-1}Ax \in \operatorname{dom} B$$

and hence

$$(BA-\lambda)\,R_\lambda x = x.$$

In particular,

$$\operatorname{dom} A \subset \operatorname{ran}(BA-\lambda) \tag{2.5}$$

and

$$R_\lambda = (BA-\lambda)^{-1}|\operatorname{dom} A. \tag{2.6}$$

Choose μ in the resolvent set of BA. Using (2.5) and (2.6) we obtain for $x \in \operatorname{ran}(BA-\lambda)$

$$\begin{aligned}(BA-\lambda)^{-1}x &= (BA-\mu)^{-1}x + (\lambda-\mu)R_\lambda\,(BA-\mu)^{-1}x\\ &= \mu\lambda^{-1}(BA-\mu)^{-1}x\\ &\quad +(\lambda-\mu)\lambda^{-1}\,[B(AB-\lambda)^{-1}]\,[A(BA-\mu)^{-1}]x. & (2.7)\end{aligned}$$

Since the operators $B(AB-\lambda)^{-1}$ and $A(BA-\mu)^{-1}$ are bounded due to the closed graph theorem, $(BA-\lambda)^{-1}$ is bounded as well. Since it is also closed and densely defined by (2.5), we obtain $\lambda \in \rho(BA)$, which proves (2.1). The formulas (2.2) and (2.3) now follow from (2.6) and (2.7), respectively.

Observe that by Lemma 2.2 and the triangle inequality,

$$\|B(AB-\lambda)^{-1}\| \leq c_1\left(\|(AB-\lambda)^{-1}\|+\|AB(AB-\lambda)^{-1}\|\right) \leq c_1 M_1(\lambda)(2+|\lambda|).$$

Interchanging the roles of A and B we obtain for $\mu \in \rho(BA)$

$$\|A(BA-\mu)^{-1}\| \leq c_2 M_2(\mu)(2+|\mu|).$$

Now, it is easy to see that these estimates, together with (2.3), imply (2.4) with $C := \max\{1, c_1 c_2\}$. □

For a proof of the following proposition see [5, Remark 2.5] and [6, Corollary 1.7].

Proposition 2.4. *Let $\rho(AB)$ and $\rho(BA)$ be non-empty. Then AB and BA are densely defined. Moreover, we have*

$$(AB)' = B'A' \quad \text{and} \quad (BA)' = A'B',$$

where $'$ denotes the Banach space adjoint of densely defined linear operators in $\mathcal{X}$ or in $\mathcal{Y}$ or between these spaces. In consequence, if $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, then

$$(AB)^* = B^*A^* \quad \text{and} \quad (BA)^* = A^*B^*.$$

Let T be a closed and densely defined linear operator in a Banach space $\mathcal{X}$. The *approximative point spectrum* $\sigma_{\mathrm{ap}}(T)$ of T is the set of all complex numbers λ for which there exists a sequence $(x_n)_{n=0}^{\infty} \subset \operatorname{dom} T$ with $\|x_n\| = 1$ and $(T-\lambda)x_n \to 0$ as $n \to \infty$. Obviously, $\sigma_{\mathrm{ap}}(T)$ is a subset of the spectrum of T. Note that

$$\lambda \notin \sigma_{\mathrm{ap}}(T) \iff \ker(T-\lambda) = \{0\} \text{ and } \operatorname{ran}(T-\lambda) \text{ is closed.}$$

The *continuous spectrum* $\sigma_{\mathrm{c}}(T)$ and the *residual spectrum* $\sigma_{\mathrm{r}}(T)$ of T are defined as usual. The operator T is called *upper* (*lower*) *semi-Fredholm* if $\operatorname{ran} T$ is closed and $\ker T$ is finite-dimensional (resp. $\operatorname{ran} T$ is finite-codimensional). The operator T is called *Fredholm* if it is both upper and lower semi-Fredholm. Note that T is upper (lower) semi-Fredholm if and only if T' is lower (resp. upper) semi-Fredholm. The *essential spectrum of* T is defined by

$$\sigma_{\mathrm{ess}}(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Fredholm}\}.$$

Theorem 2.5. *Let $\rho(AB)$ and $\rho(BA)$ be non-empty. Then for $\lambda \in \mathbb{C}\setminus\{0\}$ the following statements hold:*

(i) $\operatorname{ran}(AB-\lambda)$ *is closed if and only if* $\operatorname{ran}(BA-\lambda)$ *is closed;*
(ii) $\operatorname{ran}(AB-\lambda)$ *is dense in $\mathcal{Y}$ if and only if* $\operatorname{ran}(BA-\lambda)$ *is dense in $\mathcal{X}$;*
(iii) *$AB-\lambda$ is upper semi-Fredholm if and only if $BA-\lambda$ is upper semi-Fredholm;*
(iv) *$AB-\lambda$ is lower semi-Fredholm if and only if $BA-\lambda$ is lower semi-Fredholm.*

In consequence,

$$\sigma_{\mathrm{ap}}(AB)\setminus\{0\} = \sigma_{\mathrm{ap}}(BA)\setminus\{0\}, \quad \sigma_{\mathrm{c}}(AB)\setminus\{0\} = \sigma_{\mathrm{c}}(BA)\setminus\{0\},$$
$$\sigma_{\mathrm{r}}(AB)\setminus\{0\} = \sigma_{\mathrm{r}}(BA)\setminus\{0\}, \quad \sigma_{\mathrm{ess}}(AB)\setminus\{0\} = \sigma_{\mathrm{ess}}(BA)\setminus\{0\}.$$

Proof. Obviously, it is sufficient to prove only one of the implications in each of the points (i)–(iv).

(i) Assume that $\operatorname{ran}(BA-\lambda)$ is not closed. Then there exists a sequence $(x_n)_{n=0}^{\infty} \subset \operatorname{dom}(BA)$ with[3]

$$\operatorname{dist}(x_n, \ker(BA-\lambda)) = 1 \quad \text{and} \quad (BA-\lambda)x_n \to 0 \ \text{ as } \ n \to \infty. \tag{2.8}$$

Fix $\mu \in \rho(BA) \setminus \{0, \lambda\}$ and set

$$y_n := (\lambda-\mu)(BA-\mu)^{-1}x_n.$$

Then $y_n \in \operatorname{dom}((BA)^2)$ for every $n \in \mathbb{N}$ and

$$(BA-\lambda)y_n = (\lambda-\mu)\left(x_n + (\mu-\lambda)(BA-\mu)^{-1}x_n\right) = (\lambda-\mu)(x_n - y_n). \tag{2.9}$$

On the other hand, we have

$$(BA-\lambda)y_n = (\lambda-\mu)(BA-\mu)^{-1}(BA-\lambda)x_n \ \to 0 \ \text{ as } \ n \to \infty. \tag{2.10}$$

Consequently, $\|x_n - y_n\| \to 0$ as $n \to \infty$. Furthermore, (2.9) gives

$$BA(BA-\lambda)y_n = (\lambda-\mu)BA(x_n - y_n),$$

which also tends to zero as $n \to \infty$, by (2.8) and (2.10). By Lemma 2.2 we have

$$(AB-\lambda)Ay_n = A(BA-\lambda)y_n \ \to 0 \quad \text{as } \ n \to \infty. \tag{2.11}$$

Now we show that

$$\liminf_{n\to\infty} \operatorname{dist}(Ay_n, \ker(AB-\lambda)) > 0, \tag{2.12}$$

which will prove that $\operatorname{ran}(AB-\lambda)$ is not closed. Let us suppose that (2.12) is not true. Without loss of generality we can assume that

$$\operatorname{dist}(Ay_n, \ker(AB-\lambda)) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.13}$$

From (2.9) and (2.11) we obtain

$$Ax_n - Ay_n = \frac{1}{\lambda-\mu}(AB-\lambda)Ay_n \to 0,$$

and consequently [cf. (2.13)] $\operatorname{dist}(Ax_n, \ker(AB-\lambda)) \to 0$ as $n \to \infty$. This implies that there exists a sequence $(u_n)_{n=0}^{\infty} \subset \ker(AB-\lambda)$ with $\|Ax_n - u_n\| \to 0$ as $n \to \infty$. Since

$$(BA-\lambda)B(AB-\mu)^{-1}u_n = B(AB-\lambda)(AB-\mu)^{-1}u_n = 0,$$

we have $B(AB-\mu)^{-1}u_n \in \ker(BA-\lambda)$. As $B(AB-\mu)^{-1}$ is bounded we get

$$\operatorname{dist}\left(B(AB-\mu)^{-1}Ax_n, \ker(BA-\lambda)\right) \to 0 \quad \text{with } \ n \to \infty.$$

[3] Indeed, consider the quotient Banach space $X/\ker(BA-\lambda)$ and the injective operator C that maps the equivalence class $f + \ker(BA-\lambda)$ ($f \in \operatorname{dom}(BA)$) to $(BA-\lambda)f$. Then, the range of $BA-\lambda$ coincides with the range of C and the latter is closed if and only if C is bounded from below.

In view of

$$B(AB-\mu)^{-1}Ax_n = BA(BA-\mu)^{-1}x_n$$
$$= x_n + \mu(BA-\mu)^{-1}x_n = x_n - y_n + \frac{\lambda}{\lambda-\mu}\, y_n$$

together with $\|x_n - y_n\| \to 0$ as $n \to \infty$ we conclude that

$$\operatorname{dist}\left(x_n, \ker(BA-\lambda)\right) \;\to\; 0 \quad \text{as} \quad n \to \infty,$$

which is a contradiction to (2.8).

(ii) Let $\operatorname{ran}(AB-\lambda)$ be dense in $\mathcal{Y}$ and let $x \in \operatorname{dom} A$ be arbitrary. We will show that $x \in \overline{\operatorname{ran}(BA-\lambda)}$, which will finish the proof of (ii). By assumption there exists a sequence $(v_n)_{n=0}^{\infty} \subset \operatorname{dom}(AB)$ such that $(AB-\lambda)v_n \to Ax$ as $n \to \infty$. Fix $\mu \in \rho(BA) \setminus \{0\}$. We show now that $(BA-\lambda)u_n \to x$ as $n \to \infty$ where

$$u_n := \lambda^{-1}(BA-\mu)^{-1}\left((\lambda-\mu)Bv_n + \mu x\right) \in \operatorname{dom}(BA).$$

To obtain this, observe first that for every $u \in \operatorname{dom} A$ we have

$$(BA-\mu)^{-1}u = R_\mu u := \mu^{-1}\left(B(AB-\mu)^{-1}Au - u\right)$$

(see Theorem 2.3). Thus

$$\begin{aligned}(BA-\lambda)(BA-\mu)^{-1}u &= BA(BA-\mu)^{-1}u - \lambda(BA-\mu)^{-1}u\\ &= B(AB-\mu)^{-1}Au - \frac{\lambda}{\mu}\left(B(AB-\mu)^{-1}Au - u\right)\\ &= \frac{1}{\mu}\left((\mu-\lambda)B(AB-\mu)^{-1}Au + \lambda u\right).\end{aligned}$$

Substituting $u := u_n$ ($n \in \mathbb{N}$) above and using the fact that $R_\mu B \subset B(AB-\mu)^{-1}$ we obtain

$$\begin{aligned}(BA\ -\ \lambda)u_n &= \frac{1}{\lambda\mu}\left((\mu-\lambda)B(AB-\mu)^{-1}A + \lambda\right)\left((\lambda-\mu)Bv_n + \mu x\right)\\ &= \ x + \frac{\lambda-\mu}{\lambda\mu}\left((\mu-\lambda)B(AB-\mu)^{-1}ABv_n + \lambda Bv_n\right.\\ &\quad \left. - \mu B(AB-\mu)^{-1}Ax\right)\\ &= \ x + \frac{\lambda-\mu}{\lambda\mu}\left(\mu B(AB-\mu)^{-1}(ABv_n - Ax) - \lambda\mu R_\mu Bv_n\right)\\ &= \ x + \frac{\lambda-\mu}{\lambda}B(AB-\mu)^{-1}\left((AB-\lambda)v_n - Ax\right),\end{aligned}$$

which tends to x as $n \to \infty$, since $B(AB-\mu)^{-1}$ is bounded.

Point (iii) is an easy consequence of (i) and Lemma 2.1. To see that (iv) holds suppose that $AB-\lambda$ is lower semi-Fredhom. Then $(AB-\lambda)'$ is upper semi-Fredholm. On the other hand the latter operator equals $B'A'-\lambda$, cf. Proposition 2.4. Since $\rho(B'A') = \rho(AB) \neq \varnothing$ and $\rho(A'B') = \rho(BA) \neq \varnothing$ we conclude that $A'B'-\lambda$ is upper semi-Fredholm. Consequently, $BA-\lambda$ is lower semi-Fredholm. The remainder of the theorem follows directly from (i)–(iv) and Lemma 2.1. □

3. Local Spectral Properties of $T^{[*]}T$ and $TT^{[*]}$

For an introduction to Krein spaces and operators acting therein we refer to the monographs [1] and [4] and also to [10]. Throughout this section $(\mathcal{K}, [\cdot\,,\cdot])$ will be a Krein space and $\|\cdot\|$ will be a Banach space norm on $\mathcal{K}$, such that the indefinite inner product is continuous with respect $\|\cdot\|$. All such norms are equivalent and the calculations below do not depend on the choice of one of these norms.

In what follows T stands for a closed, densely defined linear operator in $\mathcal{K}$. The adjoint of T with respect to the indefinite inner product $[\cdot\,,\cdot]$ will be denoted by $T^{[*]}$. Observe that if $T^{[*]}T \in L(\mathcal{K})$ then $T \in L(\mathcal{K})$ as well, by the closed graph theorem. Let us also note that the operator $T^{[*]}T$ is symmetric, although not necessarily densely defined, cf. [15, Section 3]. This was a reason for introducing in [15] the additional assumptions (t1)–(t3), quoted in the introduction, on the operator T. It turns out that assuming (t1) is not necessary.

Theorem 3.1. *If T satisfies* (t2) *then it satisfies* (t1) *as well.*

Proof. Note that if the resolvent sets of both $T^{[*]}T$ and $TT^{[*]}$ are non-empty, then the domain of $T^{[*]}T$ is dense in $\mathcal{K}$, by Proposition 2.4. Let J be any fundamental symmetry of the Krein space and let A^* denote the adjoint of a densely defined operator A in the Hilbert space $(\mathcal{K}, [J\cdot,\cdot])$. Then from $A^{[*]} = JA^*J$ and Proposition 2.4 it follows that

$$(T^{[*]}T)^{[*]} = J(T^{[*]}T)^*J = JT^*(T^{[*]})^*J = (JT^*J)\left(J(T^{[*]})^*J\right) = T^{[*]}T.$$

Similarly, $(TT^{[*]})^{[*]} = TT^{[*]}$. □

Recall that a well-known sufficient condition for selfadjointness of a symmetric operator in a Krein space is that both λ and $\overline{\lambda}$ belong to its resolvent set for some $\lambda \in \mathbb{C}$. Theorem 3.1 provides another sufficient condition for selfadjointness of $T^{[*]}T$. The example below shows that (t1) and (t2) are not equivalent.

Example 2. In the following we use the abbreviations

$$L^2 := L^2(-1,1),\quad L^2_+ := L^2(0,1),\quad L^2_- := L^2(-1,0)$$

and

$$AC := AC([-1,1]),\quad AC_+ := AC([0,1]),\quad AC_- := AC([-1,0]),$$

where $AC(I)$ denotes the set of all absolutely continuous functions on the interval I. The Hilbert space scalar product on L^2 will be denoted by $(\cdot\,,\cdot)$. Let J be the operator of multiplication by $\operatorname{sgn}(x)$ in L^2, i.e.

$$(Jf)(x) := \operatorname{sgn}(x)f(x),\quad f \in L^2,\ x \in (-1,1).$$

The inner product $[\cdot\,,\cdot] := (J\cdot,\cdot)$ is then a Krein space inner product on L^2.

Let the operator T be defined as follows: $Tf := f'$, $f \in \operatorname{dom} T$, where

$$\operatorname{dom} T := \{f \in L^2 : f \in AC,\ f' \in L^2\} = H^1(-1,1).$$

The adjoint of T in L^2 is then given by $T^*f := -f'$, $f \in \operatorname{dom} T^*$, where

$$\operatorname{dom} T^* = \{f \in \operatorname{dom} T : f(-1) = f(1) = 0\},$$

see, e.g. [9, Example V.3.14]. We identify L^2 with the cartesian product $L^2_+ \times L^2_-$ and write every $f \in L^2$ as a pair (f_+, f_-) with $f_\pm \in L^2_\pm$. A straightforward calculation shows that the domains of $T^{[*]}T$ and $TT^{[*]}$ are given by

$$\begin{aligned}\operatorname{dom} T^{[*]}T = \{(f_+, f_-) \in L^2 :\ & f_\pm \in AC_\pm,\ \ f'_\pm \in AC_\pm,\ \ f''_\pm \in L^2_\pm\\ & f_-(0) = f_+(0),\ \ -f'_-(0) = f'_+(0),\\ & f'_-(-1) = f'_+(1) = 0\}\end{aligned}$$

and

$$\begin{aligned}\operatorname{dom} TT^{[*]} = \{(f_+, f_-) \in L^2 :\ & f_\pm \in AC_\pm,\ \ f'_\pm \in AC_\pm,\ \ f''_\pm \in L^2_\pm\\ & -f_-(0) = f_+(0),\ \ f'_-(0) = f'_+(0),\\ & f_-(-1) = f_+(1) = 0\}.\end{aligned}$$

Moreover, we have $T^{[*]}Tf = -f''$ for $f \in \operatorname{dom} T^{[*]}T$ and $TT^{[*]}f = -f''$ for $f \in \operatorname{dom} TT^{[*]}$.

Let $\lambda \in \mathbb{C} \setminus \{0\}$ and let $\sqrt{\lambda}$ be any square root of λ. For $x \in [-1, 1]$ we define the functions

$$f(x) := \cos(\sqrt{\lambda}) \cos(\sqrt{\lambda}x) + \sin(\sqrt{\lambda}) \operatorname{sgn}(x) \sin(\sqrt{\lambda}x)$$

and

$$g(x) := \sin(\sqrt{\lambda}) \operatorname{sgn}(x) \cos(\sqrt{\lambda}x) - \cos(\sqrt{\lambda}) \sin(\sqrt{\lambda}x).$$

It is not difficult to see that $f \in \operatorname{dom} T^{[*]}T, g \in \operatorname{dom} TT^{[*]}$ and that $T^{[*]}Tf = \lambda f, TT^{[*]}g = \lambda g$. Moreover, one easily verifies that also $\lambda = 0$ is an eigenvalue of both operators, so that

$$\sigma(T^{[*]}T) = \sigma_p(T^{[*]}T) = \mathbb{C} \quad \text{and} \quad \sigma(TT^{[*]}) = \sigma_p(TT^{[*]}) = \mathbb{C}.$$

Hence, (t2) fails to hold.

Let us show that (t1) is satisfied. To this end it suffices to prove that $JT^{[*]}T$ and $TT^{[*]}J$ are selfadjoint in $(L^2, (\cdot\,, \cdot))$. Let $p(x) := \operatorname{sgn}(x), x \in [-1, 1]$. We have

$$\begin{aligned}\operatorname{dom} JT^{[*]}T &= \{f \in L^2 : f, pf' \in AC,\ (pf')' \in L^2,\ (pf')(\pm 1) = 0\},\\ \operatorname{dom} TT^{[*]}J &= \{f \in L^2 : f, pf' \in AC,\ (pf')' \in L^2,\ f(\pm 1) = 0\},\end{aligned}$$

$JT^{[*]}Tf = -(pf')'$ for $f \in \operatorname{dom} JT^{[*]}T$ and $TT^{[*]}Jf = (-pf')'$ for $f \in \operatorname{dom} TT^{[*]}J$. Hence, these operators are ordinary Sturm–Liouville operators on $[-1, 1]$ with von Neumann and Dirichlet boundary conditions, respectively. Such operators are known to be selfadjoint in L^2.

We formulate Theorem 2.3 explicitly for the operators T and $T^{[*]}$ as a separate result.

Theorem 3.2. *Assume that* (t2) *holds. Then we have*

$$\sigma(T^{[*]}T) \setminus \{0\} = \sigma(TT^{[*]}) \setminus \{0\},$$

and there exists a constant $C > 0$ depending on T only, such that for $\lambda, \mu \in \rho(T^{[*]}T), \lambda \neq 0$, the following inequality holds

$$\|(T^{[*]}T - \lambda)^{-1}\| \leq \frac{CM_1(\lambda)M_2(\mu)}{|\lambda|} \left(|\mu| + |\lambda - \mu| \, (2 + |\lambda|)(2 + |\mu|)\right), \quad (3.1)$$

where $M_1(\lambda) := \max\{1, \|(TT^{[*]} - \lambda)^{-1}\|\}$ and $M_2(\mu) := \max\{1, \|(T^{[*]}T - \mu)^{-1}\|\}$.

Remark 3.3. It is not clear whether the condition (t1), weaker then (t2), implies that the non-zero spectra of $T^{[*]}T$ and $TT^{[*]}$ coincide. In view of Theorem 3.2 we can formulate this question as the following open problem:

> Is it possible that (t1) holds and $\rho(T^{[*]}T) = \varnothing$, while $\rho(TT^{[*]}) \neq \varnothing$?

Corollary 3.4. *Assume that* (t2) *is satisfied and that zero belongs to $\rho(T^{[*]}T) \cap \sigma(TT^{[*]})$. Then zero is a pole of order one of the resolvent of $TT^{[*]}$.*

Proof. It follows from Theorem 3.2 that zero is an isolated spectral point of $TT^{[*]}$ and therefore an isolated singularity of the resolvent of $TT^{[*]}$. Applying the estimate (3.1) for λ in a deleted neighborhood of zero we see that it is a pole of order one. □

It is well-known that the real spectrum of a definitizable selfadjoint operator in a Krein space decomposes into spectral points of positive and negative type and a finite set of critical points (see [10]). There are four ways to distinguish between these three classes of spectral points:

(1) via definitizing polynomials,
(2) via the spectral function,
(3) via the functional calculus,
(4) via approximate eigensequences.

Suppose for example that a selfadjoint operator A in the Krein space $(\mathcal{K}, [\cdot\,,\cdot])$ can be decomposed into a direct $[\cdot\,,\cdot]$-orthogonal sum $A = A_1 \,[\dotplus]\, A_2$ of a definitizable operator A_1 and a selfadjoint operator A_2. Furthermore, suppose that a compact interval Δ is of positive type with respect to A_1 and lies entirely in the resolvent set of A_2. Then the operator A is not definitizable in general, but obviously it has a local spectral function on Δ and the corresponding spectral subspaces are Hilbert spaces with respect to the inner product $[\cdot\,,\cdot]$. Hence, it still makes sense to call the points in $\sigma(A) \cap \Delta$ spectral points of positive type of A. In applications such a special decomposition of A is hardly seen to exist. Hence, the only way to identify the points in $\sigma(A) \cap \Delta$ as spectral points of positive type is the possibility (4), described below in detail.

Definition 3.5. Let A be a selfadjoint operator in the Krein space $(\mathcal{K}, [\cdot\,,\cdot])$. A point $\lambda \in \sigma(A)$ is called a spectral point of *positive* (*negative*) *type of* A if

$\lambda \in \sigma_{\mathrm{ap}}(A)$ and if for every sequence $(x_n)_{n=0}^{\infty} \subset \operatorname{dom} A$ with $\|x_n\| = 1$ and $(A-\lambda)x_n \to 0$ as $n \to \infty$ we have

$$\liminf_{n\to\infty}\,[x_n, x_n] > 0 \quad \bigl(\text{resp. } \limsup_{n\to\infty}\,[x_n, x_n] < 0\bigr).$$

We denote the set of all spectral points of positive (negative) type of A by $\sigma_{++}(A)$ [resp. $\sigma_{--}(A)$]. A set $\Delta \subset \mathbb{C}$ is said to be *of positive* (*negative*) *type with respect to* A if $\Delta \cap \sigma(A) \subset \sigma_{++}(A)$ [resp. $\Delta \cap \sigma(A) \subset \sigma_{--}(A)$]. If Δ is either of positive type or of negative type with respect to A, then we say that Δ is *of definite type with respect to* A.

It is well-known that for a selfadjoint operator A we have $\sigma(A) \cap \mathbb{R} \subset \sigma_{\mathrm{ap}}(A)$. It was shown in [2,11] that $\sigma_{\pm\pm}(A) \subset \mathbb{R}$ and that for a compact interval Δ which is of positive type with respect to A there exists an open neighborhood $\mathcal{U}$ in $\mathbb{C}$ of Δ such that

$$\mathcal{U} \cap \sigma(A) \cap \mathbb{R} \subset \sigma_{++}(A) \quad \text{ and } \quad \mathcal{U} \setminus \mathbb{R} \subset \rho(A). \tag{3.2}$$

Moreover, the resolvent $(A-\lambda)^{-1}$ for λ near Δ does not grow faster than $M/|\operatorname{Im}\lambda|$ with some constant $M > 0$. This fact gives rise to a local spectral function of A on Δ. The spectral subspaces given by this spectral function are then Hilbert spaces with respect to the inner product $[\cdot\,,\cdot]$, cf. [11]. An analogue holds for intervals of negative type with respect to A.

In [2] a larger class of spectral points of a selfadjoint operator in a Krein space was introduced, namely the spectral points of type π_+ and π_-. For example, these arise from spectral points of positive or negative type after compact perturbations of the operator (see [2, Theorem 19]). Moreover, every spectral point of a selfadjoint operator in a Pontryagin space with finite rank of negativity is of type π_+. The definition below is equivalent to that in [2], cf. [2, Theorem 14]. We write $x_n \rightharpoonup x$ as $n \to \infty$ if the sequence $(x_n)_{n=0}^{\infty} \subset \mathcal{K}$ converges weakly to some $x \in \mathcal{K}$.

Definition 3.6. Let A be a selfadjoint operator in the Krein space $(\mathcal{K}, [\cdot\,,\cdot])$. A point $\lambda \in \sigma(A)$ is called a spectral point *of type* π_+ (*type* π_-) *of* A if $\lambda \in \sigma_{\mathrm{ap}}(A)$ and if for every sequence $(x_n)_{n=0}^{\infty} \subset \operatorname{dom} A$ with $\|x_n\| = 1, x_n \rightharpoonup 0$ and $(A-\lambda)x_n \to 0$ as $n \to \infty$ we have

$$\liminf_{n\to\infty}\,[x_n, x_n] > 0 \quad \bigl(\text{resp. } \limsup_{n\to\infty}\,[x_n, x_n] < 0\bigr).$$

We denote the set of all spectral points of type π_+ (type π_-) of A by $\sigma_{\pi_+}(A)$ [resp. $\sigma_{\pi_-}(A)$]. A set $\Delta \subset \mathbb{C}$ is said to be *of type* π_+ (*resp. type* π_-) *with respect to* A if $\Delta \cap \sigma(A) \subset \sigma_{\pi_+}(A)$ [resp. $\Delta \cap \sigma(A) \subset \sigma_{\pi_-}(A)$].

It was shown in [3] (for a weaker statement see also [2]) that if Δ is a compact interval of type π_+ with respect to the selfadjoint operator A in $\mathcal{K}$ which contains an accumulation point of the resolvent set of A, then—just as in the case of an interval of definite type—there exists a neighborhood $\mathcal{U}$ of Δ in $\mathbb{C}$ such that (3.2) holds with $\sigma_{++}(A)$ replaced by $\sigma_{\pi_+}(A)$. Moreover, the set $\Delta \cap (\sigma_{\pi_+}(A) \setminus \sigma_{++}(A))$ is finite. The growth of $(A-\lambda)^{-1}$ in $\mathcal{U} \setminus \mathbb{R}$ can be estimated by some power of $|\operatorname{Im}\lambda|^{-1}$. Also in this case A possesses a local spectral function on Δ (with singularities). The spectral subspaces are Pontryagin spaces with finite rank of negativity.

The following result generalizes Proposition 5.1 of [15]. By $\mathbb{R}^{\pm}$ we denote the set $\{x \in \mathbb{R} : \pm x > 0\}$.

Proposition 3.7. *Assume that* (t2) *is satisfied. Then the following holds:*

(i) $\sigma_{\mathrm{ap}}(T^{[*]}T) \setminus \{0\} = \sigma_{\mathrm{ap}}(TT^{[*]}) \setminus \{0\}$,
(ii) $\sigma_{\pm\pm}(T^{[*]}T) \cap \mathbb{R}^+ = \sigma_{\pm\pm}(TT^{[*]}) \cap \mathbb{R}^+$,
(iii) $\sigma_{\pm\pm}(T^{[*]}T) \cap \mathbb{R}^- = \sigma_{\mp\mp}(TT^{[*]}) \cap \mathbb{R}^-$,
(iv) $\sigma_{\pi_\pm}(T^{[*]}T) \cap \mathbb{R}^+ = \sigma_{\pi_\pm}(TT^{[*]}) \cap \mathbb{R}^+$,
(v) $\sigma_{\pi_\pm}(T^{[*]}T) \cap \mathbb{R}^- = \sigma_{\pi_\mp}(TT^{[*]}) \cap \mathbb{R}^-$.

Proof. Statement (i) is a direct consequence of Theorem 2.5. To prove (ii) and (iii) consider $\lambda \in \sigma_{++}(TT^{[*]}) \setminus \{0\}$. Then $\lambda \in \sigma_{\mathrm{ap}}(T^{[*]}T)$ by (i). Let $(x_n)_{n=0}^\infty \subset \mathrm{dom}(T^{[*]}T)$ be a sequence with $\|x_n\| = 1$ and $(T^{[*]}T - \lambda)x_n \to 0$ as $n \to \infty$. Then define

$$y_n := (\lambda - \mu)(T^{[*]}T - \mu)^{-1} x_n \in \mathrm{dom}((T^{[*]}T)^2)$$

with some $\mu \in \rho(T^{[*]}T) \setminus \{0\}$. This sequence satisfies $\liminf_{n\to\infty} \|Ty_n\| > 0$ and

$$\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \|(T^{[*]}T - \lambda)y_n\| = \lim_{n\to\infty} \|(TT^{[*]} - \lambda)Ty_n\| = 0.$$

This can be seen with a very similar argumentation as in the proof of Theorem 2.5 (i) (with $A = T$ and $B = T^{[*]}$). Hence,

$$\begin{aligned}\liminf_{n\to\infty} [x_n, x_n] = \liminf_{n\to\infty} [y_n, y_n] \\ &= \frac{1}{\lambda} \liminf_{n\to\infty} \Big([Ty_n, Ty_n] - [(T^{[*]}T - \lambda)y_n, y_n] \Big) \\ &= \frac{1}{\lambda} \liminf_{n\to\infty} [Ty_n, Ty_n],\end{aligned}$$

and similarly

$$\limsup_{n\to\infty} [x_n, x_n] = \frac{1}{\lambda} \limsup_{n\to\infty} [Ty_n, Ty_n].$$

Since $\lambda \in \sigma_{++}(TT^{[*]})$, we have

$$\limsup_{n\to\infty} [Ty_n, Ty_n] \geq \liminf_{n\to\infty} [Ty_n, Ty_n] > 0.$$

This shows that $\lambda \in \sigma_{++}(T^{[*]}T)$ if $\lambda > 0$ and $\lambda \in \sigma_{--}(T^{[*]}T)$ if $\lambda < 0$.

To show that (iv) and (v) hold, let $\lambda \in \sigma_{\pi_+}(TT^{[*]}) \setminus \{0\}$. Then the same argument as above applies with the additional assumption that $(x_n)_{n=0}^\infty$ converges weakly to zero. It remains to show that $(Ty_n)_{n=0}^\infty$ (or at least a subsequence) converges weakly to zero. Since $T(T^{[*]}T - \mu)^{-1}$ is bounded, $(Ty_n)_{n=0}^\infty$ is bounded. It is therefore no restriction to assume that there exists some $v \in \mathcal{K}$ such that $Ty_n \rightharpoonup v$ as $n \to \infty$. From $x_n \rightharpoonup 0$ and $\|y_n - x_n\| \to 0$ we conclude that $y_n \rightharpoonup 0$ as $n \to \infty$. Since T is (weakly) closed, v equals zero. □

In the next definition we recall the notion of locally definitizable operators. The version below is taken from [8], see also [7]. We denote the one-point compactification of the real line and the complex plane by $\overline{\mathbb{R}}$ and $\overline{\mathbb{C}}$, respectively. Moreover, we set $\mathbb{C}^{\pm} := \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$.

Definition 3.8. Let Ω be a domain in $\overline{\mathbb{C}}$ which is symmetric with respect to $\overline{\mathbb{R}}$ with $\Omega \cap \mathbb{R} \neq \varnothing$ such that $\Omega \cap \mathbb{C}^+$ and $\Omega \cap \mathbb{C}^-$ are simply connected. A selfadjoint operator A in $\mathcal{K}$ is called *definitizable over* Ω if the following holds:

(i) The set $\sigma(A) \cap (\Omega \setminus \overline{\mathbb{R}})$ does not have any accumulation points in Ω and consists of poles of the resolvent of A.

(ii) For each closed subset Δ of $\Omega \cap \overline{\mathbb{R}}$ there exist an open neighborhood $\mathcal{U}$ of Δ in $\overline{\mathbb{C}}$ and numbers $m \geq 1, M > 0$ such that

$$\|(A-\lambda)^{-1}\| \leq M \, \frac{(1+|\lambda|)^{2m-2}}{|\operatorname{Im} \lambda|^m}, \qquad \lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}.$$

(iii) Each point $\lambda \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighborhood I_λ in $\overline{\mathbb{R}}$ such that each component of $I_\lambda \setminus \{\lambda\}$ is of definite type with respect to A.

In [8, Theorem 3.6] it was shown that a selfadjoint operator A in the Krein space $\mathcal{K}$ is definitizable if and only if it is definitizable over $\overline{\mathbb{C}}$. The following theorem was proved in [15] for the special case $\Omega = \overline{\mathbb{C}}$.

Theorem 3.9. *Assume that* (t2) *holds and let* Ω *be an open domain in* $\overline{\mathbb{C}}$ *as in Definition* 3.8*. Then* $T^{[*]}T$ *is definitizable over* Ω *if and only if* $TT^{[*]}$ *is definitizable over* Ω*.*

Proof. Let us assume that $TT^{[*]}$ is definitizable over Ω. By Theorem 3.2 and Proposition 3.7 the conditions (i) and (iii) in Definition 3.8 are easily seen to be satisfied by $T^{[*]}T$. Hence, it remains to check that condition (ii) holds for $T^{[*]}T$. Let Δ be a closed subset of $\Omega \cap \overline{\mathbb{R}}$. Then, as $TT^{[*]}$ is definitizable over Ω, there exist an open neighborhood $\mathcal{U}$ of Δ in $\overline{\mathbb{C}}$ and numbers $m \geq 1, M > 0$ such that

$$\|(TT^{[*]}-\lambda)^{-1}\| \leq M \, \frac{(1+|\lambda|)^{2m-2}}{|\operatorname{Im} \lambda|^m} \tag{3.3}$$

holds for all $\lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}$. It is obviously no restriction to assume $M \geq 1$. Moreover, as the sequence $(1+|\lambda|)^{2n-2}/|\operatorname{Im} \lambda|^n$ is monotonically increasing for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we may assume $m \geq 2$, such that the right hand side of (3.3) is not smaller than 1.

Fix some $\mu \in \rho(T^{[*]}T) \setminus \{0\}$. Then, by Theorem 3.2 we have for $\lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}$:

$$\|(T^{[*]}T-\lambda)^{-1}\| \leq \frac{D}{|\lambda|} \, \left(|\mu| + |\lambda-\mu| \, (2+|\lambda|)(2+|\mu|)\right) \, M_1(\lambda),$$

with

$$M_1(\lambda) = \max\{1, \|(TT^{[*]}-\lambda)^{-1}\|\} \leq M \, \frac{(1+|\lambda|)^{2m-2}}{|\operatorname{Im} \lambda|^m}$$

and some $D > 0$ depending on T and μ only. Hence, with $c := |\mu|$ we obtain for all $\lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}$ that

$$\begin{aligned} \|(T^{[*]}T - \lambda)^{-1}\| &\le \frac{\text{const}}{|\lambda|}\ (c + (2+c)(c+|\lambda|)\,(2+|\lambda|))\ \frac{(1+|\lambda|)^{2m-2}}{|\operatorname{Im}\lambda|^m} \\ &\le \frac{\text{const}}{|\lambda|}\ (1 + (c+|\lambda|)\,(2+|\lambda|))\ \frac{(1+|\lambda|)^{2m-2}}{|\operatorname{Im}\lambda|^m} \\ &\le \frac{\text{const}}{|\lambda|}\ \left(1 + 2\max\{1,c\}(1+|\lambda|)^2\right)\ \frac{(1+|\lambda|)^{2m-2}}{|\operatorname{Im}\lambda|^m} \\ &\le \frac{\text{const}}{|\lambda|}\,(1+|\lambda|)^2\,\frac{(1+|\lambda|)^{2m-2}}{|\operatorname{Im}\lambda|^m} \\ &\le \text{const}\,\frac{(1+|\lambda|)^{2(m+1)-2}}{|\operatorname{Im}\lambda|^{m+1}}, \end{aligned}$$

with some const > 0 which is independent of λ. □

In the following let A be a selfadjoint operator in $(\mathcal{K}, [\cdot\,,\cdot])$ which is definitizable over some domain Ω. If A is unbounded and the point ∞ belongs to Ω then we say that ∞ is a spectral point of positive (negative) type of A if both components of $I_\infty \setminus \{\infty\}$ [see Definition 3.8(iii)] are of positive (resp. negative) type. We mention that this can also be formulated with the help of approximate eigensequences (see [2]).

As is well-known (see e.g. [8]), the operator A possesses a local spectral function E on $\Omega \cap \overline{\mathbb{R}}$. The projection $E(\Delta)$ is always a bounded selfadjoint operator in the Krein space $\mathcal{K}$ and is defined for all finite unions Δ of connected subsets of $\Omega \cap \overline{\mathbb{R}}$ the endpoints of which are of definite type with respect to A. We denote this system of sets by $\mathcal{R}_\Omega(A)$. The spectral points of definite type of A can be characterized with the help of E: a point $\lambda \in \mathbb{R}$ is a spectral point of positive (negative) type of A if and only if for some open $\Delta \in \mathcal{R}_\Omega(A)$ with $\lambda \in \Delta$ the space $(E(\Delta)\mathcal{K}, [\cdot\,,\cdot])$ (resp. $(E(\Delta)\mathcal{K}, -[\cdot\,,\cdot])$) is a Hilbert space (cf. [8, Theorem 2.15]).

In analogy to definitizable operators we say that a point $\lambda \in \Omega \cap \sigma(A) \cap \mathbb{R}$ (or $\lambda = \infty \in \Omega$ if A is unbounded) is a *critical point* of A if it is not a spectral point of definite type of A. This is obviously equivalent to the fact that for any $\Delta \in \mathcal{R}_\Omega(A)$ with $\lambda \in \Delta$ the space $(E(\Delta)\mathcal{K}, [\cdot\,,\cdot])$ is indefinite. The set of critical points of A in Ω will be denoted by $c_\Omega(A)$.

The critical point λ of A is called *regular* if there exists $c > 0$ such that for some (and hence for any) $\Delta_0 \in \mathcal{R}_\Omega(A)$ with $\Delta_0 \cap c_\Omega(A) = \{\lambda\}$ we have $\|E(\Delta)\| \le c$ for all $\Delta \in \mathcal{R}_\Omega(A), \Delta \subset \Delta_0$. If λ is not regular it is called a *singular* critical point. If for any $\Delta \in \mathcal{R}_\Omega(A)$ with $\lambda \in \Delta$ the space $(E(\Delta)\mathcal{K}, [\cdot\,,\cdot])$ is not a Pontryagin space, then λ is called an *essential* critical point of A. If $\infty \in \Omega$ is a critical point then it is always essential. In [2, Theorem 26] it was shown that a critical point $\lambda \ne \infty$ is essential if and only if it is neither of type π_+ nor of type π_-.

The statements (i), (ii), (iv) and (v) of the following theorem were proved in [15] for the case that $T^{[*]}T$ and $TT^{[*]}$ are definitizable. It turns out that these results also hold when $T^{[*]}T$ and $TT^{[*]}$ are only definitizable over some domain Ω.

Theorem 3.10. *Assume that* (t2) *holds and let* $T^{[*]}T$ (*and hence also* $TT^{[*]}$) *be definitizable over some domain* Ω *as in Definition* 3.8. *Then for* $\lambda \in \mathbb{R}\backslash\{0\}$ *the following statements hold:*

- (i) λ *is a critical point of* $T^{[*]}T$ *if and only if it is a critical point of* $TT^{[*]}$*;*
- (ii) λ *is a regular critical point of* $T^{[*]}T$ *if and only if it is a regular critical point of* $TT^{[*]}$*;*
- (iii) λ *is an essential critical point of* $T^{[*]}T$ *if and only if it is an essential critical point of* $TT^{[*]}$*;*

Moreover,

- (iv) *if zero is a singular critical point of* $T^{[*]}T$ *then zero belongs to* $\sigma(TT^{[*]})$*;*
- (v) *if infinity is a critical point of* $T^{[*]}T$ *then infinity is of definite type with respect to* $TT^{[*]}$.

Proof. The assertions (i), (iii) and (v) are immediate consequences of Proposition 3.7. The proof of (ii) follows analogous lines as the proof of Theorem 4.2(iii) in [15], with the use of the local spectral function instead of the spectral function of a definitizable operator. We leave the details to the reader. Point (iv) results from the equality of non-zero spectra of $T^{[*]}T$ and $TT^{[*]}$ and from the fact that an isolated point of the spectrum cannot be a singular critical point. □

We conclude this paper with two examples. In the first one the operator $T^{[*]}T$ is easily seen to be locally definitizable, while $TT^{[*]}$ has a relatively complicated form.

Example 3. *Let* $(\mathcal{K}_0, [\cdot\,,\cdot])$ *be an infinite–dimensional Krein space and let* $\mathcal{K} = \mathcal{K}_0 \times \mathcal{K}_0$ *with the standard product indefinite metric. Let* $T_0 \in L(\mathcal{K}_0)$ *be such that* $T_0^{[*]}T_0$ *is locally definitizable over some set* Ω *(see e.g. Example 1) and let* $T_1 \in L(\mathcal{K}_0)$ *be an operator having a neutral range, which is equivalent to* $T_1^{[*]}T_1 = 0$. *Consider the operator*

$$T = \begin{pmatrix} T_0 & 0 \\ T_1 & 0 \end{pmatrix} \in L(\mathcal{K}).$$

Then

$$T^{[*]}T = \begin{pmatrix} T_0^{[*]}T_0 & 0 \\ 0 & 0 \end{pmatrix} \in L(\mathcal{K}),$$

and it is clearly locally definitizable over Ω. *Although the operator* $TT^{[*]}$ *has a more complicated form, namely*

$$TT^{[*]} = \begin{pmatrix} T_0T_0^{[*]} & T_0T_1^{[*]} \\ T_1T_0^{[*]} & T_1T_1^{[*]} \end{pmatrix} \in L(\mathcal{K}),$$

we know by Theorem 3.9 that it is locally definitizable over Ω *as well.*

In the following example we apply our results to a Sturm–Liouville operator with a $\mathcal{PT}$-symmetric potential.

Example 4. The notation below is taken from Example 2, but this time the indefinite inner product on L^2 given by the fundamental symmetry

$$\tilde{J}f := f(-x), \quad f \in L^2$$

(here and later on we will write for brevity $g(-x)$ instead of $g \circ \phi$, where $\phi(x) = -x$). Let T be the differential operator

$$\operatorname{dom} T = H^1(-1,1), \quad Tf = f' + qf, \ f \in \operatorname{dom} T,$$

where the complex-valued function q belongs to $\mathcal{C}^1([-1,1])$. Further conditions on q will be given later on. Then

$$\operatorname{dom} T^{[*]} = \{f \in \operatorname{dom} T : f(-1) = f(1) = 0\}$$

and

$$T^{[*]} f = f' + \overline{q}(-x) f, \ f \in \operatorname{dom} T^{[*]}.$$

Consequently, the operator $TT^{[*]}$ is a differential operator with Dirichlet boundary conditions:

$$\operatorname{dom} TT^{[*]} = \{f \in L^2 : f, f' \in AC, \ f'' \in L^2, \ f(-1) = f(1) = 0\},$$

and

$$TT^{[*]} f = f'' + (\overline{q}(-x) + q) f' + (-\overline{q}'(-x) + q \cdot \overline{q}(-x)) f, \quad f \in \operatorname{dom} TT^{[*]}.$$

On the other hand

$$\operatorname{dom} T^{[*]} T = \{f \in L^2 : f, f' \in AC, \ f'' \in L^2, \ f'(\pm 1) + q(\pm 1) f(\pm 1) = 0\},$$
$$T^{[*]} T f = f'' + (\overline{q}(-x) + q) f' + (q' + q \cdot \overline{q}(-x)) f, \quad f \in \operatorname{dom} T^{[*]} T.$$

Let us now consider functions q which satisfy the condition

$$\overline{q(-x)} = -q(x), \quad x \in [-1,1].$$

In this case

$$TT^{[*]} f = f'' + (-q' - q^2) f.$$

and

$$T^{[*]} T f = f'' + (q' - q^2) f.$$

Moreover, the potentials $V_1 := -q' - q^2$ and $V_2 := q' - q^2$ are $\mathcal{PT}$-symmetric, i.e. for $i = 1, 2$ we have

$$\overline{V_i(-x)} = V_i(x), \quad x \in [-1,1].$$

Hence, the operator $-TT^{[*]}$ belongs to the family of operators considered in [12], Section 4 (cf. [13] as well). It was shown therein that the spectrum of $-TT^{[*]}$ consists of isolated eigenvalues and that—with possible exception of a finite number of points—these eigenvalues form a real sequence which accumulates to $+\infty$ and alternates between eigenvalues of positive and negative type. In particular, $TT^{[*]}$ is definitizable over $\mathbb{C}$. For simplicity let us assume that

$$q(-1), q(1) \in \mathbb{R}.$$

Then the operator

$$Nf = f'', \qquad \operatorname{dom} N = \operatorname{dom} T^{[*]}T.$$

is selfadjoint in the Hilbert space L^2 and hence the resolvent set of the bounded perturbation $T^{[*]}T$ of N is nonempty. By Theorem 3.9 the operator $T^{[*]}T$ is definitizable over $\mathbb{C}$, the nonzero spectra of $T^{[*]}T$ and $TT^{[*]}$ coincide by Theorem 3.2 and the sign types of the eigenvalues of definite type of $T^{[*]}T$ in $(-\infty, 0)$ are opposite to those of $TT^{[*]}$ according to Proposition 3.7.

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