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# Harsanyi power solutions for graph-restricted games

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**Abstract** We consider *cooperative transferable utility games*, or simply TU-games, with limited communication structure in which players can cooperate if and only if they are connected in the communication graph. Solutions for such *graph games* can be obtained by applying standard solutions to a modified or restricted game that takes account of the cooperation restrictions. We discuss *Harsanyi solutions* which distribute *dividends* such that the dividend shares of players in a coalition are based on *power measures* for nodes in corresponding communication graphs. We provide axiomatic characterizations of the Harsanyi power solutions on the class of cycle-free graph games and on the class of all graph games. Special attention is given to the Harsanyi degree solution which equals the *Shapley value* on the class of complete graph games and equals the *position value* on the class of cycle-free graph games. The *Myerson value* is the Harsanyi power solution that is based on the equal power measure. Finally, various applications are discussed.

**Keywords** Cooperative TU-game · Harsanyi dividend · Communication structure · Power measure · Position value · Myerson value

**JEL codes** C71

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## 1 Introduction

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair $(N, v)$, where $N \subset \mathbb{N}$ is a finite set of players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on $N$ such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the worth of coalition $S$, i.e., the members of coalition $S$ can obtain a total payoff of $v(S)$ by agreeing to cooperate. Unless stated otherwise, we assume that $N = \{1, \ldots, n\}$ and we denote a TU-game $(N, v)$ shortly by its characteristic function $v$.

A *payoff vector* $x \in \mathbb{R}^n$ of a game $v$ is an $n$-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A payoff vector $x$ is *efficient* if it exactly distributes the worth $v(N)$ of the 'grand coalition' $N$, i.e., $\sum_{i \in N} x_i = v(N)$. A (single-valued) *solution* for TU-games is a function $f$ that assigns to every game $v$ a payoff vector $f(v) \in \mathbb{R}^n$. A solution $f$ is efficient if $f(v)$ is efficient for any game $v$.

In its classical interpretation, a TU-game describes a situation in which the players in every coalition $S$ of $N$ can cooperate to form a feasible coalition and earn the worth $v(S)$. However, one can add certain restrictions on cooperation. One of the best-known restrictions are the games with *limited communication structure* in which the members of some coalition $S$ can realize the worth $v(S)$ if and only if they are connected within a given *communication graph* on the set of players. These *graph-restricted* games were first studied in Myerson (1977). Solutions for graph-restricted games usually correspond to modified classical solutions for cooperative games.

In this paper we introduce *Harsanyi power solutions* for (communication) graph restricted games which are based on *Harsanyi solutions* for TU-games. These Harsanyi solutions are proposed as solutions for TU-games in Vasil'ev (1982, 2003) (see also Derks et al. 2000, where a Harsanyi solution is called a *sharing value*). The idea behind a Harsanyi solution is that it distributes the Harsanyi *dividends* (see Harsanyi 1959) over the players in the corresponding coalitions according to a chosen *sharing system* which assigns to every coalition $S$ a sharing vector specifying for every player in $S$ its share in the dividend of $S$. The payoff to each player $i$ is thus equal to the sum of its shares in the dividends of all coalitions of which he is a member. A famous Harsanyi solution is the *Shapley value* (1953) which distributes the dividend of each coalition equally among the players in that coalition. The novelty of the Harsanyi power solutions for graph restricted games is that we associate sharing systems with some *power measure* for the underlying communication graph, this measure being a function which assigns a nonnegative real number to every node in the graph. These numbers represent the strength or power of those nodes in the graph. Given a power measure we define the corresponding sharing system such that the share vectors for every coalition are proportional to the power measure of the corresponding subgraph.

Out of a large variety of possible power measures (and corresponding power solutions), we give special attention to the *degree measure* that assigns to every player in a communication graph the number of players with whom he is directly connected. We show that on the class of cycle-free graph games, the corresponding Harsanyi power solution is equal to the position value, introduced in Borm et al. (1992), while it equals the *Shapley value* on the class of complete graph games.

Applying the *equal power measure*, which assigns equal power to all players, we obtain the Myerson value as introduced in Myerson (1977) as the corresponding Harsanyi power solution. After weakening some of the axioms used in Borm et al. (1992) to characterize the position- and Myerson value on the class of cycle-free graph games, we generalize these axioms to obtain axiomatic characterizations of all Harsanyi power solutions. It turns out that one of the axioms is not satisfied on the class of all graph games. By replacing this axiom by two invariance axioms, we also obtain an axiomatic characterization on the class of all graph games. Finally we discuss some applications, in particular assignment games, ATM-games and auction games.

The underlying paper is organized as follows. Section 2 is a preliminary section defining cooperative TU-games, notions in graph theory, power measures on graphs and solutions for graph games. In Sect. 3 we introduce the concept of Harsanyi power solution and discuss several of its properties. In Sect. 4 we give axiomatic characterizations of the Harsanyi power solutions. In Sect. 5 we discuss applications, while Sect. 6 concludes. The appendix shows the logical independence of the axioms.

## 2 Preliminaries

### 2.1 Cooperative TU-games

A game $v$ is *monotone* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$. It is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$. Throughout the paper we assume that any game is *zero-normalized*, i.e., $v(\{i\}) = 0$ for all $i \in N$. We denote the collection of all zero-normalized games on player set $N$ by $\mathcal{G}^N$, and the collection of all non-empty subsets of $N$ by $\Omega^N$. A special class of monotone and convex games are unanimity games. For each $T \subseteq N$, the *unanimity* game $u^T$ is given by $u^T(S) = 1$ if $T \subseteq S$, and $u^T(S) = 0$ otherwise. It is well-known that the unanimity games $u^T$, $T \in \Omega^N$, form a basis for $\mathcal{G}^N$ and that for each game $v \in \mathcal{G}^N$ we have that $v = \sum_{T \in \Omega^N} \Delta_v(T) u^T$, where the coefficients $\Delta_v(T)$ are the *Harsanyi dividends*, see Harsanyi (1959). By applying the Möbius transformation we obtain that

$$\Delta_v(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T), \quad S \in \Omega^N.$$

Observe that for every $i$, $\Delta_v(\{i\}) = v(\{i\})$ and thus $\Delta_v(\{i\}) = 0$, $v \in \mathcal{G}^N$, because every $v$ in $\mathcal{G}^N$ is zero-normalized.

In this paper we consider so-called *Harsanyi solutions*, proposed by Vasil'ev (1982) (see also Vasil'ev 2003) and applied recently by van den Brink et al. (2007) to the class of line-graph games. First, a *sharing system* on $N$ is a system $p = (p^S)_{S \in \Omega^N}$, where $p^S$ is an $|S|$-dimensional vector assigning a nonnegative share $p_i^S \geq 0$ to every player $i \in S$ with $\sum_{j \in S} p_j^S = 1$, $S \in \Omega^N$. We denote the collection of all sharing systems on $N$ by $P^N$. For a game $v$ and sharing system $p \in P^N$, the corresponding *Harsanyi payoff vector* is the payoff vector $h^p(v) \in \mathbb{R}^n$ given by

$$h_i^p(v) = \sum_{S \in \Omega^N, i \in S} p_i^S \Delta_v(S),\ \ i \in N,$$

i.e., the payoff $h_i^p(v)$ to player $i \in N$ is the sum over all coalitions $S \in \Omega^N$, containing $i$, of the share $p_i^S \Delta_v(S)$ of player $i$ in the Harsanyi dividend of coalition $S$. A *Harsanyi solution* on $\mathcal{G}^N$ assigns for a given sharing system $p \in P^N$ the Harsanyi payoff vector $h^p(v)$ to each game $v$. Due to the equality $v(N) = \sum_{S \in \Omega^N} \Delta_v(S)$, we have $\sum_{i \in N} h_i^p(v) = v(N)$, and thus each Harsanyi solution is efficient. The Shapley value is the Harsanyi solution that assigns to any game $v$ the Harsanyi payoff vector $h^p(v)$ with the sharing system $p$ given by $p_i^S = \frac{1}{|S|}$, $S \in \Omega^N$, $i \in S$. Throughout the paper we denote the Shapley value of game $v$ by $Sh(v)$.

## 2.2 Notions in graph theory

An *undirected graph* is a pair $(N, L)$ where $N$ is the set of *nodes*[1] and $L \subseteq \{\{i, j\} | i, j \in N, i \neq j\}$ is a collection of edges or *links*. For $\{i, j\} \in L$, the nodes $i$ and $j$ are said to be *adjacent* (neighboring) to each other and *incident* to link $\{i, j\}$. The set of nodes adjacent to $i$ in graph $(N, L)$ is called the *neighborhood* of $i$, and it is denoted by $R_{(N,L)}(i) = \{j \in N \backslash \{i\} \mid \{i, j\} \in L\}$. The number of nodes in $R_{(N,L)}(i)$ is the *degree* of node $i$ in $(N, L)$. For graph $(N, L)$ the degree vector $d(N, L) \in \mathbb{R}^n$ is given by $d_i(N, L) = |R_{(N,L)}(i)|, i \in N$. The set $D(N, L) = \{i \in N \mid R_{(N,L)}(i) \neq \emptyset\}$ is the set of *non-isolated* nodes in $(N, L)$. The *complete graph* on $N$ is the graph $(N, L^c)$ with $L^c = \{\{i, j\} \mid i, j \in N,\ i \neq j\}$. A sequence of $k$ different nodes $(i_1, \ldots, i_k)$ is a *path* in $(N, L)$ if $\{i_h, i_{h+1}\} \in L$ for $h = 1, \ldots, k-1$. A sequence of nodes $(i_1, \ldots, i_{k+1})$ is a *cycle* in $(N, L)$ if $k \geq 3$, $(i_1, \ldots, i_k)$ is a path, $\{i_k, i_{k+1}\} \in L$ and $i_{k+1} = i_1$. A graph $(N, L)$ is *cycle-free* when it does not contain any cycle. We denote the class of all possible sets of links $L$ on $N$ by $\mathcal{L}^N$. The subclass of sets $L$ such that $(N, L)$ is cycle-free is denoted by $\mathcal{L}_{CF}^N$.

Two nodes $i, j \in N$ are *connected* in graph $(N, L)$ if there exists a path $(i_1, \ldots, i_k)$ with $i_1 = i$ and $i_k = j$. A graph $(N, L)$ is *connected* if any two nodes $i, j \in N$ are connected. For some $K \subseteq N$, the graph $(K, L(K))$ with $L(K)=\{l \in L | l \subseteq K\}$ is called a *subgraph* of $(N, L)$. The notions of degree and neighborhood are straightforwardly extended to subgraphs. For given graph $(N, L)$, a set of nodes $K$ is a *connected subset* of $N$ when the subgraph $(K, L(K))$ is connected. A subset $K$ of $N$ is a *component* in $(N, L)$ if the subgraph $(K, L(K))$ is maximally connected, i.e., $(K, L(K))$ is connected and for any $j \in N \backslash K$, the subgraph $(K \cup \{j\}, L(K \cup \{j\}))$ is not connected. Clearly, for any graph $(N, L)$, the collection of components forms a unique partition of $N$. For $K \subseteq N$, we denote by $C(K)$ the set of all components in the subgraph $(K, L(K))$. Notice that a component of the subgraph $(K, L(K))$ need not be a component of $(N, L)$. For more notions on communication graphs and general graphs we refer to, respectively, van den Nouweland (1993) and Harary (1969).

[1] Since in this paper the nodes in a graph represent the players in a game we use the same notation for the set of nodes as the set of players.

### 2.3 Power measures

A *power measure* is a function $\sigma$ that assigns to any graph $(S, L)$, $S \subseteq N$, a nonnegative vector $\sigma(S, L) \in \mathbb{R}_+^{|S|}$, yielding the nonnegative power $\sigma_i(S, L)$ of node $i \in S$ in the graph $(S, L)$. A power measure is *symmetric* if for any graph $(S, L)$, $S \subseteq N$, and $i, j \in S$ such that $R_{(S,L)}(i)\backslash\{j\} = R_{(S,L)}(j)\backslash\{i\}$, we have $\sigma_i(S, L) = \sigma_j(S, L)$. It is *positive* if for any $(S, L)$, the power of node $i$ is positive if and only if $i$ is non-isolated (and thus node $i$ has zero power if it is isolated). Throughout this paper we only consider *symmetric* and *positive* power measures. Some examples of such power measures are the following.[2]

1. A straightforward power measure is the degree measure $d$, which assigns to any graph $(S, L)$, $S \in \Omega^N$, the degree vector $d(S, L)$.
2. The $\beta - measure$ is given by $\beta_i(S, L) = \sum_{j \in R_{(S,L)}(i)} \frac{1}{|R_{(S,L)}(j)|}$, $i \in S$, $S \in \Omega^N$, i.e., any node $i$ receives an amount $\frac{1}{|R_{(S,L)}(j)|}$ of power from every neighbour $j$ (if any).[3]
3. The *positional power measure* is given by $\delta_i(S, L(S)) = |R_{(S,L)}(i)| + \frac{1}{|S|} \sum_{j \in R_{(S,L)}(i)} \delta_j(S, L)$, $i \in S$, $S \in \Omega^N$, i.e., the positional power of node $i$ is equal to the number of neighbours of $i$ plus a fraction $\frac{1}{|S|}$ of the total power of its neighbours.[4]
4. The *equal power measure* is given by $\gamma_i(S, L) = 1$ if $i \in D(S, L)$ and $\gamma_i(S, L) = 0$, otherwise, $S \in \Omega^N$.[5]

### 2.4 Graph games and solutions

We assume that there is a communication structure on the player set $N$ represented by an undirected graph $(N, L)$, with $N$ as the set of nodes and $L$ as the set of (binary communication) links on $N$. A game $v$ with *communication graph* $(N, L)$ is shortly denoted by $(v, L)$ and referred to as a *graph game*. A (single-valued) solution for graph games is a function $f$ that assigns a payoff vector $f(v, L) \in \mathbb{R}^n$ to any graph game $(v, L)$.

In the graph game $(v, L)$ a coalition $S \in \Omega^N$ can realize its worth $v(S)$ if and only if $(S, L(S))$ is a connected subgraph of $(N, L)$. Whenever this is not the case, players in $S$ can only realize the sum of the worths of the components of $(S, L(S))$.

[2] Other examples are the so-called centrality measures as considered in e.g. Gómez et al. (2003) and Monsuur and Storcken (2002).

[3] This measure is introduced in van den Brink and Gilles (2000) for directed graphs, and applied to undirected graphs in van den Brink et al. (2008).

[4] Despite its name, the positional power measure is not related to the position value. This measure is introduced in Herings et al. (2005) for directed graphs. For $S \in \Omega^N$, this power measure requires to solve an $|S|$-dimensional system of equations. In Herings et al. (2005) it is shown that the measure is well-defined, i.e., the system of equations has a unique solution with positive numbers for non-isolated nodes and zero power for isolated nodes.

[5] The more common definition is $\overline{\gamma}_i(S, L)=1$, $i \in S$, $S \in \Omega^N$, but this measure is not positive.

As introduced by Myerson (1977), this yields the *restricted game* $v^L$ given by

$$v^L(S) = \sum_{T \in C(S)} v(T), \quad S \subseteq N.$$

It is well-known that in the restricted game any unconnected coalition has zero dividend, see for instance Owen (1986), Hamiache (1999) or Bilbao (1998). Borm et al. (1992) refer to the restricted game $v^L$ as the *point game* corresponding to $(v, L)$. They also introduce the *link game* $(L, r^L)$. Its set of players is the set of links $L$ and its characteristic function is given by

$$r^L(E) = v^E(N) \quad \text{for all } E \subseteq L,$$

i.e., for every subset $E \subseteq L$ of links, the worth of $E$ is what the 'grand coalition' $N$ earns in game $(N, v)$ when $E$ is the set of communication links.[6] Two well-known solutions for graph games are the Myerson value and the position value. The *Myerson value* (Myerson 1977) of $(v, L)$, denoted by $\mu(v, L)$, is given by $\mu(v, L) = Sh(v^L)$, i.e., the Myerson value assigns to every graph game $(v, L)$ the Shapley value of the restricted game $v^L$. The *position value* (see Borm et al. 1992) of $(v, L)$, denoted by $\pi(v, L)$, is given by $\pi_i(v, L) = \sum_{l \in L_i} \frac{1}{2} Sh_l(L, r^L)$, $i \in N$, where $L_i = \{l \in L \mid i \in l\}$. So, first, the Shapley value of the link game $(L, r^L)$ is determined and then the Shapley value of each link is distributed equally among the nodes incident to it.

For ease of notation, in the sequel we denote for some $E \subseteq L$ and $l \in L$, the sets $E \cup \{l\}$ and $E \backslash \{l\}$ by $E \cup l$ and $E \backslash l$, respectively. In Borm et al. (1992) a characterization is given for both the Myerson- and the position value on the class of cycle-free graph games. To give the axioms we need the following three notions. First, link $l \in L$ is *superfluous* in $(v, L)$ if $v^E(N) = v^{E \cup l}(N)$ for all $E \subseteq L$. Second, graph game $(v, L)$ is *link anonymous* if there exists a function $g^L \colon \{0, 1, \ldots, |L|\} \to \mathbb{R}$ such that $r^L(E) = g^L(|E|)$ for all $E \subseteq L$. Graph game $(v, L)$ is *link unanimous* if $r^L = v^L(N) u^L$, i.e., $(v, L)$ is link anonymous with $g^L(|L|) = v^L(N)$ and $g^L(k) = 0$ for $k < |L|$. Third, graph game $(v, L)$ is *point anonymous* if there exists a function $g^P \colon \{1, \ldots, |D(N, L)|\} \to \mathbb{R}$ such that $v^L(S) = g^P(|S \cap D(N, L)|)$, $S \subseteq N$. Graph game $(v, L)$ is *point unanimous* if $v^L = v^L(N) u^{D(N,L)}$, i.e., it is point anonymous with $g^P(|D(N, L)|) = v^L(N)$ and $g^P(k) = 0$ for $k < |D(N, L)|$. We now state the following five axioms for a solution $f$ on the class of graph games on $N$.

**Component efficiency** For every graph game $(v, L)$ and every component $S$ in $(N, L)$, we have $\sum_{i \in S} f_i(v, L) = v(S)$.
**Additivity** For every pair of graph games $(v, L)$, $(w, L)$, we have[7] $f(v + w, L) = f(v, L) + f(w, L)$.
**Superfluous link property** If $l \in L$ is a superfluous link in graph game $(v, L)$, then $f(v, L) = f(v, L \backslash l)$.

[6] In Borm et al. (1992) this game is called the arc game. Here we follow Slikker (2005) and call this game the link game.

[7] For two games $v, w \in \mathcal{G}^N$ the sum game is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$.

**Degree measure property** If graph game $(v, L)$ is link unanimous, then there is $\alpha \in \mathbb{R}$ such that $f(v, L) = \alpha d(N, L)$.
**Communication ability property** If graph game $(v, L)$ is point unanimous, then there is $\alpha \in \mathbb{R}$ such that $f_i(v, L) = \alpha$ for all $i \in D(N, L)$, and $f_i(v, L) = 0$ for every $i \in N \backslash D(N, L)$.

For the proof of the following results, we refer to Borm et al. (1992).

**Proposition 2.1**

(i) *The position value satisfies component efficiency, additivity, the superfluous link property and the degree measure property on the class of all graph games. Moreover, on the class of cycle-free graph games it is the unique solution satisfying these four properties.*
(ii) *The Myerson value satisfies component efficiency, additivity, the superfluous link property and the communication ability property on the class of all graph games. Moreover, on the class of cycle-free graph games it is the unique solution satisfying these four properties.*

Observe that the position value does not satisfy the communication ability property and that the Myerson value does not satisfy the degree measure property. It should be noticed that the proofs given in Borm et al. (1992) use stronger versions of the degree measure property and the communication ability property by requiring that these properties hold for any link anonymous, respectively point anonymous graph game. These stronger properties hold on the class of all graph games, but the weaker properties as defined above are enough to prove the characterization statements of Proposition 2.1 on the class of cycle-free graph games.

## 3 Harsanyi power solutions

Given a positive power measure $\sigma$ we define the corresponding *Harsanyi power solution*, denoted by $\varphi^{\sigma}$, on the class of all graph games on player set $N$ by

$$\varphi^{\sigma}(v, L) = h^{p(\sigma)}(v^L)$$

with sharing system $p(\sigma) = (p^S(\sigma))_{S \in \Omega^N}$ given by

$$p_i^S(\sigma) = \frac{\sigma_i(S, L(S))}{\sum_{j \in S} \sigma_j(S, L(S))}, \text{ for all } i \in S \text{ whenever } \sum_{j \in S} \sigma_j(S, L(S)) \neq 0,$$

and $p_i^S(\sigma) = \frac{1}{|S|}$ for all $i \in S$ whenever $\sum_{j \in S} \sigma_j(S, L(S)) = 0$. So, the Harsanyi power solution assigns to each graph game $(v, L)$ the Harsanyi solution of the corresponding restricted game $v^L$ with the sharing system $p(\sigma)$ determined by the power measure $\sigma$ such that the distribution of any dividend $\Delta_{v^L}(S)$ of coalition $S$ in the restricted game $v^L$ is proportional to the powers of the players in the subgraph $(S, L(S))$. Observe that the shares do not matter when all powers are zero, because that can only happen when all players of $S$ are isolated in $(S, L(S))$, and thus the dividend of $S$ in $v^L$ is equal to zero. We call the Harsanyi power solution the *Harsanyi degree solution*, denoted by $\varphi^d$, if we take the degree measure $d$ to distribute the dividends.

*Example 3.1* Consider the graph game $(v, L)$ on $N = \{1, 2, 3, 4\}$ given by $v = u^{\{1,2,3\}}$ and $L = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}\}$. The payoffs assigned to this graph game by the position value, the Myerson value and the Harsanyi degree solution, respectively, are

$$\pi(v, L) = \frac{1}{24}(11, 4, 7, 2), \ \ \mu(v, L) = \frac{1}{3}(1, 1, 1, 0) \quad \text{and } \varphi^d(v, L) = \frac{1}{4}(2, 1, 1, 0).$$

This graph game is taken from Borm et al. (1992), Example 6.1, who use it to motivate the position value compared to the Myerson value. The position value and the Harsanyi degree solution have in common that they reward player 1 for having a 'central' position in the communication graph by giving player 1 a higher payoff than players 2 and 3. The Myerson value assigns the players in the unanimity coalition $\{1, 2, 3\}$ equal payoffs. The Harsanyi degree solution and the Myerson value have in common that they both assign zero payoff to player 4. Observe that 4 is a null player in the game $v$ and is not needed to connect the players in the unanimity coalition $\{1, 2, 3\}$. However, the links $\{1, 4\}$ and $\{2, 4\}$ are not superfluous, which is why player 4 gets a positive payoff in the position value. □

Clearly, when we take the equal power measure $\gamma$, the corresponding Harsanyi power solution distributes all (nonzero) dividends of the restricted game $v^L$ equally among the players of the coalitions.

**Corollary 3.2** *For every graph game $(v, L)$ on $N$, $\varphi^\gamma(v, L) = \mu(v, L)$.*

In case the power measure is symmetric, the corresponding Harsanyi power solution $\varphi^\sigma$ extends the Shapley value to the class of graph games in the sense that it yields the Shapley value of game $v$ whenever the graph is complete.

**Proposition 3.3** *For a symmetric positive power measure $\sigma$, $\varphi^\sigma(v, L^c) = Sh(v)$ for all $v \in \mathcal{G}^N$.*

*Proof* If $\sigma$ is symmetric then $\frac{\sigma_i(S, L^c(S))}{\sum_{j \in S} \sigma_j(S, L^c(S))} = \frac{1}{|S|}$ for all $i \in S \subseteq N$. Moreover, $\Delta_v(S) = \Delta_{v^{L^c}}(S)$ for all $S \in \Omega^N$. Thus $\varphi_i^\sigma(v, L^c) = \sum_{\{S \subseteq N | i \in S\}} \frac{\sigma_i(S, L^c(S))}{\sum_{j \in S} \sigma_j(S, L^c(S))} \Delta_{v^{L^c}}(S) = \sum_{\{S \subseteq N | i \in S\}} \frac{1}{|S|} \Delta_v(S) = Sh_i(v)$ for all $i \in N$. □

In the remainder of this section we state several properties of Harsanyi power solutions. To do so, we first generalize the communication ability property to the class of positive power measures.

**$\sigma$-Communication ability property** If $(v, L)$ is point unanimous, then there is $\alpha \in \mathbb{R}$ such that $f(v, L) = \alpha\sigma(N, L)$.

Observe that the $\sigma$-communication ability property reduces to the communication ability property when we take the equal power measure $\sigma = \gamma$. We now have the following proposition.

**Proposition 3.4** *For a positive power measure $\sigma$, the Harsanyi power solution $\varphi^\sigma$ satisfies component efficiency, additivity and the $\sigma$-communication ability property on the class of all graph games.*

*Proof* Since $\Delta_{v^L}(S) = 0$ if $S$ is not connected in $(N, L)$, for every component $T \in C(N)$ in $(N, L)$ we have

$$\begin{aligned}\sum_{i\in T}\varphi_i^\sigma(v, L) &= \sum_{i\in T} h_i^{p(\sigma)}(N, v^L) = \sum_{i\in T}\sum_{S\subseteq N, i\in S} p_i^S(\sigma)\Delta_{v^L}(S)\\ &= \sum_{i\in T}\sum_{S\subseteq T, i\in S} p_i^S(\sigma)\Delta_{v^L}(S) = \sum_{S\subseteq T}\sum_{i\in S} p_i^S(\sigma)\Delta_{v^L}(S)\\ &= \sum_{S\subseteq T}\Delta_{v^L}(S) = v^L(T),\end{aligned}$$

showing that $\varphi^\sigma$ satisfies component efficiency. For all $v, w \in \mathcal{G}^N$ and all $L \in \mathcal{L}^N$,

$$(v+w)^L(S) = \sum_{T\in C(S)}(v+w)(T) = \sum_{T\in C(S)}(v(T)+w(T)) = v^L(S) + w^L(S)$$

and thus $\Delta_{(v+w)^L}(S) = \Delta_{v^L}(S) + \Delta_{w^L}(S)$ for all $S \subseteq N$. Then

$$\begin{aligned}\varphi_i^\sigma(N, v+w, L) &= \sum_{S\subseteq N, i\in S} p_i^S(\sigma)\Delta_{(v+w)^L}(S)\\ &= \sum_{S\subseteq N, i\in S} p_i^S(\sigma)(\Delta_{v^L}(S) + \Delta_{w^L}(S)) = \varphi_i^\sigma(v, L) + \varphi_i^\sigma(w, L),\end{aligned}$$

showing that $\varphi^\sigma$ satisfies additivity. Finally, when $(v, L)$ is point unanimous, then $v^L = v^L(N)u^{D(N,L)}$ and $\varphi^\sigma$ is obtained by distributing the unique nonzero dividend $\Delta_{v^L}(D(N, L))$ among the players in $D(N, L)$ according to the $\sigma$-measure, showing that $\varphi^\sigma$ satisfies the $\sigma$-communication ability property. □

Next we generalize the degree measure property.

**$\sigma$-Measure property** If $(v, L)$ is link unanimous, then there is $\alpha \in \mathbb{R}$ such that $f(v, L) = \alpha\sigma(N, L)$.

Observe that the $\sigma$-measure property yields the degree measure property when we take $\sigma = d$. The next lemma implies that the $\sigma$-measure property and the $\sigma$-communication ability property are related.

**Lemma 3.5**

(i) *Let $(v, L)$ be a graph game. If $(v, L)$ is link unanimous, then $(v, L)$ is also point unanimous.*
(ii) *Let $(v, L)$ be a cycle-free graph game. Then $(v, L)$ is link unanimous if and only if $(v, L)$ is point unanimous.*

*Proof* (i) If $D(N, L) = \emptyset$, then $L = \emptyset$ implying that $v^L(S) = 0$ for all $S \subseteq N$, and thus (i) holds. Next, suppose $D(N, L) \neq \emptyset$. By definition of link unanimity, $r^L = v^L(N)u^L$ and thus $v^{L\backslash l}(N) = r^L(L\backslash l) = 0$ for all $l \in L$. For $S \subseteq N$, we distinguish two cases.

1. For $S$ such that $D(N, L) \not\subseteq S$, denote by $C'(N)$ the set of components in $(N, L(S))$. Then

$$v^{L(S)}(N) = \sum_{T \in C'(N)} v(T) = \sum_{T \in C'(S)} v(T) + \sum_{i \in N \backslash S} v(\{i\}) \\ = v^{L(S)}(S) = v^L(S),$$

where the second equality follows from the fact that all nodes outside $S$ are singletons in the set of components in $(N, L(S))$, and the third equality follows from $v$ being zero-normalized. From $r^L = v^L(N)u^L$ it further follows that $v^{L(S)}(N) = 0$ since $L(S)$ is a proper subset of $L$. Hence

$$v^L(S) = v^{L(S)}(N) = 0. \tag{3.1}$$

2. For $S$ such that $D(N, L) \subseteq S$, by definition of $v^L$,

$$v^L(S) = \sum_{T \in C(S)} v(T) = \sum_{T \in C(S), |T|=1} v(T) \\ + \sum_{T \in C(S), |T| \geq 2} v(T) = \sum_{T \in C(S), |T| \geq 2} v(T) \\ = \sum_{T \in C(S), T \subseteq D(N,L)} v(T) = \sum_{T \in C(N), |T| \geq 2} v(T) = v^L(N). \tag{3.2}$$

From Eqs. 3.1 and 3.2 it follows that $v^L = v^L(N)u^{D(N,L)}$, which proves (i).

(ii) The 'only if' part follows from (i). To prove the 'if' part, note that if we delete a link $l$ in a cycle-free graph $(N, L)$, then the set $D(N, L)$ is not connected in $(N, L\backslash l)$. Thus, if $v^L = v^L(N)u^{D(N,L)}$, then $r^L(L) = v^L(N)$ and $r^L(E) = v^E(N) = 0$ for all $E \subset L$, implying that $r^L = v^L(N)u^L$. □

Part (i) of the lemma implies that for general graph games the $\sigma$-communication ability property implies the $\sigma$-measure property. Part (ii) of the lemma implies that on the class of cycle-free graph games the two properties are equivalent. The next corollary follows immediately from Proposition 3.4 and Lemma 3.5.

**Corollary 3.6** *For a positive power measure $\sigma$, the Harsanyi power solution $\varphi^\sigma$ satisfies the $\sigma$-measure property on the class of all graph games.*

On the class of cycle-free graph games we further have the next lemma.

**Lemma 3.7** *For a positive power measure $\sigma$, the Harsanyi power solution $\varphi^\sigma$ satisfies the superfluous link property on the class of cycle-free graph games.*

*Proof* Recall that link $l \in L$ is superfluous if $v^E(N) = v^{E \cup l}(N)$ for all $E \subseteq L$. Since it is assumed that any game is zero-normalized, it follows straightforwardly from the definition of the restricted game that this condition holds if and only if $v^L = v^{L \backslash l}$. Hence, when $l$ is superfluous,

$$\Delta_{v^L}(S) = \Delta_{v^{L \backslash l}}(S), \quad S \in \Omega^N.$$

When $l \not\subseteq S$, then $(L \backslash l)(S) = L(S)$ and so $\sigma_i(S, L(S)) = \sigma_i(S, (L \backslash l)(S))$ for all $i \in S$, implying that the share of $i \in S$ in $\Delta_{v^L}(S)$ is equal to the share of $i$ in $\Delta_{v^{L \backslash l}}(S)$. When $l \subseteq S$, we have that $\Delta_{v^{L \backslash l}}(S) = 0$ because $(N, L)$ is cycle-free. So, $\Delta_{v^L}(S) = \Delta_{v^{L \backslash l}}(S) = 0$ and the shares don't matter. Hence

$$\varphi^\sigma(v, L) = \varphi^\sigma(v, L \backslash l),$$

showing that $\varphi^\sigma$ satisfies the superfluous link property on the class of cycle-free graph games. □

From Proposition 3.4, Corollary 3.6 and Lemma 3.7, it follows that $\varphi^d$ satisfies component efficiency, additivity, the degree measure property and the superfluous link property on the class of cycle-free graph games. From Proposition 2.1.(i) we have that on the class of cycle-free graph games the position value is characterized by these four properties. Therefore, on this class the Harsanyi degree solution is equal to the position value.

**Corollary 3.8** *If $L \in \mathcal{L}^N_{CF}$ then $\varphi^d(v, L) = \pi(v, L)$ for any $v \in \mathcal{G}^N$.*

The corollary shows that to define and compute the position value for cycle-free graph games we do not need to introduce the link game as done in Borm et al. (1992), since it is a Harsanyi solution applied to the restricted game $v^L$. However, for arbitrary graph games the position value is not equal to $\varphi^d$ and the characterization in Borm et al. (1992) does not work either. Clearly, since the degree measure is symmetric, by Proposition 3.3 it follows that $\varphi^d$ yields the Shapley value if $(N, L)$ is the complete graph. It is well-known that the position value does not generalize the Shapley value, i.e., in general $\pi(v, L^c)$ is not equal to $Sh(v)$. In fact, the position value even may give a payoff vector outside the *Harsanyi set* of the corresponding restricted game. The *Harsanyi set* or *Selectope* of a game $v$, independently introduced by Vasil'ev (1978), and Hammer et al. (1977), respectively (see also Vasil'ev and van der Laan 2002 and Derks et al. 2000), is the set $H(v) = \{h^p(v) | p \in P^N\}$ of all Harsanyi payoff vectors of that game.[8]

**Proposition 3.9** *Let $L \in \mathcal{L}^N$. Then $\pi(v, L) \in H(v^L)$ for all $v \in \mathcal{G}^N$ if and only if $L \in \mathcal{L}^N_{CF}$.*

[8] Although a Harsanyi solution selects for any game the payoff vector in the Harsanyi set corresponding to a fixed sharing system, a solution that always selects a payoff vector from the Harsanyi set need not be a Harsanyi solution, since for different games it might need different sharing systems to obtain a Harsanyi payoff vector.

*Proof* The 'if' part follows from Corollary 3.8 and the fact that by definition $\varphi^d(v, L)$ is a Harsanyi payoff vector of the game $v^L$ for any $L \in \mathcal{L}^N$. To prove the 'only if' part, suppose that $L \in \mathcal{L}^N \backslash \mathcal{L}^N_{CF}$. Then $(N, L)$ contains a cycle, say $(i_1, \ldots, i_k, i_{k+1})$, for some $k \geq 3$. Take $v = u^{\{i_1,i_2\}}$ being the unanimity game of two neighboring nodes in the cycle. Since $\{i_1, i_2\} \in L$, we have that $v^L = v = u^{\{i_1,i_2\}}$, and thus all $i \in N\backslash\{i_1, i_2\}$ are null players in $v^L$, i.e., $v^L(S) = v^L(S\backslash\{i\})$ for all $i \in N\backslash\{i_1, i_2\}$. Since null players earn a zero payoff in any Harsanyi payoff vector we have

$$x_i = 0 \quad \text{for all } x \in H(v^L) \quad \text{and} \quad i \in N\backslash\{i_1, i_2\}. \tag{3.3}$$

However, since $(i_1, \ldots, i_{k+1})$ is a cycle we have that $r^L(E) - r^L(E\backslash\{i_2, i_3\}) = 1$ for $E = \{\{i_j, i_{j+1}\} \mid j = 2, \ldots, k\}$. Hence the link $\{i_2, i_3\}$ is not a null player in the link game $(L, r^L)$. Moreover, since $v = u^{\{i_1,i_2\}}$ is monotone, it follows that also $r^L$ is monotone, and thus the Shapley value of the link game satisfies $Sh_{\{i_2,i_3\}}(L, r^L) > 0$ and $Sh_l(L, r^L) \geq 0$ for all $l \in L$. But then $\pi_{i_3}(v, L) \geq \frac{1}{2} Sh_{\{i_2,i_3\}}(L, r^L) > 0$. With (3.3) it then follows that $\pi(v, L) \notin H(v^L)$. □

## 4 Axiomatizations of Harsanyi power solutions

In this section we give axiomatic characterizations of Harsanyi power solutions, first for the subclass of cycle-free graph games and subsequently for general graph games.

### 4.1 Axiomatization on the class of cycle-free graph games

According to statement (i) of Proposition 2.1 and Corollary 3.8, on the class of cycle-free graph games the Harsanyi power solution with $\sigma = d$ is the unique solution satisfying component efficiency, additivity, the superfluous link property and the degree-measure property. The next proposition generalizes this result to general positive power measures.

**Proposition 4.1** *Let $\sigma$ be a positive power measure. Then on the class of cycle-free graph games the Harsanyi power solution $\varphi^\sigma$ is the unique solution satisfying component efficiency, additivity, the superfluous link property and the $\sigma$-measure property.*

*Proof* According to Proposition 3.4, Corollary 3.6 and Lemma 3.7 the Harsanyi power solution $\varphi^\sigma$ satisfies the four properties on the class of cycle-free graph games. Uniqueness follows along similar lines as the uniqueness proof for the position value in Borm et al. (1992). Here we give a brief outline. Consider graph game $(cu^T, L)$, $c \in \mathbb{R}$, $T \in \Omega^N$, with $L$ cycle-free. If there is no component $S$ in $(N, L)$ such that $T \subseteq S$ then all links are superfluous, and the *superfluous link property* implies that $f(cu^T, L) = f(cu^T, L^\emptyset)$ with $L^\emptyset = \emptyset$. *Component efficiency* uniquely determines $f(cu^T, L^\emptyset)$. On the other hand, if there is a component $S$ in $(N, L)$ such that $T \subseteq S$ then all links outside the connected hull $H(T) = \{h \in N \mid$ there exist $i, j \in T$ such that $h$ belongs to the path between $i$ and $j\}$ of $T$ are superfluous, and thus the *superfluous link property* implies that $f(cu^T, L) = f(cu^T, L(H(T)))$. (Note that in a cycle-free graph there is exactly one path between any pair of connected nodes.) Since $(cu^T, L(H(T)))$ is link unanimous, the $\sigma$ *− measure property* and *component efficiency* uniquely determine $f(cu^T, L(H(T)))$. Finally, for arbitrary $v \in \mathcal{G}^N$ uniqueness follows from *additivity*. □

Logical independence of the axioms in Proposition 4.1 is shown in the appendix. In this axiomatization, component efficiency works similarly as efficiency for TU-games, component efficiency with the superfluous link property work similarly as the null player property, and additivity for graph games works similarly as additivity for TU-games. The $\sigma$-measure property replaces the symmetry axiom. Although symmetry seems reasonable for TU-games, it does not for graph-games because of the asymmetric positions of players in the communication graph, even when they are symmetric in the game. The $\sigma$-measure property is the only axiom in Proposition 4.1 where the power measure $\sigma$ appears, and it shows how symmetry can be replaced by an axiom that takes account of the power measures of the players in the communication graph.

Recall from the previous section that the $\sigma$-degree measure property and the $\sigma$-communication ability property are equivalent to each other on the class of cycle-free graph games. So, in Proposition 4.1 we can replace the $\sigma$-measure property by the $\sigma$-communication ability property. This gives the following corollary (the logical independence of the axioms is again shown in the appendix.)

**Corollary 4.2** *Let $\sigma$ be a positive power measure. Then on the class of cycle-free graph games the Harsanyi power solution $\varphi^\sigma$ is the unique solution satisfying component efficiency, additivity, the superfluous link property and the $\sigma$-communication ability property.*

Proposition 4.1 generalizes the characterization of the position value on the class of cycle-free graphs using the degree measure property, see Proposition 2.1.(i), to the characterization of the Harsanyi power solution for any positive power measure. Taking the equal power measure $\gamma$, we then obtain a characterization of the Myerson value using the equal power measure property as an alternative for the characterization of the Myerson value by using the communication ability property, see Proposition 2.1.(ii). On the other hand, on the class of cycle-free graph games by Corollary 4.2 the position value also can be characterized using some $\sigma$-communication ability property, namely the degree-communication ability property. This gives the following corollary, being the counterpart of the two statements of Proposition 2.1.

**Corollary 4.3**
- (i) *On the class of cycle-free graph games, the position value is the unique solution satisfying component efficiency, additivity, the superfluous link property, and the degree-communication ability property.*
- (ii) *On the class of cycle-free graph games, the Myerson value is the unique solution satisfying component efficiency, additivity, the superfluous link property, and the equal power measure property.*

In Borm et al. (1992) the position value is characterized by using the degree measure property and the Myerson value by using the communication ability property. Here we have generalized both characterizations so that they both include characterizations of the position value and the Myerson value, taking the appropriate power measure. So, the view that the difference between the position value and Myerson value (on cycle-free graph games) is about using the degree measure property (which is defined using link unanimous graph games) or the communication ability property (which is defined using point unanimous graph games) has to be reconsidered. Both values satisfy a $\sigma$-measure property and a $\sigma$-communication ability property, but the difference is with respect to which power measure $\sigma$ to use, the degree measure or the equal power measure.

## 4.2 Axiomatization on the class of all graph games

In Proposition 3.4 we saw that on the class of all graph games the Harsanyi power solutions satisfy component efficiency, additivity and the $\sigma$-communication ability property. On the class of cycle-free graph games these solutions also satisfy the superfluous link property, see Lemma 3.7. However, this property does not hold on the class of all graph games (see Example 5.4 on assignment games). To characterize the Harsanyi power solutions on the class of all graph games we replace the superfluous link property by the following two invariance properties.

**Inessential link property** When $T \subseteq N$ is a connected nonempty coalition, $v = cu^T$, $c \in \mathbb{R}$, and $l \in L$ contains a player not in $T$, then $f(v, L) = f(v, L\backslash l)$.
**Connectedness** When $v^L = w^L$, then $f(v, L) = f(w, L)$.

The inessential link property states that when $v$ is a unanimity game on a connected nonempty coalition $T$, then the solution does not depend on links containing at least one player not in $T$. Such links are called *inessential* in that unanimity graph game. The connectedness property states that the solution only depends on the worths of the connected coalitions. The following proposition characterizes the Harsanyi power solutions on the class of all graph games.

**Proposition 4.4** *Let $\sigma$ be a positive power measure. On the class of all graph games, the Harsanyi power solution $\varphi^\sigma$ is the unique solution satisfying component efficiency, additivity, the $\sigma$-communication ability property, the inessential link property and connectedness.*

*Proof* From Proposition 3.4 we know that $\varphi^\sigma$ satisfies the first three properties. Recall that for $L \in \mathcal{L}^N$, $L(T) = \{l \in L | l \subseteq T\}$ is the set of links between players in $T$. To show that $\varphi^\sigma$ satisfies the inessential link property, consider $v = cu^T$ for some nonempty connected $T$, $c \in \mathbb{R}$ and $l \not\subseteq T$. Then, $\Delta_{v^L}(T) = \Delta_{v^{L\backslash l}}(T) = c$ and $\Delta_{v^L}(S) = \Delta_{v^{L\backslash l}}(S) = 0$ for any $S \neq T$. Moreover, for every $l \not\subseteq T$, $(T, L(T)) = (T, (L\backslash l)(T))$, and so $\sigma_i(T, L(T)) = \sigma_i(T, (L\backslash l)(T))$ for all $i \in T$. Hence, for $v = cu^T$ and $l \not\subseteq T$ we have

$$\varphi^\sigma(v, L) = \varphi^\sigma(v, L\backslash l),$$

which proves the inessential link property. The connectedness property follows straightforwardly from the fact that $\Delta_{v^L}(S) = \Delta_{w^L}(S)$ for all $S$ when $v^L = w^L$.

To prove uniqueness, we first consider $v = cu^T$ for some $c \in \mathbb{R}$ and $T \in \Omega^N$. We distinguish two cases.

1. Let $T$ be connected in $(N, L)$. Then $(cu^T)^L = cu^T$. From the *inessential link property*, $f(cu^T, L) = f(cu^T, L(T))$. Since $(cu^T, L(T))$ is a point unanimous graph game, the $\sigma -$ *communication ability property* implies that $f(cu^T, L) = f(cu^T, L(T)) = \alpha\sigma(N, L(T))$ for some $\alpha \in \mathbb{R}$. *Component efficiency* then determines $\alpha$.
2. Suppose that $T$ is not connected in $(N, L)$. Let $\mathcal{T}$ be a collection of connected subsets of $N$ that contain $T$, and let $\delta_S$, $S \in \mathcal{T}$, be numbers such that $(cu^T)^L = \sum_{S\in\mathcal{T}} \delta^S u^S$ (see Hamiache 1999 for the existence of such a $\mathcal{T}$ and numbers $\delta_S$). Since all $S \in \mathcal{T}$ are connected in $(N, L)$, we know from case 1 that $f(\delta^S u^S, L) = f(\delta^S u^S, L(S))$ is uniquely determined for every $S \in \mathcal{T}$. By *additivity* we then have that $f(cu^T, L) = \sum_{S\in\mathcal{T}} f(\delta^S u^S, L(S))$ is uniquely determined.

It further follows that *additivity* uniquely determines $f(v, L) = \sum_{T\in\Omega^N} f(\Delta_v(T) u^T, L)$ for any graph game $(v, L)$. □

Logical independence of the axioms in Proposition 4.4 is again shown in the appendix. Note that in the proof of this proposition we used that a graph game $(u^T, L(T))$ is point unanimous if $T$ is connected. However, such a graph game is not necessarily link unanimous (as can be seen from the graph game $(v, L)$ on $N = \{1, 2, 3\}$ with $v = u^{\{1,2,3\}}$ and $L = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, which is point unanimous but not link unanimous). Consequently, the $\sigma$-measure property does not imply the $\sigma$-communication ability property (while it follows from Lemma 3.5.(i) that the implication holds the other way around).

## 5 Applications

### 5.1 Assignment games

The assignment game, introduced by Shapley and Shubik (1972), is a game in which the player set $N$ is partitioned in two sets, say the set $V$ of *sellers* and the set $W$ of *buyers*. Any pair $\{i, j\}, i \in V, j \in W$, can realise a nonnegative surplus $a_{i,j}$ from trade. However, any seller $i \in V$ can trade with only one buyer $j \in W$, and the other way around. A *matching* on a subset $S \subseteq N$ of players is a collection $M$ of subsets $\{i, j\} \subseteq S, i \in V, \ j \in W$, such that for any $i \in V, |\{\{h, j\} \in M | h = i\}| \leq 1$ and for any $j \in W, |\{\{i, h\} \in M | h = j\}| \leq 1$. For $S \subseteq N$, let $\mathcal{M}(S)$ be the set of all matchings on $S$. Then the maximum surplus that can be obtained by a coalition $S \subseteq N$ is given by

$$v(S) = \max_{M\in\mathcal{M}(S)} \sum_{\{i,j\}\in M} a_{i,j}$$

with $v(S) = 0$ when $\mathcal{M}(S) = \emptyset$, i.e., when $S \subseteq V$ or $S \subseteq W$.

We now consider the communication graph on $N$ in which the links reflect all matching possibilities, so the graph on $N$ is the bipartite graph $(N, L)$ with $\{i, j\} \in L$ if and only if $i \in V$ and $j \in W$. For this graph $(N, L)$ we have that $v^L = v$ and therefore we denote also the restricted game by $v$. Since any coalition only containing either sellers or buyers is unconnected, it follows that $\Delta_v(S) = 0$ if $S \subseteq V$ or $S \subseteq W$. For all connected coalitions the dividends are given by

$$\Delta_v(S) = \sum_{\{T \subseteq S \mid \min(|T \cap V|, |T \cap W|) \geq 1\}} (-1)^{|S|-|T|} v(T).$$

*Example 5.1* Consider the assignment game with one seller $V = \{1\}$, two buyers $W = \{2, 3\}$ and with $a_{1,2} = 1, a_{1,3} = 2$. Then $L = \{\{1, 2\}, \{1, 3\}\}$ and $v$ is given by

$$v(S) = \begin{cases} 1 & \text{if } S = \{1, 2\}, \\ 2 & \text{if } S \in \{\{1, 3\}, \{1, 2, 3\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

The dividends are

$$\Delta_v(S) = \begin{cases} 1 & \text{if } S = \{1, 2\}, \\ 2 & \text{if } S = \{1, 3\}, \\ -1 & \text{if } S = \{1, 2, 3\}, \\ 0 & \text{otherwise.} \end{cases}$$

The degree measures on the subgraphs of the coalitions with nonzero dividends are $d_1(\{1, 2\}, L(\{1, 2\})) = d_2(\{1, 2\}, L(\{1, 2\})) = d_1(\{1, 3\}, L(\{1, 3\})) = d_3(\{1, 3\}, L(\{1, 3\})) = d_2(N, L) = d_3(N, L) = 1$ and $d_1(N, L) = 2$. This gives the Harsanyi degree solution payoffs $\varphi^d(v, L) = (1, \frac{1}{4}, \frac{3}{4})$. Since, the graph is cycle-free, these payoffs are equal to the position value payoffs of $(v, L)$.

As an alternative solution, the $\beta$-measure yields $\beta_1(\{1, 2\}, L(\{1, 2\})) = \beta_2(\{1, 2\}, L(\{1, 2\})) = \beta_1(\{1, 3\}, L(\{1, 3\})) = \beta_3(\{1, 3\}, L(\{1, 3\})) = 1$, $\beta_1(N, L) = 2$ and $\beta_2(N, L) = \beta_3(N, L) = \frac{1}{2}$. The resulting Harsanyi power solution yields payoffs $\varphi^\beta(v, L) = (\frac{5}{6}, \frac{1}{3}, \frac{5}{6})$.

*Example 5.2* Consider the assignment game with $V = \{1, 2\}$, $W = \{3, 4\}$ and $a_{1,3}=1$, $a_{1,4} = 3$, $a_{2,3} = 4$ and $a_{2,4} = 5$. Then $L = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$ and the non-zero dividends of $v^L = v$ are given by

$$\Delta_v(S) = \begin{cases} a_{i,j} & \text{if } S = \{i, j\},\ i \in V, j \in W, \\ -\min[a_{1,j}, a_{2,j}] & \text{if } S = \{1, 2, j\}, j \in W, \\ -\min[a_{i,3}, a_{i,4}] & \text{if } S = \{i, 3, 4\}, i \in V, \\ 3 & \text{if } S = V \cup W. \end{cases}$$

For these coalitions the degree measures on the subgraphs are $d_i(S, L(S)) = d_j(S, L(S)) = 1$ if $S = \{i, j\}, i \in V, j \in W$, $d_1(S, L(S)) = d_2(S, L(S)) = \frac{1}{2} d_j(S, L(S)) = 1$ if $S = \{1, 2, j\}$ with $j \in W$, $d_3(S, L(S)) = d_4(S, L(S)) = \frac{1}{2} d_i(S, L(S)) = 1$ if $S = \{i, 3, 4\}$ with $i \in V$, and $d_i(N, L) = d_j(N, L) = 2$ for $i \in V, j \in W$ if $N = V \cup W$. Distributing the dividends according to the degrees, we obtain the Harsanyi degree solution payoffs $\varphi^d(v, L) = (\frac{5}{4}, \frac{9}{4}, \frac{3}{2}, 2)$.

In this case the graph is not cycle-free and the Harsanyi degree solution is not equal to the position value. It follows that the Shapley value of the link game $(L, r^L)$ is given by $Sh_{\{1,3\}}(L, r^L) = \frac{1}{3}$, $Sh_{\{1,4\}}(L, r^L) = 2$, $Sh_{\{2,3\}}(L, r^L) = \frac{5}{2}$ and $Sh_{\{2,4\}}(L, r^L) = \frac{13}{6}$. This yields the position value $\pi(v, L) = (\frac{7}{6}, \frac{7}{3}, \frac{17}{12}, \frac{25}{12})$. □

Although Example 5.2 shows that in the assignment game the position value is not equal to the Harsanyi degree solution, in both solutions the total payoff to the sellers is equal to the total payoff to the buyers. Clearly, in the communication graph as defined above each link is between a seller and a buyer. So, the position value is obtained by distributing the Shapley payoff of each link in the link game equally between a seller and a buyer. For the Harsanyi degree solution we have that in each connected coalition the sum of the degrees of the sellers is equal to the sum of the degrees of the buyers, so any dividend is equally shared between sellers and buyers. Since both solutions are component efficient, we have the following corollary.

**Corollary 5.3** *Let $v$ be an assignment game on the player set $N = V \cup W$, and let $(N, L)$ be the corresponding bipartite graph with $L = \{\{i, j\} | i \in V,\ j \in W\}$. Then*

$$\sum_{i\in V} \varphi_i^d(v, L) = \sum_{j\in W} \varphi_j^d(v, L) = \sum_{i\in V} \pi_i(v, L) = \sum_{j\in W} \pi_j(v, L) = \frac{1}{2} v(N).$$

*Moreover, $\varphi^d(v, L) = \pi(v, L)$ if $|V| = 1$ or $|W| = 1$.*

The next example shows that the Harsanyi degree solution does not satisfy the superfluous link property when the graph is not cycle-free.

*Example 5.4* Consider the assignment game given in Example 5.1 and suppose now that also the two buyers 2 and 3 can communicate, i.e., the communication graph is given by $L^c = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\} = L \cup \{\{2, 3\}\}$ with $L$ the graph in Example 5.1. Clearly $v^E(N) = v^{E\cup\{2,3\}}(N)$ for all $E \subset L^c$, so $\{2, 3\}$ is superfluous in $L^c$. Hence, according to the superfluous link property we have that $\pi(v, L^c) = \pi(v, L) = (1, \frac{1}{4}, \frac{3}{4})$. However, when distributing the dividends according to the degree measure, each player has degree 2 in the grand coalition $N$, so the Harsanyi degree solution yields $\varphi^d(v, L^c) = (\frac{7}{6}, \frac{1}{6}, \frac{4}{6})$ (being the Shapley value of $v$), which is not equal to $\varphi^d(v, L)$. The communication possibility between the two buyers decreases their payoffs. It shows that communication might be harmful, because it may give larger shares in negative dividends. □

We end this subsection by considering the case that buyers and sellers cannot trade directly with each other, but need intermediaries. Then the set $N$ is partitioned in a set $V$ of sellers, a set $W$ of buyers and a set $I$ of intermediaries, and the communication graph on $N$ is the graph $(N, L)$ in which every intermediary is connected to every buyer and seller, i.e., $L = \{\{i, j\} \mid i \in I,\ j \in V \cup W\}$. Again $v^L = v$.

*Example 5.5* Extend Example 5.1 with a single intermediary player, labeled 4. So $I = \{4\}$ and $L = \{\{i, 4\} \mid i = 1, 2, 3\}$. Now the game $v$ follows from the assignment game in Example 5.1 but with player 4 being necessary to obtain a positive worth, and thus is given by

$$v(S) = \begin{cases} 1 & \text{if } S = \{1, 2, 4\}, \\ 2 & \text{if } S \in \{\{1, 3, 4\}, \{1, 2, 3, 4\}\}, \\ 0 & \text{otherwise,} \end{cases}$$

with dividends

$$\Delta_v(S) = \begin{cases} 1 & \text{if } S = \{1, 2, 4\}, \\ 2 & \text{if } S = \{1, 3, 4\}, \\ -1 & \text{if } S = \{1, 2, 3, 4\}, \\ 0 & \text{otherwise.} \end{cases}$$

For any coalition $S$ with nonzero dividend, the degree of $i$ in $(S, L(S))$ is 1 if $i \neq 4$, while the degree of player 4 is equal to $|S| - 1$. From this it follows that the Harsanyi degree solution yields $\varphi^d(v, L) = (\frac{7}{12}, \frac{1}{12}, \frac{4}{12}, 1)$. Since, the graph is cycle-free, this solution is equal to the position value. □

Observe that the graph $(N, L)$ is not cycle-free when $|I| \geq 2$, and thus the position value differs from the Harsanyi degree solution when there are multiple intermediaries. However, also if $|I| \geq 2$, any link in the graph connects one of the intermediaries with either a buyer or a seller. So, in both the position value and the Harsanyi degree solution the total payoff to the intermediaries is equal to the total payoff to the sellers and the buyers together. Since both solutions are component efficient, we have the following corollary.

**Corollary 5.6** *Let $v$ be an assignment game with intermediaries on $N = V \cup W \cup I$, and let $(N, L)$ be the corresponding graph with $L = \{\{i, j\} \mid i \in I, j \in V \cup W\}$. Then*

$$\sum_{i \in I} \varphi_i^d(v, L) = \sum_{j \in V \cup W} \varphi_j^d(v, L) = \sum_{i \in I} \pi_i(v, L) = \sum_{j \in V \cup W} \pi_j(v, L) = \frac{1}{2} v(N).$$

*Moreover, $\varphi^d(v, L) = \pi(v, L)$ if $|I| = 1$.*

## 5.2 ATM games

In this subsection we consider ATM games as introduced in Bjorndal et al. (2004). An ATM-game models a situation of $n$ banks on a single location, where some banks have an Automated Teller Machine (money dispenser) and others do not. The banks may agree to cooperate, meaning that customers of banks not having an ATM are allowed to make use of the ATMs of the other banks, resulting in cost savings because using ATMs is a relatively cheap way of cash withdrawals.

We first consider a situation when there is only one single bank having an ATM. Specifically, let the banks not having an ATM be indexed by $i = 2, \ldots, n$ and let player

1 be the only bank that has an ATM. The number of visits of customers of bank $i \neq 1$ to the ATM of bank 1 is given by $\omega_i$. We assume that each visit yields a cost saving of one. So, coalition $S = \{1, i\}$ can realize the non-negative worth $\omega_i$, $i = 2, \ldots, n$ and thus the corresponding game is given by

$$v(S) = \begin{cases} \sum\limits_{i \in S \setminus \{1\}} \omega_i, & \text{if } 1 \in S, \\ 0, & \text{otherwise.} \end{cases}$$

It follows straightforwardly that

$$\Delta_v(S) = \begin{cases} \omega_i, & \text{if } S = \{1, i\},\ i = 2, \ldots, m, \\ 0, & \text{otherwise.} \end{cases}$$

From van den Nouweland et al. (1996) (Theorem 4.3) we know that when only two-player coalitions have nonzero dividends, the Shapley value $Sh$ coincides with the $\tau$-value (see Tijs 1981) and the nucleolus $\eta$ (see Schmeidler 1969). We now model the ATM game as a graph game with $(N, L)$ the star graph given by $L = \{\{1, i\} \mid i = 2, \ldots, n\}$, i.e., the single ATM bank is linked with any other bank. Since only the two-player coalitions $\{1, i\}$, $i \neq 1$, have nonzero dividends, it follows again that $v^L = v$. Further, in any subgraph on a two-player coalition $\{1, i\}$, $i \neq 1$, both players have degree one and also both players have equal $\beta$-power. Therefore, the Harsanyi degree solution, the Harsanyi $\beta$-measure solution, the Shapley value and the Myerson value coincide. Since the graph is cycle-free, we also have that the position value equals this solution. Hence

$$\pi(v, L) = \varphi^d(v, L) = \varphi^\beta(v, L) = \mu(v, L) = Sh(v) = \tau(v) = \eta(v).$$

In all these solutions the payoff to the ATM bank 1 is $\sum_{i=2}^n \frac{1}{2}\omega_i$ and the payoff to bank $i$ is $\frac{1}{2}\omega_i$, $i = 2, \ldots, n$. Thus, all these solutions satisfy the *equal split property* (see Bjorndal et al. 2004) meaning that the cost savings $\omega_i$ obtained from the cooperation between bank $i \neq 1$, and ATM bank 1 is equally distributed between these two banks. Finally, because all dividends are nonnegative, this payoff vector also belongs to the core of the game.

Next we consider the case when there are multiple banks having an ATM, but there is only one bank without an ATM. Let $\{1, \ldots, n-1\}$ be the set of banks who possess ATMs and let bank $n$ be the bank without ATM. The value of any coalition containing bank $n$ and at least one other bank equals the total number of customers $\omega_n$ of bank $n$. So, the game is given by

$$v(S) = \begin{cases} \omega_n, & \text{if } n \in S \ \text{ and } \ |S| \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the dividends are given by

$$\Delta_v(S) = \begin{cases} \omega_n, & \text{if } n \in S \text{ and } |S| \geq 2 \text{ is even,} \\ -\omega_n, & \text{if } n \in S \text{ and } |S| \geq 2 \text{ is odd,} \\ 0, & \text{if } S \subseteq N \backslash \{n\}. \end{cases}$$

Note that this ATM game is equivalent to the assignment game with $n-1$ sellers (the banks with ATMs) and one buyer (bank $n$ without ATM), which can realise the surplus $\omega_n$ with anyone of the sellers. Taking $(N, L)$ as the star graph given by $L = \{\{i, n\})|i = 1, \ldots, n-1\}$, again $v^L = v$. Since $(N, L)$ is cycle-free, it follows that the Harsanyi degree solution is equal to the position value. Furthermore, from Corollary 5.3 it follows that the payoff to player $n$ according to these solutions is given by

$$\pi_n(v, L) = \varphi_n^d(v, L) = \frac{\omega_n}{2}.$$

Hence, by symmetry and component efficiency of $\varphi^d$ and $\pi$ it follows that

$$\pi_i(v, L) = \varphi_i^d(v, L)) = \frac{\omega_n}{2(n-1)}, \quad i = 1, \ldots, n-1.$$

It is instructive to find out what distribution of the total value $\omega_n$ other solution concepts prescribe. The marginal contribution of player $n$ is $\omega_n$ in any permutation except those $(n-1)!$ ones where he enters first. Hence,

$$Sh_n(v) = \frac{n! - (n-1)!}{n!}\omega_n = \frac{n-1}{n}\omega_n$$

and, by symmetry and efficiency of the Shapley value, we obtain $Sh_i(v) = \frac{1}{n(n-1)}\omega_n$ for all $i \neq n$. The Myerson value $\mu(v, L)$ is equal to $Sh(v)$. Further, for $n \geq 3$ it has been shown in Bjorndal et al. (2004) that $\tau(v) = \eta(v) = x^*$ with $x^*$ the single element in the core of the game given by $x_n^* = \omega_n$ and $x_i^* = 0, i \neq n$. Summarizing these observations, we obtain the following relations (in case $n \geq 3$)

$$\begin{aligned} \omega_n &= \tau_n(v) = \eta_n(v) > Sh_n(v) = \mu_n(v, L) > \varphi_n^d(v, L), \\ 0 &= \tau_i(v) = \eta_i(v) < Sh_i(v) = \mu_i(v, L) < \varphi_i^d(v, L) \\ &= \pi_i(v, L),\ i \in \{1, \ldots, n-1\}. \end{aligned}$$

The Harsanyi degree solution shares the cost savings equally between the banks with ATMs and the single bank without ATM, whereas the $\tau$-value and the nucleolus assign all the value to the bank without ATM, implying that this bank can use the money dispensers of the other banks for free. The Shapley value is between the nucleolus and the Harsanyi degree solution, converging to the former when $n$ goes to infinity.

In Bjorndal et al. (2004) a single solution concept is proposed for both situations with one bank with ATM and situations with a single bank without ATM. They call this solution the *aggregate allocation solution*. This solution is given by the nucleolus which in case of a single ATM shares the surplus equally, but it gives all the surplus to the bank without ATM in case of a single bank without ATM. In contrast, the analysis above shows that the Harsanyi degree solution yields an equal split in both situations. In reality, banks cooperate using each other's ATM's by agreeing on a fee to be paid by the bank without ATM for each withdrawal of one of its customers at an ATM bank. Typically this fee is uniform and does not depend on whether or not there is a single ATM. When the fee is taken to be half of the cost savings, the Harsanyi degree solution yields the resulting payoffs in case of a single ATM. In case of multiple ATMs, the Harsanyi degree solution gives the expected payoff to the ATM-banks when the customers of bank $n$ choose randomly between the available ATMs.

### 5.3 Auction games

Consider a second-price sealed bid auction with $n$ bidders. Suppose that their private valuations are arranged in a non-increasing order $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n-1} \geq \theta_n > 0$, and let us assume that the seller attaches utility $\theta_{n+1} \in [0, \theta_n)$ to the object which serves as a reservation price. If all $n$ bidders collude and reveal their private valuations,[9] they can earn as mush as $\theta_1 - \theta_{n+1}$. How should they share this surplus?

This can be modeled as a TU-game $v$ in wich any coalition not including player 1 (player with the highest private valuation) generates zero worth and for any coalition $S$ that includes player 1 the worth is equal to $\theta_1 - \theta_{k+1}$, where $k+1 = \min\{j \in N | j \notin S\}$. Taking the line-graph $L = \{\{i, i+1\} \mid i \in \{1, \ldots, n-1\}\}$ as the communication graph, the worth of $S$ including player 1 is determined by its largest connected part $[1, k]$, where $[i, j] = \{i, i+1, \ldots, j-1, j\}$ denotes the coalition of consecutive players from $i$ to $j$. From van den Brink et al. (2007) [Formula (8)], it follows that

$$\Delta_v(S) = \begin{cases} \theta_k - \theta_{k+1}, & \text{if } S = [1, k]\,, \\ 0, & \text{otherwise.} \end{cases}$$

For any connected coalition $S = [i, j]$ in a line-graph, the degree measure is given by

$$d_k\left(S, L\left(S\right)\right) = \begin{cases} 1, & \text{if } k \in \{i, j\} \\ 2, & \text{if } k \in S \backslash \{i, j\}. \end{cases}$$

From this it follows that the Harsanyi degree solution is given by

$$\varphi_1^d(v, L) = \Delta_v(\{1\}) + \frac{1}{2} \sum_{k=2}^{n} \frac{\Delta_v([1, k])}{k-1} = (\theta_1 - \theta_2) + \frac{1}{2}\left(\sum_{k=2}^{n} \frac{\theta_k - \theta_{k+1}}{k-1}\right),$$

[9] In second-price auctions collusion is supported by Graham and Marshall (1987) who provide a simple incentive-compatible mechanism that induces bidders to disclose their private information about valuations and fosters collusive behavior in such auctions.

$$\varphi_i^d(v, L) = \frac{1}{2(i-1)}\Delta_v([1, i]) + \sum_{k=i+1}^{n} \frac{\Delta_v([1, k])}{k-1} = \frac{\theta_i - \theta_{i+1}}{2(i-1)} + \sum_{k=i+1}^{n} \frac{\theta_k - \theta_{k+1}}{k-1}$$

for any $i \in \{2, \ldots, n-1\}$, and

$$\varphi_n^d(v, L) = \frac{1}{2(n-1)}\Delta_v([1, n]) = \frac{\theta_n - \theta_{n+1}}{2(n-1)}.$$

Since the graph is cycle-free this solution coincides with the position value. The Shapley value, proposed for this type of games in Graham et al. (1990), is given by $Sh_i(v) = \sum_{k=i}^{n} \frac{\Delta_v([1,k])}{k}$. It is easily shown that if $n \geq 3$, $Sh_1(v) > \varphi_1^d(v, L)$, and $Sh_n(v) > \varphi_n^d(v, L)$ for the bidder with the lowest valuation. Thus, the Harsanyi degree solution gives more to 'central' players at the expense of the 'end' ones.

## 6 Concluding remarks

In this paper we studied Harsanyi power solutions for TU-games in which the cooperation possibilities are restricted by a communication graph. In such solutions the sharing system that is used in distributing the Harsanyi dividends in the restricted game is determined by a power measure for communication graphs. Although any positive power measure can be applied, we gave special attention to the degree measure and the equal power measure. On the class of cycle-free graph games, the Harsanyi degree solution is equal to the position value, while it equals the Shapley value on the class of complete graph games. The Harsanyi equal power solution is always equal to the Myerson value.

We gave two axiomatic characterizations of the Harsanyi power solutions on the class of cycle-free graph games. One axiomatization uses the $\sigma$-measure property, and the other uses the $\sigma$-communication ability property. Both give characterizations for the position value and the Myerson value as special cases.[10] So, the difference between the position value and the Myerson value (on cycle-free graph games) is not about using the degree measure property or the communication ability property, but about which power measure to use. In fact, on the class of cycle-free graph games these two properties are equivalent given the same power measure. This is not the case for the class of all graph games where the $\sigma$-communication ability property implies the $\sigma$-measure property. On the class of all graph games the Harsanyi power solutions satisfy all properties used in these characterizations for cycle-free graph games except the superfluous link property. Replacing this property by two invariance properties, we gave an axiomatic characterization for the Harsanyi power solutions on the class of all graph games using the $\sigma$-communication ability property.

We also applied Harsanyi power solutions to some specific classes of games, in particular to assignment games, ATM games and auction games. Finally, we mention that the results of this paper can be restated for antisymmetric directed graphs.

[10] Recently another approach unifying the Myerson value and the Shapley value has been given in Gómez et al. (2004).

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## Appendix: Logical independence

In this appendix we show the logical independence of the axioms in the various characterizations. First we consider the characterization of the Harsanyi power solution on the class of cycle-free graph games given in Proposition 4.1. Each of the next four solutions satisfies three of the four axioms, but differs from the Harsanyi power solution.

1. Let solution $f^1$ be given by $f_i^1(v, L) = 0$ for all $i \in N$. Then $f^1$ satisfies additivity, the superfluous link property and the $\sigma$-measure property (take $\alpha = 0$), but $f^1$ does not satisfy component efficiency.
2. For any cycle-free graph game $(v, L)$, let $L^s \subseteq L$ be the set of links that are not superfluous in $(v, L)$. Let $f^2$ be given by $f^2(v, L) = h^{\sigma(N,L^s)}(v^{L^s})$, i.e., the dividends in the restricted game on the graph $(N, L^s)$ are allocated proportional to the $\sigma$-measure on $(N, L^s)$. Then $f^2$ satisfies component efficiency, the superfluous link property and the $\sigma$-measure property, but it does not satisfy additivity.
3. Let $f^3$ be given by $f^3(v, L) = h^{\sigma(N,L)}(v^L)$, i.e., the dividend of coalition $S$ in the restricted game is allocated proportional to the $\sigma$-measures of the players in $S$ in the graph $(N, L)$ (instead of their powers on the subgraph $(S, L(S))$). Then $f^3$ satisfies component efficiency, additivity, and the $\sigma$-measure property, but it does not satisfy the superfluous link property.
4. For any power measure $\overline{\sigma} \neq \alpha\sigma$ for any $\alpha > 0$, define $f^4$ as the Harsanyi power solution with $\overline{\sigma}$. Then $f^4$ satisfies component efficiency, additivity and the superfluous link property, but does not satisfy the $\sigma$-measure property.

   Since for cycle-free graphs, the $\sigma$-communication ability property is equivalent with the $\sigma$-measure property, the same four solutions also show the logical independence of the axioms in Corollary 4.2.

   It remains to consider the logical independence of the five axioms used in Proposition 4.4 to characterize the Harsanyi power solution on the class of all graph games. On that class each of the solutions $f^1$, $f^3$ and $f^4$ defined above satisfies the five axioms of Proposition 4.4, except component efficiency, the inessential link property and the $\sigma$-communication ability property, respectively. It remains to consider the other two axioms.
5. On the class of all graph games, let $f^5$ be defined similarly as $f^2$, but now by deleting links such that all coalitions containing such a link has zero dividend, i.e., $f^5(v, L) = h^{\sigma(N,L^e)}(v^{L^e})$ with $L^e = \{l \in L \mid \text{there is an } S \subseteq N \text{ with } l \subseteq S \text{ and } \Delta_v(S) \neq 0\}$. This solution satisfies all five axioms, except additivity.
6. Let $f^6$ be defined by $f_i^6(v, L) = \frac{\sigma_i(N,L^e)}{\sum_{j \in B_i} \sigma_j(N,L^e)} v(B_i)$ with $B_i$ being the component in $(N, L^e)$ containing player $i$. Then $f^6$ satisfies all five axioms, except connectedness.

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