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Colorings of simplicial complexes and vector bundles over Davis–Januszkiewicz spaces

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Abstract We show that coloring properties of a simplicial complex K are reflected by splitting properties of a bundle over the associated Davis–Januszkiewicz space whose Chern classes are given by the elementary symmetric polynomials in the generators of the Stanley-Reisner algebra of K.

Keywords Davis–Januszkiewicz spaces · Vector bundle · Characteristic classes · Colorings · Simplicial complexes

Mathematics Subject Classification (2000) 55R10 · 57R22 · 05C15

1 Introduction

For a simplicial complex K, Davis and Januszkiewicz constructed a family of spaces, all of which are homotopy equivalent, and whose integral cohomology is isomorphic to the associated Stanley-Reisner algebra $\mathbb{Z}[K]$ [2, Sect. 4]. We denote a generic model for this homotopy type by $DJ(K)$. In the above mentioned influential paper, Davis and Januszkiewicz also constructed a particular complex vector bundle λ over $DJ(K)$ whose Chern classes are given by the elementary symmetric polynomials in the generators of $\mathbb{Z}[K]$ [2, Sect. 6]. This vector bundle is of particular interest. For example, if K is the dual of the boundary of a simple polytope P, then the associated moment angle complex Z_K is a manifold and the realification $\lambda_{\mathbb{R}}$ of λ is stably isomorphic to the bundle given by applying the Borel construction to the tangent bundle of Z_K. And if M^{2n} is a quasitoric manifold over P, then again the Borel construction applied to the tangent bundle of M^{2n} produces a vector bundle stably isomorphic to $\lambda_{\mathbb{R}}$ [2, Theorem 6.6, Lemma 6.5].

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Davis and Januszkiewicz also noticed that, if K is the dual of the boundary of a polytope of dimension n and admits a coloring with n colors, the bundle λ splits into a direct sum of n complex line bundles and a trivial bundle [2, Sect. 6.2]. We are interested in generalizations of this observation. In fact, we will show that a simplicial complex admits a coloring with r colors precisely when λ splits stably into a direct sum of r linear complex bundles and a trivial bundle. We will also show that a similar result holds for the realification of λ.

To make our statements more precise we have to fix notation and recall some basic constructions. Let $[m] := \{1, \ldots, m\}$ be the set of the first m natural numbers. A finite abstract simplicial complex K on $[m]$ is given by a set of faces $\alpha \subseteq [m]$ which is closed under the formation of subsets. We consider the empty set $\emptyset$ as a face of K. The dimension $\dim \alpha$ of a face α is given in terms of its cardinality by $|\alpha| - 1$, and the dimension $\dim K$ of K is the maximum of the dimensions of its faces. The most basic examples are given by full simplices. For $\alpha \subseteq [m]$ we denote by $\Delta[\alpha]$ the simplicial complex which consists of all possible subsets of α. Then $\Delta[\alpha]$ is an $(|\alpha| - 1)$-dimensional simplex. The full simplex $\Delta[m]$ contains K as a subcomplex, and if $\alpha \in K$ then $\Delta[\alpha] \subset K$ is a subcomplex as well.

A *regular r-paint coloring*, an *r-coloring* for short, of a simplicial complex K is a non degenerate simplicial map $g\colon K \longrightarrow \Delta[r]$, i.e. g maps each face of K isomorphically on a face of $\Delta[r]$. The inclusion $K \subset \Delta[m]$ always provides an m-coloring. If $\dim(K) = n - 1$, then K may only allow r-colorings for $r \geq n$.

For a commutative ring R with unit we denote by $R[m] := R[v_1, \ldots, v_m]$ the graded polynomial algebra generated by the algebraically independent elements $v_1, \ldots, v_m$ of degree two, one for each vertex of K. For each subset $\alpha \subseteq [m]$ we denote by $v_\alpha := \prod_{j \in \alpha} v_j$ the square-free monomial whose factors are in one-to-one correspondence with vertices contained in α. The graded Stanley-Reisner algebra $R[K]$ associated with K is defined as the quotient $R[K] := R[m]/I_K$, where $I_K \subset R[m]$ is the ideal generated by all elements v_μ such that $\mu \subseteq [m]$ is not a face of K.

Since BT^m is an Eilenberg-MacLane space realizing the polynomial algebra $\mathbb{Z}[m]$, the projection $\mathbb{Z}[m] \longrightarrow \mathbb{Z}[K]$ can be realized by a map $f : DJ(K) \longrightarrow BT^m$. We can think of T^m as the maximal torus of the unitary group $U(m)$. The pull back along the composition $DJ(K) \longrightarrow BT^m \longrightarrow BU(m)$ of the universal bundle over $BU(m)$ gives a vector bundle $\lambda \downarrow DJ(K)$. This is the vector bundle studied by Davis and Januszkiewicz and mentioned above. The total Chern class $c(\lambda) = 1 + \sum_i c_i(\lambda)$ of λ is then given by $c(\lambda) = \prod_{i=1}^m (1 + v_i) \in \mathbb{Z}[K]$.

The realification of a complex vector bundle ξ is denoted by $\xi_\mathbb{R}$. Confusing notation we will denote by $\mathbb{C}$ and $\mathbb{R}$ a 1-dimensional trivial complex or real vector bundle over a space X. Now we can state our main theorem.

Theorem 1.1 *Let K be an finite simplicial complex over the vertex set $[m]$. Then the following conditions are equivalent.*

- (i) *K admits an r-coloring $K \longrightarrow \Delta[r]$.*
- (ii) *The vector bundle λ splits into a direct sum $(\bigoplus_{i=1}^r \nu_i) \oplus \mathbb{C}^{m-r}$ of r complex line bundles ν_i and a trivial $(m-r)$-dimensional complex bundle.*
- (iii) *The realification $\lambda_\mathbb{R}$ of λ splits into direct a sum $(\bigoplus_{i=1}^r \theta_i) \oplus \mathbb{R}^{2(m-r)}$ of r 2-dimensional real bundles θ_i and a trivial $2(m-r)$-dimensional real bundle.*
- (iv) *The vector bundle λ is stably isomorphic to a direct sum $\bigoplus_{i=1}^r \nu_i$ of r complex line bundles.*
- (v) *The realification $\lambda_\mathbb{R}$ is stably isomorphic to a direct sum $\bigoplus_{i=1}^r \theta_i$ of r 2-dimensional real bundles.*

Several of our vector bundles will be constructed as homotopy orbit spaces. For a compact Lie group G and a G-space X, the *Borel construction* or *homotopy orbit space* $EG \times_G X$ will be denoted by X_{hG}. If $\eta \downarrow X$ is an n-dimensional G-vector bundle over X with total space $E(\eta)$, the Borel construction establishes a fibre bundle $E(\eta)_{hG} \longrightarrow X_{hG}$. In fact, this is an n-dimensional vector bundle over X_{hG}, denoted by η_{hG}. For definitions and details see [4].

Let M^{2n} be a quasitoric manifold over the simple polytope P. That is that M^{2n} carries a T^n-action, which is locally standard and that the orbit space $M^{2n}/T^n = P$ is a simple polytope. The Borel construction produces a space $(M^{2n})_{hT^n}$, which is homotopy equivalent to $DJ(K_P)$, where K_P is the simplicial complex dual to the boundary of P. For details see [2, Sect. 4.2]. Let τ_M denote the tangent bundle of M^{2n}. Davis and Januszkiewicz showed that the vector bundle $(\tau_M)hT^n \downarrow DJ(K)$ and $\lambda_{\mathbb{R}}$ are stably isomorphic as real vector bundles over $DJ(K_P)$ [2, Sect. 6]. We can draw the following corollary of Theorem 1.1.

Corollary 1.2 *Let M^{2n} be a quasitoric manifold over a simple polytope P. Let K_P be the simplicial complex dual to the boundary of P. If the tangent bundle τ_M of M^{2n} is stably equivariantly isomorphic to a direct sum of r 2-dimensional real equivariant T^n-bundles over M^{2n}, then K_P admits an r-coloring.*

The paper is organized as follows. For the proof of our main theorem we will need two different models for $DJ(K)$. They are discussed in the next section. In Sect. 3, we will use some geometric constructions to produce a splitting of λ from a given coloring. The final section contains the proof of Theorem 1.1.

If not specified otherwise, K will always denote an $(n-1)$-dimensional finite simplicial complex with m-vertices.

2 Models for $DJ(K)$

Let $\text{CAT}(K)$ denote the category whose objects are the faces of K and whose arrows are given by the subset relations between the faces. $\text{CAT}(K)$ has an initial object given by the empty face. Given a pair (X, Y) of pointed topological space we can define covariant functors

$$X^K,\ (X,Y)^K : \text{CAT}(K) \longrightarrow \text{TOP}.$$

The functor X^K assigns to each face α the cartesian product X^α and to each morphism $i_{\alpha,\beta}$ the inclusion $X^\alpha \subset X^\beta$ where missing coordinates are set to the base point $*$. If $\alpha = \emptyset$, then X^α is a point. And $(X, Y)^K$ assigns to α the product $X^\alpha \times Y^{[m]\backslash\alpha}$ and to $i_{\alpha,\beta}$ the coordinate wise inclusion $X^\alpha \times Y^{[m]\backslash\alpha} \subset X^\beta \times Y^{[m]\backslash\beta}$. The inclusions $X^\alpha \subset X^{[m]} = X^m$ and $X^\alpha \times Y^{[m]-\alpha} \subset X^m$ establish inclusions

$$\operatorname{colim}_{\text{CAT}(K)} X^K \longrightarrow X^m,\ \ \operatorname{colim}_{\text{CAT}(K)} (X,Y)^K \longrightarrow X^m.$$

We are interested in two particular cases, namely the functor X^K for the classifying space $BT = \mathbb{C}P^\infty$ of the 1-dimensional circle T and the functor $(X, Y)^K$ for the pair (D^2, S^1). The colimit

$$Z_K := \operatorname{colim}_{\text{CAT}(K)} \left(D^2, S^1\right)^K$$

is called the *moment angle complex* associated to K. The inclusions $Z_K \subset (D^2)^m \subset \mathbb{C}^m$ allow to restrict the standard T^m-action on $\mathbb{C}^m$ to Z_K. The Borel construction produces a fibration

$$q_K : (Z_K)_{hT^m} \longrightarrow BT^m$$

with fiber Z_K. Moreover, $B_T K := (Z_K)_{hT^m}$ is a realization of the Stanley-Reisner algebra $\mathbb{Z}[K]$ and a model for $DJ(K)$. That is there exists an isomorphism $H^*(B_T K; \mathbb{Z}) \cong \mathbb{Z}[K]$ such that the map $H^*(q_K; \mathbb{Z})$ can be identified with the map $\mathbb{Z}[m] \longrightarrow \mathbb{Z}[K]$ [2, Theorem 4.8]. We will use this model for geometric construction with our vector bundles.

Buchstaber and Panov gave a different construction for $DJ(K)$. They showed that $c(K) := \operatorname{colim}_{\mathrm{CAT}(K)} BT^K$ is homotopy equivalent to $B_T K$ and that the inclusion

$$c(K) \longrightarrow BT^m$$

is homotopic to q_K [1, Theorem 6.29]. In particular, each face $\alpha \in K$ defines a map $h_\alpha \colon BT^\alpha \longrightarrow c(K)$. The model $c(K)$ will be used to produce a coloring from a stable splitting of λ.

Remark 2.1 If K is the triangulation of an $(n-1)$-dimensional sphere, the moment angle complex Z_K is a manifold. In this case, the tangent bundle τ_Z is a $(m+n)$-dimensional T^m-equivariant vector bundle, which satisfies the analogue of Corollary 1.2. If τ_Z is stably equivariantly isomorphic to a direct sum of r 2-dimensional real equivariant T^m-bundles over Z_K, then K admits an r coloring. Again this follows from the fact that $(\tau_Z)_{hT^m}$ and $\lambda_\mathbb{R}$ are stably isomorphic [2, Sect. 6].

3 Geometric constructions

The m-dimensional torus T^m acts coordinate wise on $\mathbb{C}^m$. And the diagonal action of T^m on $\mathbb{C}^m \times Z_K$ makes the projection $\mathbb{C}^m \times Z_K \longrightarrow Z_K$ onto the second factor into a T^m-equivariant complex vector bundle over Z_K, denoted by λ'. An application of the Borel construction produces the bundle $\lambda := \lambda'_{hT^m} \downarrow B_T K$ over $B_T K$ whose total Chern class is given by $c(\lambda) = \prod_i (1 + v_i) \in \mathbb{Z}[K]$ and whose classifying map is the composition $B_T K \xrightarrow{q_K} BT^m \longrightarrow BU(m)$. Since T^m acts coordinatewise on $\mathbb{C}^m$, both bundles, λ' and λ split into a direct sum of (equivariant) line bundles. Let $\mathbb{C}_j$ denote the jth component of $\mathbb{C}^m$. in particular, T^m acts on $\mathbb{C}_j$ via the projection $T^m \longrightarrow T^{\{j\}}$ onto the jth component of T^m. The vector bundle $\lambda'_j := \mathbb{C}_j \times Z_K$ is T^m-equivariant, and $\lambda_j := (\lambda'_j)_{hT^m}$ is a 1-dimensional complex vector bundle over $B_T K$. We have $\lambda' \cong \bigoplus_j \lambda'_j$ and $\lambda \cong \bigoplus_j \lambda_j$. All this can be found in [2, Sect. 6].

If $g \colon K \longrightarrow \Delta[r]$ is an r-coloring we want to construct an equivariant splitting of $\lambda' \downarrow Z_K$ into a direct sum of T^m-equivariant complex line bundles and a trivial bundle $\mathbb{C}^{m-r}$. We will use ideas of Davis and Januszkiewicz discussed in [2, Sect. 6.2]. For each $i \in [r]$ we denote by $S_i := g^{-1}(i) \subset [m]$ the preimage of i and by $s_i := |S_i|$ the order of S_i. There are two vector bundles associated with S_i, namely the tensor product $\nu_i := \bigotimes_{j \in S_i} \lambda'_j$ of all complex line bundles associated to the vertices contained in S_i and the direct sum $\eta_i := \bigoplus_{j \in S_i} \lambda'_j$ of all these line bundles. Both are T^m-equivariant vector bundles over $\mathbb{Z}_K$.

Lemma 3.1 *For all $i \in [r]$, there exists an T^m-equivariant vector bundle isomorphism $\nu_i \oplus \mathbb{C}^{s_i - 1} \longrightarrow \eta_i$.*

For simplicial complexes dual to the boundary of simple polytopes the claim is already stated in [2, Sect. 6.2]. We will give here a different proof.

Proof For simplification we drop the subindex i in the notation and assume that $S = [s]$. We will think of $\mathbb{C}^{s-1} \subset \mathbb{C}^s$ as the subspace given by $\{(x_1, \ldots, x_s) \in \mathbb{C}^s | \sum_k x_k = 0\}$. We define a map

$$f : \mathbb{C} \times \mathbb{C}^{s-1} \times Z_K \longrightarrow \mathbb{C}^s \times Z_K$$

by $f(y, x, z) := (u, z)$ where the jth coordinate u_j of u is given by $u_j := y \prod_{k \neq j, k \in [s]} \bar{z}_k + z_j x_j$. Here, $\bar{z}_k$ denotes the complex conjugate of z_k. If T^m acts on $\mathbb{C}$ via the map $t \mapsto \prod_{j \in [s]} t_j$, trivially on $\mathbb{C}^{s-1}$ and on $\mathbb{C}^s$ via the projection $T^m \longrightarrow T^s$ onto the first s coordinates, one can easily show that this map is T^m-equivariant. Moreover, with these actions, the source is the total space of the bundle $\nu \oplus \mathbb{C}^{s-1} \downarrow Z_K$ and the target the total space of $\eta \downarrow Z_K$. Since both sides have the same dimension, it is only left to show that f is fiber wise a monomorphism.

By construction, any subset $\{j, k\} \subset [s]$ is a missing face in K. Since $Z_K = \bigcup_{\alpha \in K} (D^2)^\alpha \times (S^1)^{[m] \setminus \alpha}$, the space $(D^2)^{\{j,k\}} \times (S^1)^{[m] \setminus \{i,j\}}$ is not contained in Z_K and for $z = (z_1, \ldots, z_m) \in Z_K$ there is at most one coordinate among $z_1, \ldots, z_s$ which is trivial.

Now we assume that $f(y, x, z) = (0, z)$. In particular, we have $x_j z_j = -y \prod_{k \neq j} \bar{z}_k$. If one of the coordinates z_j vanishes, say $z_1 = 0$, then $z_j \neq 0$ for $j \neq 1$ and hence $y = 0$ as well as $x_j = 0$ for $j \neq 1$. Since $\sum_j x_j = 0$, we also have $x_1 = 0$.

If $z_j \neq 0$ for all j, then $x_j = y \prod_{k \neq j} \bar{z}_k / z_j$ and $0 = \sum_j x_j = \sum_j y \prod_{k \neq j} \bar{z}_k / z_j = y \sum_j \prod_{k \neq j} \bar{z}_k / z_j$. Multiplying with $\prod_j z_j$ shows that $y \sum_j \prod_{k \neq j} \bar{z}_k z_k = 0$ and hence that $y = 0$ as well as $x_j = 0$ for all j. This shows that f is a fiber wise monomorphism and finishes the proof. □

Corollary 3.2 *Let $K \longrightarrow \Delta[r]$ be an r-coloring of a finite simplicial complex. Then the following holds:*

(i) *The bundle $\lambda' \downarrow Z_K$ splits equivariantly into a direct sum of r equivariant complex line bundles and a trivial bundle.*
(ii) *The bundle $\lambda \downarrow DJ(K)$ splits into a direct sum of r complex line bundles and a trivial bundle.*

Proof By Proposition 3.1 we have

$$\lambda' \cong \bigoplus_{j=1}^{r} \bigoplus_{i \in S_j} \lambda'_i \cong \bigoplus_{j=1}^{r} \left(\nu'_j \oplus \mathbb{C}^{s_i - 1} \right) \cong \left(\bigoplus_{j=1}^{r} \nu'_j \right) \oplus \mathbb{C}^{m-r}.$$

This proves the first part, the second follows from the first by applying the Borel construction. □

4 Proof of Theorem 1.1

The proof needs some preparation. For topological spaces X and Y we denote by $[X, Y]$ the set of homotopy classes of maps from X to Y and for two compact Lie groups G and H by $\mathrm{hom}(H, G)$ the set of Lie group homomorphism $H \longrightarrow G$.

Let G be a compact connected Lie group with maximal torus $j : T_G \hookrightarrow G$ and Weyl group W_G. Since the action of W_G on T_G is induced by conjugation with elements of G, the composition of $w \in W_G$ and j induces a map between the classifying spaces homotopic to Bj. And passing to classifying spaces followed by composing with Bj induces a map $\mathrm{hom}(H, T_G) \longrightarrow [BH, BG]$ which factors through the orbit space of the W_G-action on $\mathrm{hom}(H, T_G)$ and provides a map $\mathrm{hom}(H, T_G)/W_G \longrightarrow [BH, BG]$. The following two facts may be found in [3] and are needed for the proof of our main theorem.

Theorem 4.1 [3] *Let G be a connected compact Lie group and S a torus.*

(i) *The map* $\mathrm{hom}(S, T_G)/W_G \longrightarrow [BS, BG]$ *is a bijection.*
(ii) *The map* $[BS, BG] \longrightarrow \mathrm{Hom}(H^*(BG;\mathbb{Q}), H^*(BS;\mathbb{Q}))$ *is an injection.*

The rational cohomology $H^*(BG;\mathbb{Q}) \cong H^*(BT_G,\mathbb{Q})^{W_G}$ is the ring of polynomial invariants of the induced W_G-action on the polynomial algebra $H^*(BT_G;\mathbb{Q})$. For $G = SO(2k+1)$ the maximal torus $T_{SO(2k+1)} = T^k$ is an k-dimensional torus and we can identify $H^*(BT_{SO(2k+1)});\mathbb{Z})$ with $\mathbb{Z}[k] = \mathbb{Z}[v_1, ..., v_k]$. The Weyl group $W_{SO(2k+1)}$ is the wreath product $\mathbb{Z}/2 \wr \Sigma_l$ where $(\mathbb{Z}/2)^k$ acts on T^k via coordinate wise complex conjugation and Σ_k via permutations of the coordinates. The rational cohomology of $BSO(2k+1)$ is then given by

$$H^*(BSO(2k+1);\mathbb{Q}) \cong \mathbb{Q}[k]^{\mathbb{Z}/2\wr\Sigma_k} \cong \mathbb{Q}[p_1, \ldots, p_k].$$

The classes p_i are already defined over $\mathbb{Z}$. On the one hand $p_i \in H^{4i}(BSO(2k+1);\mathbb{Z})$ is the universal ith Pontrjagin class for oriented bundles and on the other hand $p_i = (-1)^i\sigma_i(v_1^2, ..., v_k^2) \in \mathbb{Z}[k]^{\mathbb{Z}/2\wr\Sigma_k}$ is up to a sign the ith elementary symmetric polynomial in the squares of the generators of $\mathbb{Z}[k]$. In particular, for an oriented $(2k+1)$-dimensional real vector bundle ρ over a space X, the total Pontrjagin class $p(\rho) = 1 + \sum_{i=1}^k p_i(\rho)$ determines completely the map $H^*(BSO(2k+1);\mathbb{Q}) \longrightarrow H^*(X;\mathbb{Q})$ induced by the classifying map $\rho\colon X \longrightarrow BSO(2k+1)$.

Example 4.2 Let $\rho\colon BT^s \longrightarrow BSO(2k+1)$ be the composition of a coordinate wise inclusion $\hat\rho\colon BT^s \longrightarrow BT^k$ followed by the maximal torus inclusion $BT^k \longrightarrow BSO(2k+1)$. Then the total Pontrjagin class of ρ is given by $p(\rho) = \prod_{i=1}^s(1 - v_i^2)$, where we identify $H^*(BT^s;\mathbb{Z})$ with $\mathbb{Z}[v_1, \ldots v_s]$.

By Theorem 4.1, up to homotopy every vector bundle $\omega\colon BT^s \longrightarrow BSO(2k+1)$ is the composition of a lift $\hat\omega\colon BT^s \longrightarrow BT^k$ and Bj. If $p(\omega) = p(\rho)$ then both maps ρ and ω are homotopic and the underlying homomorphisms $j_\omega, j_\rho\colon T^s \longrightarrow T^k$ of the lifts $\hat\omega$ and $\hat\rho$ differ only by an element of the Weyl group (Theorem 4.1). In particular, since j_ρ is given by a coordinatewise inclusion, the homomorphism j_ω also is a coordinate wise inclusion possibly followed by complex conjugation on some coordinates.

Proof of Theorem 1.1 If $K \longrightarrow \Delta[r]$ is an r-coloring, Corollary 3.2 provides the appropriate splitting for λ. The splitting conditions on λ can be put into a hierarchy, a splitting of λ establishes a splitting of $\lambda_{\mathbb{R}}$ and a stable isomorphism between λ and a direct sum of r complex line bundles, and the two latter conditions a stable isomorphism between $\lambda_{\mathbb{R}}$ and a direct sum of r 2-dimensional real bundles. It is only left to show that this last stable isomorphism allows to construct a coloring.

We will work with the model $c(K)$ for $DJ(K)$ and again describe vector bundles over $c(K)$ by their classifying maps. In particular, for $t \geq 2m$ the real vector bundle $\rho_t := \lambda_{\mathbb{R}} \oplus R^{t-2m}$ is a map $\rho_t : c(K) \longrightarrow BO(t)$. Since we are considering stable splittings, we can pass from ρ_t to ρ_{t+1}, if necessary, and assume that $t = 2s+1$ is odd. This will simplify the discussion. For example, if t is odd, we have $BO(t) \simeq BSO(t) \times B\mathbb{Z}/2$. And since $c(K)$ is simply connected, the bundle ρ_t has a unique orientation given by the first coordinate of the map $\rho_t\colon c(K) \longrightarrow BSO(t) \times B\mathbb{Z}/2$. We also denote this map by ρ_t. The total Pontrjagin class of ρ_t is given by $p(\rho_t) = \prod_{i=1}^m(1 - v_i^2)$ [2, Sect. 6].

Let $\phi\colon BT^r \longrightarrow BSO(t)$ be the map induced by the composition of the coordinate wise inclusion $T^r \subset T^s$ into the first r coordinates followed by the maximal torus inclusion $T^s = T_{SO(t)} \subset SO(t)$. A splitting $\rho_t \cong (\bigoplus_{j=1}^r \theta_j) \oplus \mathbb{R}^{t-2r}$ establishes a map $\hat\rho_t\colon c(K) \longrightarrow BT^r$ such that $\phi\hat\rho_t \simeq \rho_t$.

Now let $\alpha \in K$ be a face and $h_\alpha \colon BT^\alpha \longrightarrow c(K)$ the associated map. The composition $\hat{\rho}_t h_\alpha \colon BT^\alpha \longrightarrow BT^r$ determines a unique homomorphism $j_\alpha \colon T^\alpha \longrightarrow T^r$. The total Pontrjagin class of $\phi\hat{\rho}_t h_\alpha$ is given by $p(\phi\hat{\rho}_t h_\alpha) = \prod_{i\in\alpha}(1 - v_i^2)$. Example 4.2 shows that j_α is a coordinate wise inclusion $T^\alpha \longrightarrow T^k$ possibly followed by complex conjugation on some coordinates. The coordinate wise inclusion defines an injection $\alpha \longrightarrow [r]$. Since for any inclusion of faces $\beta \subset \alpha$ the restriction $(\hat{\rho}_t h_\alpha)|_{BT^\beta}$ equals the composition $\hat{\rho}_t h_\beta$, the underlying homomorphisms satisfies the formula $j_\alpha|_\beta = j_\beta$. We can conclude that the collection of all these maps defines a map $[m] \longrightarrow [r]$, whose restriction to any face of K is an injection. This establishes a non degenerate simplicial map $K \longrightarrow \Delta[r]$ which is an r-coloring for K. □

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