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Right Invertible Multiplication Operators and Stable Rational Matrix Solutions to an Associate Bezout Equation, I: The Least Squares Solution

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Abstract. In this paper a state space formula is derived for the least squares solution X of the corona type Bezout equation $G(z)X(z) = I_m$. Here G is a (possibly non-square) stable rational matrix function. The formula for X is given in terms of the matrices appearing in a state space representation of G and involves the stabilizing solution of an associate discrete algebraic Riccati equation. Using these matrices, a necessary and sufficient condition is given for right invertibility of the operator of multiplication by G. The formula for X is easy to use in Matlab computations and shows that X is a rational matrix function of which the McMillan degree is less than or equal to the McMillan degree of G.

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1. Introduction

Throughout this paper G is a stable rational $m \times p$ matrix function. Here *stable* means that G has all its poles in $|z| > 1$, infinity included. In particular, G is a rational matrix-valued H^∞ function. In general, p will be larger than m, and thus G will be a “fat” non-square matrix function. We shall be dealing with the corona type Bezout equation

$$G(z)X(z) = I_m, \quad z \in \mathbb{D}. \tag{1.1}$$

Equation (1.1)—for arbitrary H^∞ functions—has a long and interesting history, starting with Carleson’s corona theorem [4] (for the case when $m = 1$)

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and Fuhrmann's extension to the matrix-valued case [10]. The topic has beautiful connections with operator theory (see [1,22,24], the books [15,19–21], and the more recent papers [26–28]). Rational matrix equations of the form (1.1) also play an important role in solving systems and control theory problems, in particularly, in problems involving coprime factorization, see, e.g., [30, Section 4.1], [13, Section A.2], [31, Chapter 21]). For matrix polynomials (1.1) is closely related to the Sylvester resultant; see, e.g., Section 3 in [12] and the references in this paper.

The operator version of the corona theorem tells us that (1.1) has a $p\times m$ matrix valued H^∞ solution X if and only if the operator M_G of multiplication by G mapping the Hardy space $H^2(\mathbb{C}^p)$ into the Hardy space $H^2(\mathbb{C}^m)$ is right invertible. The necessity of this condition is trivial; sufficiency can be proved by using the commutant lifting theorem (see, e.g., [21, Theorem 3.6.1]). In our case, because $G(z)$ is rational, a simple approximation argument (see the paragraph after Proposition 2.1 below) shows that the existence of a H^∞ solution implies the existence of a rational H^∞- solution.

Assuming that M_G is right invertible, let X be the $p\times m$ matrix function defined by

$$X(\cdot)y = M_G^*(M_G M_G^*)^{-1}Ey, \quad y\in\mathbb{C}^m. \tag{1.2}$$

Here E is the canonical embedding of $\mathbb{C}^m$ into $H^2(\mathbb{C}^m)$, that is, $(Ey)(z) = y$ for each $z\in\mathbb{D}$ and $y\in\mathbb{C}^m$. We shall see (Theorem 1.1 or Proposition 2.1 below) that the function X determined by (1.2) is a stable rational matrix function satisfying (1.1). Note that the operator $M_G^*(M_G M_G^*)^{-1}$ is the unique (Moore-Penrose) right inverse of M_G mapping $H^2(\mathbb{C}^m)$ onto the orthogonal complement of $\operatorname{Ker} M_G$ in $H^2(\mathbb{C}^p)$. This implies that the solution X of (1.1) defined by (1.2) has an additional minimality property, namely given a stable rational matrix solution V of (1.1) we have

$$\|X(\cdot)u\|_{H^2(\mathbb{C}^p)} \le \|V(\cdot)u\|_{H^2(\mathbb{C}^p)} \quad \text{for each } u \text{ in } \mathbb{C}^m, \tag{1.3}$$

or equivalently

$$\frac{1}{2\pi}\int_0^{2\pi} X(e^{it})^* X(e^{it})\,dt \le \frac{1}{2\pi}\int_0^{2\pi} V(e^{it})^* V(e^{it})\,dt. \tag{1.4}$$

Moreover, equality holds in (1.4) if and only if $V = X$. For this reason we refer to the matrix function X defined by (1.2) as the *least squares solution* of (1.1). We note that the use of the Moore-Penrose right inverse $M_G^*(M_G M_G^*)^{-1}$ is not uncommon in the analysis of the corona problem (see, e.g., Section 1 in [27]).

Let us now describe the main result of the present paper. The starting point is a state space representation of G. As is well-known from mathematical systems theory, the fact that G is a stable rational matrix function, allows us (see, e.g., Chapter 1 of [5] or Chapter 4 in [2]) to write G in the following form:

$$G(z) = D + zC(I_n - zA)^{-1}B. \tag{1.5}$$

Here A, B, C, D are matrices of appropriate sizes, I_n is an identity matrix of order n, and the $n \times n$ matrix A is *stable*, that is, A has all its eigenvalues in the open unit disc $\mathbb{D}$. In the sequel we shall refer to the right hand side of (1.5) as a *stable state space representation*. State space representations are not unique. By definition the smallest n for which G has a stable state space representation of the form (1.5) is called the *McMillan degree* of G, denoted by $\delta(G)$. From the stability of the matrix A in (1.5) it follows that the symmetric Stein equation

$$P - APA^* = BB^* \tag{1.6}$$

has a unique solution P. Given this $n \times n$ matrix P we introduce two auxiliary matrices:

$$R_0 = DD^* + CPC^*, \quad \Gamma = BD^* + APC^*. \tag{1.7}$$

The following theorem is our main result.

Theorem 1.1. *Let G be the $m \times p$ rational matrix function given by the stable state space representation* (1.5). *Let P be the unique solution of the Stein equation* (1.6), *and let the matrices R_0 and Γ be given by* (1.7). *Then equation* (1.1) *has a stable rational matrix solution if and only if*

(i) *the discrete algebraic Riccati equation*

$$Q = A^*QA + (C - \Gamma^*QA)^*(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA) \tag{1.8}$$

has a (unique) stabilizing solution Q, that is, Q is an $n \times n$ matrix with the following properties:

  (a) *$R_0 - \Gamma^*Q\Gamma$ is positive definite,*
  (b) *Q satisfies the Riccati equation* (1.8),
  (c) *the matrix $A - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA)$ is stable;*

(ii) *the matrix $I_n - PQ$ is non-singular.*

Moreover, (i) *and* (ii) *are equivalent to M_G being right invertible. Furthermore, if* (i) *and* (ii) *hold, then the $p \times m$ matrix-valued function X defined by* (1.2) *is a stable rational matrix solution of* (1.1) *and X admits the following the state space representation:*

$$X(z) = \big(I_p - zC_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}B\big)\, D_1, \tag{1.9}$$

where

$$\begin{aligned} A_0 &= A - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA),\\ C_1 &= D^*C_0 + B^*QA_0, \ \textit{with}\ C_0 = (R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA),\\ D_1 &= (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1} + C_1(I_n - PQ)^{-1}PC_0^*. \end{aligned}$$

Finally, X is the least squares solution of (1.1), *the McMillan degree of X is less than or equal to the McMillan degree of G, and*

$$\frac{1}{2\pi}\int_0^{2\pi} X(e^{it})^*X(e^{it})\,dt = D_1^*\big(I_p + B^*Q(I_n - PQ)^{-1}B\big)\, D_1. \tag{1.10}$$

The necessary and sufficient state space conditions for the existence of a stable rational matrix solution and the formula for the least squares solution given in the above theorem are new. They resemble analogous conditions and formulas appearing in the state space theory of discrete H^2 and H^∞ optimal control; see [16,17,23], Chapter 21 in the book [31], see also [6] for the continuous time analogues. However, the algebraic Riccati equation in Theorem 1.1 is of the stochastic realization type with the solution Q being positive semidefinite, while the H^∞ or H^2 control Riccati equations in the mentioned references are of the LQR type again with the solutions being positive semidefinite (see, e.g., [14, Chapter 5] for the LQR type, and [14, Chapter 6] for the stochastic realization type). It is easy to rewrite the stochastic realization Riccati equation into the LQR type, but then the condition on the stabilizing solution being positive semidefinite changes into negative semidefinite. As far as we know there is no direct way to reduce the problem considered in the present paper to a standard H^2 control problem or to a coprime factorization problem. Concerning the latter, the discrete time analogue of the coprime method employed in [30, Section 4.1] could be used to obtain a parametrization of all stable rational solutions of (1.1). However, minimal H^2-solutions are not considered in [30], and to the best of our knowledge coprime factorization does not provide a method to single out such a solution. Moreover, it is not clear whether or not the minimal H^2-solution X considered in the present paper does appear among the solutions obtained by using the discrete time analogue of the state space formulas given in [30, Section 4.2]; see the final part of Example 2 in Sect. 5 for a negative result in this direction.

We remark that Theorem 1.1 provides a computationally feasible way to check whether or not for a given $m \times p$ stable rational matrix function G the multiplication operator M_G is right invertible and to obtain the least squares solution in that case. Indeed, first one constructs a realization (1.5) in the standard way. Next, one solves (1.6) for P, for instance by using the Matlab command `dgram` or `dlyap`. With P one constructs the matrices R_0 and Γ as in (1.7). Then solve the algebraic Riccati equation (1.8) for Q, either using the Matlab command `dare` or an iterative method. Finally, one checks that one is not an eigenvalue of PQ. Continuing in this way one also computes the least squares solution X given by (1.9).

In the subsequent paper [9], assuming M_G is right invertible, we shall present a state space description of the set of all stable rational matrix solutions of equation (1.1) and a full description of the null space of M_G. In that second paper we shall also discuss the connection with the related Tolokonnikov lemma [25] for the rational case.

The paper consists of five sections, the first being the present introduction. Sections 2 and 3 have a preparatory character. The basic operator theory results on which Theorem 1.1 is based are presented in Sect. 2. In Sect. 3 we explain the role of the stabilizing solution Q of the Riccati equation appearing in Theorem 1.1. Also a number of auxiliary state space formulas are presented in this third section. The proof of Theorem 1.1 is given in Sect. 4. In Sect. 5 we present two examples, and illustrate the comment on MatLab procedures made above.

2. The Underlying Operator Theory Results

We begin with some terminology and notation. Let F be any $m\times p$ matrix-valued function of which the entries are essentially bounded on the unit circle $\mathbb{T}$. Recall (see, e.g., Chapter XXIII in [11]) that the block Toeplitz operator defined by F is the operator T_F given by

$$T_F = \begin{bmatrix} F_0 & F_{-1} & F_{-2} & \cdots \\ F_1 & F_0 & F_{-1} & \cdots \\ F_2 & F_1 & F_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2_+(\mathbb{C}^p) \to \ell^2_+(\mathbb{C}^m). \tag{2.1}$$

Here $\ldots, F_{-1}, F_0, F_1, \ldots$ are the block Fourier coefficients of F. By H_F we denote the block Hankel operator determined by the block Fourier coefficients F_j with $j = 1, 2, \ldots$, that is,

$$H_F = \begin{bmatrix} F_1 & F_2 & F_3 & \cdots \\ F_2 & F_3 & F_4 & \cdots \\ F_3 & F_4 & F_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2_+(\mathbb{C}^p) \to \ell^2_+(\mathbb{C}^m). \tag{2.2}$$

We shall write $\tilde{E}$ for the canonical embedding of $\mathbb{C}^m$ onto the first coordinate space of $\ell^2_+(\mathbb{C}^m)$. Note that $T_F^*\tilde{E}$ is just equal to the operator from $\mathbb{C}^m$ into $\ell^2_+(\mathbb{C}^p)$ defined by the first column of T_F^*. The identity operator on $\ell^2_+(\mathbb{C}^m)$ or $\ell^2_+(\mathbb{C}^p)$ will be denoted by I. The symbol I_n stands $n \times n$ identity matrix or the identity operator on $\mathbb{C}^n$.

Let G be a stable rational $m \times p$ matrix function. In this case H_G is an operator of finite rank and its rank is equal to the McMillan degree $\delta(G)$. Furthermore, the multiplication operator M_G used in the previous section is unitarily equivalent to the block Toeplitz operator T_G. In fact, $M_G\mathcal{F}_{\mathbb{C}^p} = \mathcal{F}_{\mathbb{C}^m}T_G$, where for each positive integer k the operator $\mathcal{F}_{\mathbb{C}^k}$ is the Fourier transform mapping $\ell^2_+(\mathbb{C}^k)$ onto the Hardy space $H^2(\mathbb{C}^k)$. In what follows it will be more convenient to work with T_G than with M_G. Note that $\mathcal{F}_{\mathbb{C}^m}\tilde{E} = E$, where E is the embedding operator appearing in (1.2). Furthermore, the expression $M_G(M_GM_G^*)^{-1}$, also appearing in (1.2), can be derived from

$$M_G^*(M_GM_G^*)^{-1}\mathcal{F}_{\mathbb{C}^m} = \mathcal{F}_{\mathbb{C}^p}T_G^*(T_GT_G^*)^{-1}. \tag{2.3}$$

The following result provides the operator theory background for the proof of Theorem 1.1.

Proposition 2.1. *Let G be a stable rational $m \times p$ matrix function, and let R be the rational $m \times m$ matrix function given by $R(z) = G(z)G^*(z)$. Then the following four statements are equivalent.*

- (a) *The equation $GX = I$ has a stable rational matrix solution.*
- (b) *The Toeplitz operator T_G is right invertible.*
- (c) *The Toeplitz operator T_R is invertible and the same holds true for the operator $I - H_G^*T_R^{-1}H_G$.*

Moreover, if one of these conditions is satisfied, then $T_G T_G^*$ *is invertible, its inverse is given by*

$$(T_G T_G^*)^{-1} = T_R^{-1} + T_R^{-1} H_G (I - H_G^* T_R^{-1} H_G)^{-1} H_G^* T_R^{-1}, \tag{2.4}$$

and the function $X = \mathcal{F}_{\mathbb{C}^p} T_G^* (T_G T_G^*)^{-1} \tilde{E}$ *is a stable rational matrix function satisfying* (1.1)*.*

We note that the equivalence of (a) and (b) in the above proposition is known. In fact, (a) implies (b) is trivial, and (b) implies the existence of a H^∞ solution. But if (1.1) has an H^∞ solution, then it also has a stable rational matrix solution. The latter follows from a simple approximation argument. To see this, given an H^∞ function F and $0 < r < 1$, let us write F_r for the function $F_r(z) = F(rz)$. Now assume that X is an H^∞ solution of (1.1). Then $G_r(z) X_r(z) = I_m$, and hence

$$G(z) X_r(z) = I_m - (G(z) - G_r(z))\, X_r(z), \quad |z| < 1.$$

Since G is rational, $G_r(z) \to G(z)$ uniformly on $|z| \leq 1$ for $r \to \infty$. Furthermore, $\|X_r\|_\infty \to \|X\|_\infty$ for $r \to \infty$, and the sequence $\{\|X_r\|_\infty\}_{r \geq 1}$ is uniformy bounded. Thus there exists $r_\circ$ such that $\|(G - G_{r_\circ}) X_{r_\circ}\|_\infty < 1/2$. Since $X_{r_\circ}(z)$ is continuous on $|z| \leq 1$, there exists a stable rational matrix function $\tilde{X}$ such that $\|X_{r_\circ} - \tilde{X}\|_\infty$ is strictly less than $(4 + 4\|G\|_\infty)^{-1}$. Now note that

$$G(z)\tilde{X}(z) = I_m + (G(z) - G_{r_\circ}(z))\, X_{r_\circ}(z) - G(z)\left(X_{r_\circ}(z) - \tilde{X}(z)\right), \quad |z| < 1.$$

Moreover, $\|(G - G_{r_\circ}) X_{r_\circ}\|_\infty + \|G(X_{r_\circ} - \tilde{X})\|_\infty < 3/4$. Hence $G\tilde{X}$ is a stable rational matrix function which has a stable rational matrix inverse. This implies that $\tilde{X}(G\tilde{X})^{-1}$ is a stable rational matrix solution of (1.1).[1]

In order to prove Proposition 2.1 it will be convenient to prove the following lemma first.

Lemma 2.2. *Let* G *be a stable rational* $m \times p$ *matrix function, and let* R *be the rational* $m \times m$ *matrix function given by* $R(z) = G(z) G^*(z)$*. Assume* T_R *is invertible. Then* T_G *has closed range, the spectrum of* $H_G^* T_R^{-1} H_G$ *is contained in the closed interval* $[0, 1]$*, and*

$$\dim \operatorname{Ker} T_G^* = \dim \operatorname{Ker} \left(I - H_G^* T_R^{-1} H_G\right) < \infty. \tag{2.5}$$

In particular, T_G *is semi-Fredholm.*

Proof. We shall need the identity

$$T_R = T_{GG^*} = T_G T_G^* + H_G H_G^*. \tag{2.6}$$

This identity can be found, for example, in [11], see formula (4) in Section XXIII.4 of [11]. It was proved there for the case when the entries of T_G and H_G are square matrices, but the general case can be reduced to the square case by adding zero rows or columns to the entries. Since T_R is assumed to be invertible, (2.6) yields

$$T_G T_G^* = T_R - H_G H_G^* = T_R \left(I - T_R^{-1} H_G H_G^*\right). \tag{2.7}$$

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[1] We thank the referee for providing the above argument.

Recall that H_G has finite rank. Thus the first equality in (2.7) shows that $T_G T_G^*$ is a finite rank perturbation of an invertible operator. Hence $T_G T_G^*$ is a Fredholm operator of index zero. As is well-known, the latter implies that T_G has closed range (cf., Exercise 2 on page 283 of [11]).

Next we use the fact that $\operatorname{Ker} T_G$ is perpendicular to $\operatorname{Im} T_G^*$. This implies that the operator T_G is one-to-one on $\operatorname{Im} T_G^*$, and therefore $\operatorname{Ker} T_G^* = \operatorname{Ker} T_G T_G^*$. Since is $\dim \operatorname{Ker} T_G T_G^*$ is finite, the same holds true for $\dim \operatorname{Ker} T_G^*$. Furthermore, we can use the second identity in (2.7) to show that

$$\begin{aligned}\dim \operatorname{Ker} T_G^* &= \dim \operatorname{Ker} T_G T_G^* = \dim \operatorname{Ker} T_R \left(I - T_R^{-1} H_G H_G^*\right)\\ &= \dim \operatorname{Ker} \left(I - T_R^{-1} H_G H_G^*\right)\\ &= \dim \operatorname{Ker} \left(I - H_G^* T_R^{-1} H_G\right).\end{aligned}$$

This proves (2.5).

It remains to prove that the spectrum of $H_G^* T_R^{-1} H_G$ is contained in the closed interval $[0, 1]$. Since $H_G^* T_R^{-1} H_G$ is selfadjoint, it suffices to show that the spectral radius of $H_G^* T_R^{-1} H_G$ is at most one. To do this we use the fact that T_R is strictly positive, which implies that T_R factors as $T_R = \Lambda^* \Lambda$, with Λ being an invertible operator. For instance, for Λ we can take the square root of T_R. Multiplying (2.6) from the left by Λ^{-1} and from the right by Λ^{-*} yields the identity

$$I - \Lambda^{-*} H_G H_G^* \Lambda^{-1} = \Lambda^{-*} T_G T_G^* \Lambda^{-1}. \tag{2.8}$$

The right hand side of the latter identity is non-negative, and hence the operator $\Lambda^{-*} H_G H_G^* \Lambda^{-1}$ is a contraction. In particular, its spectrum is in the closed unit disc, that is, $r_{\text{spec}}(\Lambda^{-*} H_G H_G^* \Lambda^{-1}) \leq 1$. Here $r_{\text{spec}}(K)$ stands for the spectral radius of the operator K. But the spectral radius of a product of two operators is independent of the order of the operators. Thus

$$\begin{aligned}r_{\text{spec}}(H_G^* T_R^{-1} H_G) &= r_{\text{spec}}\left((H_G^* \Lambda^{-1})(\Lambda^{-*} H_G)\right)\\ &= r_{\text{spec}}(\Lambda^{-*} H_G H_G^* \Lambda^{-1}).\end{aligned} \tag{2.9}$$

We conclude $r_{\text{spec}}(H_G^* T_R^{-1} H_G) \leq 1$, as desired. □

Proof of Proposition 2.1. We split the proof into three parts. The equivalence (a) ⇒ (b) is trivial. The first part of the proof deals with (b) ⇒ (c). In the second part, assuming (c) holds, we derive (2.4), and in the third part, again assuming (c), we prove the final statement of the theorem and (c) ⇒ (a). On the way we give a new proof of (b) ⇒ (a) not using the corona theorem as was done in the paragraph directly after Proposition 2.1.

Part 1. Assume that T_G is right invertible. Then $T_G T_G^*$ is strictly positive. As $H_G H_G^*$ is non-negative, it follows from (2.6) that T_R is strictly positive. In particular, T_R is invertible. Since $H_G^* T_R^{-1} H_G$ is a finite rank operator, $I - H_G^* T_R^{-1} H_G$ is invertible if and only if $I - H_G^* T_R^{-1} H_G$ is one-to-one. The fact that T_G is right invertible implies that $\operatorname{Ker} T_G^*$ consists of the zero element only, and hence formula (2.5) shows that $I - H_G^* T_R^{-1} H_G$ is indeed one-to-one. Thus $I - H_G^* T_R^{-1} H_G$ invertible, and (c) is proved.

Part 2. In this part we assume (c) and derive (2.4). Assume that T_R is invertible and that the same holds true for the operator $I - H_G^* T_R^{-1} H_G$. Hence we can apply Lemma 2.2 to show that T_G^* is a one-to-one operator with closed range. This implies that T_G is surjective, and hence $T_G T_G^*$ is invertible. But then we can use (2.7) to show that

$$\begin{aligned}(T_G T_G^*)^{-1} &= \left(I - T_R^{-1} H_G H_G^*\right)^{-1} T_R^{-1} \\ &= \left(I + T_R^{-1} H_G (I - H_G^* T_R^{-1} H_G)^{-1} H_G^*\right) T_R^{-1} \\ &= T_R^{-1} + T_R^{-1} H_G (I - H_G^* T_R^{-1} H_G)^{-1} H_G^* T_R^{-1}.\end{aligned}$$

Thus the inverse of $T_G T_G^*$ is given by (2.4). Note that the above also shows (c) $\Rightarrow$ (b), and thus (b) and (c) are equivalent.

Part 3. In this part we assume (c) holds and derive (a). To do this it remains to prove the final statement of the theorem. For this purpose we need the following terminology. A vector x in $\ell_+^2(\mathbb{C}^m)$ is said to be a *rational vector* whenever $\mathcal{F}_{\mathbb{C}^m} x$ is a stable rational $m \times 1$ matrix function. If F is a rational $m \times p$ matrix function without poles on the unit circle $\mathbb{T}$, then T_F maps rational vectors in $\ell_+^2(\mathbb{C}^p)$ into rational vectors in $\ell_+^2(\mathbb{C}^m)$ and the range of H_F consists of rational vectors only. These facts are well-known; for the statement about the range of H_F see the remark made at the end of the second paragraph of Sect. 3.

We first show that $(T_G T_G^*)^{-1}$ maps rational vectors into rational vectors. To do this, let x be a rational vector in $\ell_+^2(\mathbb{C}^m)$. Put

$$y = H_G (I - H_G^* T_R^{-1} H_G)^{-1} H_G^* T_R^{-1} x.$$

Thus $(T_G T_G^*)^{-1} x = T_R^{-1}(x + y)$. Since G is a stable rational matrix function and y is in the range of H_G, we know (see the previous paragraph) that y is a rational vector. Thus we have to show $T_R^{-1}(x+y)$ is a rational vector. Note that $x + y$ is a rational vector. As R is positive definite on the unit circle, R admits a spectral factorization relative to the unit circle. It follows that T_R^{-1} can be written as $T_R^{-1} = TT^*$ where T is a Toeplitz operator defined by a stable rational matrix function (see Theorem 3.2 below for more details). Thus both T and T^* are Toeplitz operators defined by a rational matrix function without poles on the unit circle. But such Toeplitz operators map rational vectors into rational vectors (see the previous paragraph). We conclude that $T_R^{-1}(x + y)$ is a rational vector, and thus $(T_G T_G^*)^{-1} x$ is a rational vector.

Now put

$$\tilde{\Xi} = (T_G T_G^*)^{-1} \tilde{E} \quad \text{and} \quad \tilde{X} = T_G^* (T_G T_G^*)^{-1} \tilde{E}.$$

From the result of the previous paragraph we know that for each u in $\mathbb{C}^p$ the vector $\tilde{\Xi} u$ is a rational vector in $\ell_+^2(\mathbb{C}^p)$. Note that $\tilde{X} u = T_G^* \tilde{\Xi} u$, and recall that a Toeplitz operator defined by a rational matrix function maps rational vectors into rational vectors. Hence $\tilde{X} u$ is also a rational vector. This implies that $X = \mathcal{F}_{\mathbb{C}^p} \tilde{X}$ is a stable rational matrix function. From

$$T_G \tilde{X} = T_G T_G^* (T_G T_G^*)^{-1} \tilde{E} = \tilde{E},$$

it follows that $G(z)X(z) = I_m$. Thus (a) holds and the final statements of the theorem are proved. □

Both Proposition 2.1 and Lemma 2.2 hold in greater generality. For instance, Lemma 2.2 remains true when G is an $m \times p$ matrix-valued H^∞ function continuous on the closed unit disk. Also the equivalence of (a), (b) and (c) in Proposition 2.1 as well as formula (2.4) remain true for such a function G, provided one allows in (a) for H^∞ solutions.

3. Preliminaries About the Riccati Equation

In this section we clarify the role of the Riccati equation (1.8), and present some auxiliary state space formulas. Throughout this and the following sections we assume that G is given by the stable state space representation (1.5). With this representation we associate the operators

$$W_{obs} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} : \mathbb{C}^n \to \ell^2_+(\mathbb{C}^m) \tag{3.1}$$

$$W_{con} = \begin{bmatrix} B & AB & A^2B & A^3B & \cdots \end{bmatrix} : \ell^2_+(\mathbb{C}^p) \to \mathbb{C}^n. \tag{3.2}$$

The fact that the matrix A is stable implies that these operators are well-defined and bounded. We call W_{obs} the *observability operator* and W_{con} the *controllability operator* corresponding to the state space representation (1.5).

Since for $j = 1, 2, \ldots$ the j-th Taylor coefficient of G at zero is given by $CA^{j-1}B$ it follows from (3.1) and (3.2) that

$$H_G = \begin{bmatrix} G_1 & G_2 & G_3 & \cdots \\ G_2 & G_3 & G_4 & \cdots \\ G_3 & G_4 & G_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = W_{obs}W_{con}. \tag{3.3}$$

From (3.3) it is clear that $\operatorname{rank} H_G$ is finite and the range of H_G consists of rational vectors.

Recall that P is the unique solution of the Stein equation

$$P - APA^* = BB^*. \tag{3.4}$$

Thus $P = \sum_{\nu=0}^{\infty} A^\nu BB^*(A^*)^\nu = W_{con}W_{con}^*$, where W_{con} is defined by (3.2). Recall that P is unique because A is stable.

Lemma 3.1. *Let G be the $m \times p$ rational matrix function given by the stable state space representation (1.5), and let P be the unique solution of the Stein equation (3.4). Put $R(z) = G(z)G^*(z)$, where $G^*(z) = G(\bar{z}^{-1})^*$. Then R admits the following representation*

$$R(z) = zC(I_n - zA)^{-1}\Gamma + R_0 + \Gamma^*(zI_n - A^*)^{-1}C^*, \tag{3.5}$$

where

$$R_0 = DD^* + CPC^*, \quad \Gamma = BD^* + APC^*. \tag{3.6}$$

Proof. From (1.5) we see that

$$R(z) = G(z)G^*(z) = G(z)D^* + G(z)B^*(zI_n - A^*)^{-1}C^*. \tag{3.7}$$

We first prove that

$$G(z)B^*(zI_n - A^*)^{-1} = C(I_n - zA)^{-1}P + \Gamma^*(zI_n - A^*)^{-1}. \tag{3.8}$$

To do this observe that

$$\begin{aligned} G(z)B^*(zI_n - A^*)^{-1} \;=\; & DB^*(zI_n - A^*)^{-1} \\ & + zC(I_n - zA)^{-1}BB^*(zI_n - A^*)^{-1}. \end{aligned}$$

From (3.4) we see that $zBB^* = P(zI_n - A^*) + (I_n - zA)PA^*$, and thus

$$z(I_n - zA)^{-1}BB^*(zI_n - A^*)^{-1} = (I_n - zA)^{-1}P + PA^*(zI_n - A^*)^{-1}.$$

Inserting the latter identity in the formula for $G(z)B^*(zI_n - A^*)^{-1}$ we obtain

$$\begin{aligned} G(z)B^*(zI_n - A^*)^{-1} \;=\; & DB^*(zI_n - A^*)^{-1} \\ & + C(I_n - zA)^{-1}P + CPA^*(zI_n - A^*)^{-1}. \end{aligned}$$

From the second identity in (3.6) we know that $\Gamma^* = DB^* + CPA^*$. Thus (3.8) holds.

Using the representation (1.5) and inserting (3.8) in (3.7) yields

$$\begin{aligned} G(z)G^*(z) &= G(z)D^* + C(I_n - zA)^{-1}PC^* + \Gamma^*(zI_n - A^*)^{-1}C^* \\ &= G(z)D^* + CPC^* + zC(I_n - zA)^{-1}APC^* + \Gamma^*(zI_n - A^*)^{-1}C^* \\ &= DD^* + CPC^* + zC(I_n - zA)^{-1}(BD^* + APC^*) \\ &\quad + \Gamma^*(zI_n - A^*)^{-1}C^*. \end{aligned}$$

But $DD^* + CPC^* = R_0$ and $BD^* + APC^* = \Gamma$ by (3.6). Thus (3.5) is proved. □

Following [8] we associate with the representation (3.5) the discrete algebraic Riccati equation

$$Q = A^*QA + (C - \Gamma^*QA)^*(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA). \tag{3.9}$$

Note that this is precisely the Riccati equation appearing Theorem 1.1. Using the symmetric version of Theorem 1.1 in [8] (see Section 14.7 in [3] or Sections 10.2 and 10.2 in [7]) we know that $R(z) = G(z)G^*(z)$ is positive definite for each z on the unit circle $\mathbb{T}$ if and only if the Riccati equation (3.9) has a solution Q satisfying

(a) $R_0 - \Gamma^*Q\Gamma$ is positive definite,
(b) Q satisfies the Riccati equation (3.9),
(c) the matrix $A - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA)$ is stable.

Moreover, this solution is unique and hermitian. In fact,

$$Q = W_{obs}^* T_R^{-1} W_{obs}. \tag{3.10}$$

Here T_R is the block Toeplitz operator on $\ell_+^2(\mathbb{C}^p)$ defined by the matrix function R, and W_{obs} is defined by (3.2). The solution Q satisfying (a), (b), (c) above will be called the *stabilizing solution* of (3.9), cf., Section 13.5 in [18].

In sequel, given the stabilizing solution Q of (3.9), we write

$$A_0 = A - \Gamma C_0, \quad \text{where} \quad C_0 = (R_0 - \Gamma^* Q\Gamma)^{-1}(C - \Gamma^* QA). \tag{3.11}$$

Note that (c) tells us that A_0 is stable.

When (3.9) has a stabilizing Q, then (and only then) the function R admits a right spectral factorization relative to the unit circle $\mathbb{T}$. Moreover, in that case, a right spectral factorization $R(z) = \Phi^*(z)\Phi(z)$ is obtained (see, e.g., Section 14.7 in [3]) by taking

$$\Phi(z) = \Delta + z\Delta C_0(I_n - zA)^{-1}\Gamma, \quad \text{where} \quad \Delta = (R_0 - \Gamma^* Q\Gamma)^{1/2}. \tag{3.12}$$

Note that Δ is invertible, because $R_0 - \Gamma^* Q\Gamma$ is invertible. The first identity in (3.11) then implies (cf., Theorem 2.1 in [2]) that

$$\Phi(z)^{-1} = \Delta^{-1} - zC_0(I_n - zA_0)^{-1}\Gamma\Delta^{-1}. \tag{3.13}$$

Since A and A_0 are both stable, (3.12) and (3.13) both present stable state space representations, and hence Φ is invertible outer. (We call a square matrix-valued H^∞ function F *invertible outer* whenever $F(z)^{-1}$ exists and is again an H^∞ function. Thus a square stable rational matrix function F is invertible outer whenever $F(z)^{-1}$ exists and is a stable rational matrix function, i.e., F is invertible in the algebra of stable square rational matrix functions.) Given the right spectral factorization $R(z) = \Phi^*(z)\Phi(z)$ with Φ given by (3.12), the block Toeplitz operator T_R factors as $T_R = L^*L$, where $L = T_\Phi$. Note that both T_Φ and T_Φ^{-1} are block lower triangular. We summarize the above results in the following theorem.

Theorem 3.2. *Let G be given by (1.5) with A stable, and put $R(z) = G(z)G^*(z)$. Then T_R is invertible if and only if the Riccati equation (3.9) has a stabilizing solution Q. In that case, Q is uniquely determined by (3.10) and the inverse of T_R is given by $T_R^{-1} = T_\Psi T_\Psi^*$. Here T_Ψ is the block lower triangular Toeplitz operator on $\ell_+^2(\mathbb{C}^m)$ defined by the stable rational matrix function*

$$\Psi(z) = \left(I_m - zC_0(I_n - zA_0)^{-1}\Gamma\right)\Delta^{-1}, \quad \textit{where } \Delta = (R_0 - \Gamma^* Q\Gamma)^{1/2}. \tag{3.14}$$

The following result is an addition to Lemma 2.2.

Lemma 3.3. *Let G be given by (1.5) with A stable, and let P be the unique solution of the Stein equation (3.4). Put $R(z) = G(z)G^*(z)$, and assume that T_R is invertible, or equivalently, assume that the Riccati equation (3.9) has a stabilizing solution Q. Then the $n \times n$ matrix PQ has all its eigenvalues in the closed interval* $[0, 1]$, *and*

$$\dim \operatorname{Ker} T_G^* = \dim \operatorname{Ker} (I_n - PQ). \tag{3.15}$$

Proof. Recall that $H_G = W_{obs}W_{con}$ and $Q = W_{obs}^* T_R^{-1} W_{obs}$; see (3.3) and (3.10). Using these identities we see that

$$H_G^* T_R^{-1} H_G = W_{con}^* W_{obs}^* T_R^{-1} W_{obs} W_{con} = W_{con}^* Q W_{con}. \tag{3.16}$$

Next we use $P = W_{con}W_{con}^*$ and the identity (2.5). It follows that

$$\begin{aligned}\dim \operatorname{Ker} T_G^* &= \dim \operatorname{Ker}\,(I - H_G^* T_R^{-1} H_G) = \dim \operatorname{Ker}\,(I_n - W_{con}^* Q W_{con})\\ &= \dim \operatorname{Ker}\,(I_n - W_{con}W_{con}^* Q) = \dim \operatorname{Ker}\,(I_n - PQ).\end{aligned}$$

This proves (3.15). By Lemma 2.2 the spectral radius of $I - H_G^* T_R^{-1} H_G$ is at most one. Hence (3.16) yields

$$1 \geq r_{\rm spec}(I - W_{con}^* Q W_{con}) = r_{\rm spec}(I_n - PQ).$$

Finally, note that the non-zero eigenvalues of PQ are equal to the non-zero eigenvalues of $P^{1/2}QP^{1/2}$. But the latter matrix is nonnegative (because Q is nonnegative by (3.10)), and thus all the eigenvalues of PQ belong to $[0, 1]$, as desired. □

The following lemma will be useful in the next sections.

Lemma 3.4. *Let G be given by (1.5) with A stable, and let P be the unique solution of the Stein equation (3.4). Assume that $R(z) = G(z)G^*(z)$ is positive definite for each z on $\mathbb{T}$, and let Q be the stabilizing solution of the Riccati equation (3.9). Then the following identities hold:*

$$G^*(z)C_0(I_n - zA_0)^{-1} = C_1(I_n - zA_0)^{-1} + B^*(zI_n - A^*)^{-1}Q, \qquad (3.17)$$

$$G(z)C_1(I_n - zA_0)^{-1} = C(I_n - zA)^{-1}(I_n - PQ), \qquad (3.18)$$

$$R(z)C_0(I_n - zA_0)^{-1} = C(I_n - zA)^{-1} + \Gamma^*(zI_n - A^*)^{-1}Q. \qquad (3.19)$$

Here A_0 and C_0 are given by (3.11), the matrix Γ is defined by the second identity in (3.6), and C_1 is given by

$$C_1 = D^*C_0 + B^*QA_0. \qquad (3.20)$$

Furthermore, we have

$$\begin{aligned} BC_1 &= A(I_n - PQ) - (I_n - PQ)A_0, &(3.21)\\ DC_1 &= C(I_n - PQ), &(3.22)\\ C_1^*C_1 &= (Q - QPQ) - A_0^*(Q - QPQ)A_0. &(3.23)\end{aligned}$$

Proof. We begin the proof with the last three identities and then we proceed with the first three. Using the definition of A_0 and C_0 in (3.11) together with the fact that Q is a hermitian matrix satisfying (3.9) we see that

$$Q = A^*QA_0 + C^*C_0. \qquad (3.24)$$

The latter identity will play an important role in deriving (3.17) and (3.23). □

Proof of (3.21). Using the definition of C_1 in (3.20) and the Stein equation (3.4), we have

$$\begin{aligned}BC_1 &= BD^*C_0 + BB^*QA_0 = BD^*C_0 + (P - APA^*)QA_0\\ &= BD^*C_0 + PQA_0 - APA^*QA_0\\ &= BD^*C_0 + PQA_0 - AP(Q - C^*C_0) \quad [\text{by } (3.24)]\\ &= (BD^* + APC^*)C_0 + PQA_0 - APQ\\ &= \Gamma C_0 + PQA_0 - APQ = A - A_0 + PQA_0 - APQ\\ &= A(I_n - PQ) - (I_n - PQ)A_0.\end{aligned}$$

□

Proof of (3.22). Notice that

$$\begin{aligned}DC_1 &= DD^*C_0 + DB^*QA_0\\ &= DD^*C_0 + (\Gamma^* - CPA^*)QA_0 \quad [\text{by the second identity in } (3.6)]\\ &= DD^*C_0 + \Gamma^*QA_0 - CPA^*QA_0\\ &= DD^*C_0 + \Gamma^*Q(A - \Gamma C_0) - CP(Q - C^*C_0)\\ &= (DD^* + CPC^*)C_0 + \Gamma^*QA - \Gamma^*Q\Gamma C_0 - CPQ\\ &= (R_0 - \Gamma^*Q\Gamma)C_0 + \Gamma^*QA - CPQ \quad [\text{by the first identity in } (3.6)\\ &= C - \Gamma^*QA + \Gamma^*QA - CPQ \quad [\text{by the second identity in } (3.11)]\\ &= C(I_n - PQ).\end{aligned}$$

□

Proof of (3.23). We use $C_1^* = C_0^*D + A_0^*QB$ and the previous identities for BC_1 and DC_1 above. This yields

$$\begin{aligned}C_1^*C_1 &= C_0^*DC_1 + A_0^*QBC_1\\ &= C_0^*C(I_n - PQ) + A_0^*Q\left(A(I_n - PQ) - (I_n - PQ)A_0\right)\\ &= (C_0^*C + A_0^*QA)(I_n - PQ) - A_0^*Q(I_n - PQ)A_0\\ &= Q(I_n - PQ) - A_0^*Q(I_n - PQ)A_0 \quad [\text{by } (3.24)]\\ &= Q - QPQ - A_0^*(Q - QPQ)A_0.\end{aligned}$$

□

Proof of (3.17). Using the representation of $G(z)$ given by (1.5), we obtain

$$\begin{aligned}G^*(z)C_0(I_n - zA_0)^{-1} &= D^*C_0(I_n - zA_0)^{-1}\\ &\quad + B^*(zI_n - A^*)^{-1}C^*C_0(I_n - zA_0)^{-1}.\end{aligned}$$

According to (3.24), we have $C^*C_0 = Q - A^*QA_0$. It follows that

$$C^*C_0 = (zI_n - A^*)QA_0 + Q(I_n - zA_0).$$

This yields

$$(zI_n - A^*)^{-1}C^*C_0(I_n - zA_0)^{-1} = QA_0(I_n - zA_0)^{-1} + (zI_n - A^*)^{-1}Q.$$

By using the latter identity in the formula for $G^*(z)C_0(I_n - zA_0)^{-1}$ above we obtain

$$\begin{aligned} G^*(z)C_0(I_n - zA_0)^{-1} &= D^*C_0(I_n - zA_0)^{-1} \\ &\quad + B^*QA_0(I_n - zA_0)^{-1} + B^*(zI_n - A^*)^{-1}Q. \end{aligned}$$

As $C_1 = D^*C_0 + B^*QA_0$, we have proved (3.17). □

Proof of (3.18). Note that

$$\begin{aligned} G(z)C_1(I_n - zA_0)^{-1} &= DC_1(I_n - zA_0)^{-1} \\ &\quad + zC(I_n - zA)^{-1}BC_1(I_n - zA_0)^{-1}. \end{aligned}$$

Using (3.21) we have

$$\begin{aligned} zBC_1 &= zA(I_n - PQ) - z(I_n - PQ)A_0 \\ &= (I_n - PQ)(I_n - zA_0) - (I_n - zA)(I_n - PQ). \end{aligned}$$

This yields

$$\begin{aligned} &z(I_n - zA)^{-1}BC_1(I_n - zA_0)^{-1} \\ &\quad = (I_n - zA)^{-1}(I_n - PQ) - (I_n - PQ)(I_n - zA_0)^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} G(z)C_1(I_n - zA_0)^{-1} &= DC_1(I_n - zA_0)^{-1} \\ &\quad + C(I_n - zA)^{-1}(I_n - PQ) \\ &\quad - C(I_n - PQ)(I_n - zA_0)^{-1}. \end{aligned}$$

Now (3.22) shows that $DC_1(I_n - zA_0)^{-1} - C(I_n - PQ)(I_n - zA_0)^{-1} = 0$. Thus (3.18) holds. □

Proof of (3.19). Using (3.17) and (3.18) we have

$$\begin{aligned} R(z)C_0(I_n - zA_0)^{-1} &= G(z)G^*(z)C_0(I_n - zA_0)^{-1} \\ &= G(z)C_1(I_n - zA_0)^{-1} + G(z)B^*(zI_n - A^*)^{-1}Q \\ &= C(I_n - zA)^{-1}(I_n - PQ) + G(z)B^*(zI_n - A^*)^{-1}Q. \end{aligned} \tag{3.25}$$

Inserting the identity for $G(z)B^*(zI_n - A^*)^{-1}$ given by (3.8) into (3.25) we obtain (3.19). □

4. Proof of Theorem 1.1

It will be convenient to prove the following result first.

Theorem 4.1. *Let G be given by* (1.5) *with A stable, and let P be the unique solution of the Stein equation* (3.4). *Then the operator T_G is right invertible if and only if*

(i) *the Riccati equation* (3.9) *has a stabilizing solution Q and*
(ii) *the matrix $I_n - PQ$ is non-singular.*

In that case the operator $T_G T_G^*$ *is invertible and its inverse is given by*

$$(T_G T_G^*)^{-1} = T_\Psi T_\Psi^* + K(I_n - PQ)^{-1} P K^*. \tag{4.1}$$

Here T_Ψ *is the block lower triangular Toeplitz operator on* $\ell^2_+(\mathbb{C}^m)$ *defined by the stable rational matrix function* (3.14)*, and* K *is the observability operator defined by*

$$K = W_{0,\,obs} = \begin{bmatrix} C_0 \\ C_0 A_0 \\ C_0 A_0^2 \\ \vdots \end{bmatrix} : \mathbb{C}^n \to \ell^2_+(\mathbb{C}^m). \tag{4.2}$$

In that case $\Xi = \mathcal{F}_{\mathbb{C}^m}(T_G T_G^*)^{-1}\tilde{E}$ *is a stable rational* $m \times m$ *matrix function, and* Ξ *admits the following state space representation:*

$$\Xi(z) = D_0 + zC_0(I_n - zA_0)^{-1} B_0, \tag{4.3}$$

where A_0 *and* C_0 *are given by* (3.11)*, and*

$$B_0 = A_0(I_n - PQ)^{-1} P C_0^* - \Gamma(R_0 - \Gamma^* Q \Gamma)^{-1}, \tag{4.4}$$

$$D_0 = C_0(I_n - PQ)^{-1} P C_0^* + (R_0 - \Gamma^* Q \Gamma)^{-1}. \tag{4.5}$$

Finally, it is noted that D_0 *is strictly positive.*

Proof. By Proposition 2.1 and Lemma 2.2 the operator T_G is right invertible if and only if T_R is invertible and $\dim \operatorname{Ker} T_G^* = 0$. But T_R being invertible is equivalent to the requirement that the Riccati equation (3.9) has a stabilizing solution Q, and in that case, Lemma 2.2 tells us that $\dim \operatorname{Ker} T_G^* = 0$ if and only if $I_n - PQ$ is non-singular. This proves the necessity and sufficiency of the conditions (i) and (ii).

Now, assume that these two conditions are fulfilled. Then we know that $T_G T_G^*$ is invertible and its inverse is given by (2.4). We have to transform (2.4) into (4.1). Note that (3.19) tells us that $T_R W_{0,\,obs} = W_{obs}$. It follows that

$$T_R^{-1} H_G = T_R^{-1} W_{obs} W_{con} = W_{0,\,obs} W_{con}.$$

We already know that $H_G^* T_R^{-1} H_G = W_{con}^* Q W_{con}$; see (3.16). Since $P = W_{con} W_{con}^*$, we obtain

$$\begin{aligned} &T_R^{-1} H_G (I - H_G^* T_R^{-1} H_G)^{-1} H_G^* T_R^{-1} \\ &\quad = W_{0,\,obs} W_{con} (I - W_{con}^* Q W_{con})^{-1} W_{con}^* W_{0,\,obs}^* \\ &\quad = W_{0,\,obs} (I_n - W_{con} W_{con}^* Q)^{-1} W_{con} W_{con}^* W_{0,\,obs}^* \\ &\quad = W_{0,\,obs} (I_n - PQ)^{-1} P W_{0,\,obs}^* \\ &\quad = K (I_n - PQ)^{-1} P K^*. \end{aligned}$$

This takes care of the second term in the right hand side of (4.1). The first term in the right hand side of (4.1) follows by applying Theorem 3.2 to the first term in the right hand side of (2.4).

It remains to derive the formula for $\Xi = \mathcal{F}_{\mathbb{C}^m}(T_G T_G^*)^{-1}\tilde{E}$. To do this we use (4.1). From (4.2) is clear that $K^*\tilde{E} = C_0^*$. We conclude that

$$\begin{aligned}\left(\mathcal{F}_{\mathbb{C}^m} K(I_n - PQ)^{-1} PK^*\tilde{E}\right)(z) &= \left(\mathcal{F}_{\mathbb{C}^m} K(I_n - PQ)^{-1} PC_0^*\right)(z)\\ &= C_0(I_n - zA_0)^{-1}(I_n - PQ)^{-1}PC_0^*\\ &= C_0(I_n - PQ)^{-1}PC_0^* +\\ &\quad + zC_0(I_n - zA_0)^{-1}A_0(I_n - PQ)^{-1}PC_0^*.\end{aligned} \tag{4.6}$$

Now consider $\mathcal{F}_{\mathbb{C}^m} T_\Psi T_\Psi^* \tilde{E}$. Since T_Ψ^* is block upper triangular with the matrix $(R_0 - \Gamma^* Q\Gamma)^{-1/2}$ on the main diagonal, $T_\Psi^*\tilde{E} = \tilde{E}(R_0 - \Gamma^* Q\Gamma)^{-1/2}$. Finally, because T_Ψ is the block Toeplitz operator defined by Ψ, we obtain

$$\begin{aligned}(\mathcal{F}_{\mathbb{C}^m} T_\Psi T_\Psi^* \tilde{E})(z) &= \Psi(z)(R_0 - \Gamma^* Q\Gamma)^{-1/2}\\ &= (R_0 - \Gamma^* Q\Gamma)^{-1} - zC_0(I_n - zA_0)^{-1}\Gamma(R_0 - \Gamma^* Q\Gamma)^{-1}.\end{aligned} \tag{4.7}$$

By adding (4.6) and (4.7) we see that $\Xi = \mathcal{F}(T_G T_G^*)^{-1}\tilde{E}$ has the desired state space representation.

To complete the proof, it is noted that

$$C_0(I_n - PQ)^{-1}PC_0^* = C_0 P^{1/2}(I_n - P^{1/2}QP^{1/2})^{-1}P^{1/2}C_0^*$$

is positive. Since $(R_0 - \Gamma^* Q\Gamma)^{-1}$ is strictly positive, it follows that D_0 is strictly positive. □

Corollary 4.2. *Let G be given by (1.5) with A stable. Then M_G is right invertible if and only if G can be written as $G(z) = DV(z)$, where $D = G(0)$ has full row rank and V is an invertible outer stable rational matrix function. Moreover, in that case one can take for V the function given by*

$$V(z) = I_p + zC_1(I_n - PQ)^{-1}(I_n - zA)^{-1}B. \tag{4.8}$$

Here P and Q are as in Theorem 4.1 *and C_1 is defined by* (3.20).

Proof. Assume $G(z) = DV(z)$ for some invertible outer stable rational matrix function V, and let D^+ be any right inverse of D. Put $U(z) = V(z)^{-1}D^+$. Then $G(z)U(z) = DV(z)V(z)^{-1}D^+ = I_m$ for each $|z| \leq 1$. Thus $M_G M_U = I$, and M_G is right invertible.

Conversely, assume M_G is right invertible. Let P and Q be as in Theorem 4.1. Then $I_n - PQ$ is invertible. Let V be defined by (4.8). By consulting (3.22), we obtain $C = DC_1(I_n - PQ)^{-1}$. Thus

$$\begin{aligned}G(z) &= D + zC(I_n - zA)^{-1}B\\ &= D + zDC_1(I_n - PQ)^{-1}(I_n - zA)^{-1}B\\ &= DV(z).\end{aligned}$$

It remains to show that V is invertible outer. We have

$$V(z)^{-1} = I_p - zC_1(I_n - PQ)^{-1}(I_n - zA^\times)^{-1}B,$$

where

$$\begin{aligned} A^{\times} &= A - BC_1(I_n - PQ)^{-1} \\ &= A - \left(A(I_n - PQ) - (I_n - PQ)A_0\right)(I_n - PQ)^{-1} \quad \text{[by (3.21)]} \\ &= (I_n - PQ)A_0(I_n - PQ)^{-1}. \end{aligned}$$

Therefore $A^{\times}$ is similar to the stable matrix A_0, and hence $A^{\times}$ is stable. It follows that both $V(z)$ and $V(z)^{-1}$ are stable rational matrix functions. Thus V is invertible outer. □

Proof of Theorem 1.1. In view of Theorem 4.1 we only have to derive the formula for $X = M_G^*(M_G M_G^*)^{-1}E$ and to prove the statements in the final paragraph of the theorem.

From (2.3) we see that $M_G^*(M_G M_G^*)^{-1}E = \mathcal{F}_{\mathbb{C}^p} T_G^*(T_G T_G^*)^{-1}\tilde{E}$. It follows that $X = \mathcal{F}_{\mathbb{C}^p} T_G^* \tilde{\Xi}$, where $\tilde{\Xi} = (T_G T_G^*)^{-1}\tilde{E}$. Put $\Xi = \mathcal{F}_{\mathbb{C}^m}\tilde{\Xi}$. According to Theorem 4.1, the function Ξ is given by (4.3). Note that

$$X = \mathcal{F}_{\mathbb{C}^p} T_{G^*}\tilde{\Xi} = \mathcal{F}_{\mathbb{C}^p} T_{G^*\Xi}\tilde{E}.$$

Lets us compute $G^*(z)\Xi(z)$. Using the state space representation (1.5) for G and the identity (3.17) we have

$$\begin{aligned} G^*(z)\Xi(z) &= G^*(z)D_0 + zG^*(z)C_0(I_n - zA_0)^{-1}B_0 \\ &= D^*D_0 + B^*(zI_n - A_0^*)^{-1}C^*D_0 \\ &\quad + zC_1(I_n - zA_0)^{-1}B_0 + zB^*(zI_n - A^*)^{-1}QB_0 \\ &= D^*D_0 + B^*QB_0 + zC_1(I_n - zA_0)^{-1}B_0 \\ &\quad + B^*(zI_n - A_0^*)^{-1}C^*D_0 + B^*(zI_n - A^*)^{-1}A^*QB_0. \end{aligned} \tag{4.9}$$

It follows that

$$X(z) = (\mathcal{F}_{\mathbb{C}^p} T_{G^*\Xi}\tilde{E})(z) = D^*D_0 + B^*QB_0 + zC_1(I_n - zA_0)^{-1}B_0. \tag{4.10}$$

Recall that the operators D_0 and B_0 are given by (4.5) and (4.4), respectively. Since $C_1 = D^*C_0 + B^*QA_0$, it is clear that $D^*D_0 + B^*QB_0 = D_1$, where D_1 is defined in Theorem 1.1.

The next step is to show that $B_0 = -(I_n - PQ)^{-1}BD_1$. To accomplish this we compute BD_1. Let us set $\Lambda = (R_0 - \Gamma^*Q\Gamma)^{-1}$. Then

$$\begin{aligned} BD_1 &= B(D^* - B^*Q\Gamma)\Lambda + BC_1(I_n - PQ)^{-1}PC_0^* \\ &= BD^*\Lambda - BB^*Q\Gamma\Lambda + BC_1(I_n - PQ)^{-1}PC_0^* \\ &= BD^*\Lambda - PQ\Gamma\Lambda + APA^*Q\Gamma\Lambda + BC_1(I_n - PQ)^{-1}PC_0^* \\ &= (I_n - PQ)\Gamma\Lambda + (BD^* - \Gamma)\Lambda + APA^*Q\Gamma\Lambda + BC_1(I_n - PQ)^{-1}PC_0^*. \end{aligned}$$

We proceed with

$$\begin{aligned}
&(BD^* - \Gamma)\Lambda + APA^*Q\Gamma\Lambda + BC_1(I_n - PQ)^{-1}PC_0^*\\
&\quad = -APC^*\Lambda + APA^*Q\Gamma\Lambda + BC_1(I_n - PQ)^{-1}PC_0^*\\
&\quad = -APC_0^* + BC_1(I_n - PQ)^{-1}PC_0^*\\
&\quad = \left(BC_1(I_n - PQ)^{-1} - A\right)PC_0^*\\
&\quad = (BC_1 - A(I_n - PQ))\,(I_n - PQ)^{-1}PC_0^* \quad [\text{by } (3.21)]\\
&\quad = (A(I_n - PQ) - A(I_n - PQ) - (I_n - PQ)A_0)\,(I_n - PQ)^{-1}PC_0^*\\
&\quad = -(I_n - PQ)A_0(I_n - PQ)^{-1}PC_0^*.
\end{aligned}$$

Thus

$$\begin{aligned}
BD_1 &= (I_n - PQ)\Gamma\Lambda - (I_n - PQ)A_0(I_n - PQ)^{-1}PC_0^*\\
&= -(I_n - PQ)(-\Gamma\Lambda + A_0(I_n - PQ)^{-1}PC_0^*)\\
&= -(I_n - PQ)B_0.
\end{aligned}$$

We conclude with the statements in the final paragraph of the theorem. First we prove the result about McMillan degrees. To do this assume that the number n in the state space representation (1.5) is chosen as small as possible. In that case, $\delta(G) = n$. Since the matrix A_0 in the state space representation of X has the same size as A, we conclude that $\delta(X) \le n$. Thus $\delta(X) \le \delta(G)$, as desired.

Finally, we prove (1.10). The left hand side of (1.10) can be written as $D_1^*ND_1$, where

$$N = I_p + B^*(I_n - QP)^{-1}\left(\sum_{\nu=0}^{\infty}(A_0^*)^\nu C_1^*C_1A_0^\nu\right)(I_n - PQ)^{-1}B.$$

From (3.23) we know that $\sum_{\nu=0}^{\infty}(A_0^*)^\nu C_1^*C_1A_0^\nu = Q - QPQ$. It follows that

$$\begin{aligned}
N &= I_p + B^*(I_n - QP)^{-1}(Q - QPQ)(I_n - PQ)^{-1}B\\
&= I_p + B^*Q(I_n - PQ)^{-1}B.
\end{aligned}$$

Thus $D_1^*ND_1$ is equal to the right side of (1.10). □

A direct proof that X is a solution of (1.1)**.** Let X be as in Theorem 1.1. From our construction of X we know that X is a solution of (1.1). This fact can also be checked directly by using (3.18) and (3.22). To see this, recall that X is given by (1.9). By using (3.18) we compute that

$$\begin{aligned}
G(z)X(z) &= G(z)D_1 - zG(z)C_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}BD_1\\
&= DD_1 + zC(zI_n - A)^{-1}BD_1 - zC(zI_n - A)^{-1}BD_1\\
&= DD_1.
\end{aligned}$$

It remains to show $DD_1 = I_m$. For this purpose we use (3.22). As before put $\Lambda = (R_0 - \Gamma^* Q\Gamma)^{-1}$. We compute

$$\begin{aligned} DD_1 &= (DD^* - DB^*Q\Gamma)\Lambda + DC_1(I - PQ)^{-1}PC_0^* \\ &= (DD^* - DB^*Q\Gamma)\Lambda + CPC_0^* \\ &= (DD^* - \Gamma^*Q\Gamma + CPA^*Q\Gamma)\Lambda + CP(C^* - A^*Q\Gamma)\Lambda \\ &= (DD^* + CPC^* - \Gamma^*Q\Gamma)\Lambda \\ &= (R_0 - \Gamma^*Q\Gamma)\Lambda = I_m. \end{aligned}$$

Hence $DD_1 = I_m$, and $G(z)X(z) = I_m$.

5. Two Examples

In this section we present two examples. The first is a simple example for which all computations can be carried out by hand. For the second example we use MatLab procedures to obtain the desired formulas.

Example **1.** Consider the 1×2 matrix function $G(z) = [1+z \quad -z]$. Obviously,

$$[1 + z \quad - z]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1.$$

Hence the equation $G(z)X(z) = 1$ has a stable rational matrix solution. The solution $[1 \quad 1]^{\mathrm{T}}$ in the above equation is not the least squares solution but it is the optimal corona solution (that is, the solution of minimal H^∞ norm); see [29]. We shall use Theorem 1.1 to compute the least squares solution.

A minimal realization of G is given by

$$A = 0, \quad B = [1 \quad -1], \quad C = 1, \quad D = [1 \quad 0].$$

Solving the symmetric Stein equation (3.4) for this case, we see that $P = 2$. Since $G(z)G^*(z) = 3 + z + z^{-1}$, we have $R_0 = 3$ and $\Gamma = 1$. The Riccati equation (3.9) now becomes

$$Q = \frac{1}{3 - Q},$$

and the stabilizing solution is given by $q = \frac{1}{2}(3-\sqrt{5})$. We see that $qP = 3-\sqrt{5}$ is in the open unit disc, as expected.

Inserting this data into the formulas for C_0 and A_0 in (3.11), we obtain $C_0 = q$ and $A_0 = -q$. Computing C_1 and D_1 from Theorem 1.1, and using the fact that $q = 1/(3 - q)$, we arrive at

$$C_1 = \begin{bmatrix} q \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} q^2 = q\begin{bmatrix} 1-q \\ q \end{bmatrix}, \tag{5.1}$$

$$D_1 = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} q\right) q + q\begin{bmatrix} 1-q \\ q \end{bmatrix}\frac{2}{1-2q}q = \frac{q}{1-2q}\begin{bmatrix} 1-q \\ q \end{bmatrix} \tag{5.2}$$

It follows that $-(1 - Pq)^{-1}BD_1 = -(1 - Pq)^{-1}q = -q(1 - 2q)^{-1}$. Using Theorem 1.1, we see that for this case the least squares solution X of (1.1) is given by

$$\begin{aligned} X(z) &= D_1 - \frac{z}{1+zq} C_1 (1-Pq)^{-1} B D_1 \\ &= \frac{q}{1-2q} \begin{bmatrix} 1-q \\ q \end{bmatrix} - \frac{z}{1+zq} \left(\frac{q^2}{1-2q} \right) \begin{bmatrix} 1-q \\ q \end{bmatrix} \\ &= \frac{1}{1-2q} \begin{bmatrix} 1-q \\ q \end{bmatrix} \left(q - \frac{q^2 z}{1+zq} \right). \end{aligned}$$

In other words,

$$X(z) = \frac{q}{1-2q} \begin{bmatrix} 1-q \\ q \end{bmatrix} (1+zq)^{-1}, \quad \text{where } q = \frac{1}{2}(3-\sqrt{5}).$$

Let us check directly that X is indeed a solution of (1.1):

$$\begin{aligned} \begin{bmatrix} 1+z & -z \end{bmatrix} X(z) &= \frac{q}{1-2q} \left((1+z)(1-q) - zq \right) (1+zq)^{-1} \\ &= \frac{q}{1-2q} (1+z-q-2qz)(1+zq)^{-1} \\ &= \frac{q}{1-2q} \left((1-2q)z + (1-q) \right) (1+zq)^{-1} \\ &= qz(1+zq)^{-1} + \frac{q-q^2}{1-2q}(1+zq)^{-1} = 1. \end{aligned}$$

The last equality holds because $(q-q^2)/(1-2q) = 1$. To obtain this identity recall that q satisfies $q = 1/(3-q)$ or $q^2 - 3q + 1 = 0$.

Example **2.** Consider the 2×3 matrix function $G(z)$ given by

$$G(z) = \begin{bmatrix} 1 & z+z^2 & z^2 \\ 0 & 1+z & z \end{bmatrix}. \tag{5.3}$$

We have

$$\begin{bmatrix} 1 & z+z^2 & z^2 \\ 0 & 1+z & z \end{bmatrix} \begin{bmatrix} 1 & -z \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = I_2. \tag{5.4}$$

Hence the equation $G(z)X(z) = I_2$ has a stable rational matrix solution. Our aim is to compute the least squares solution. To do this we apply the method provided by Theorem 1.1.

A minimal realization for G is given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = I_2, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5.5}$$

For this case the solution of the symmetric Stein equation (3.4) is given by

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

Furthermore, one computes that

$$R_0 = DD^* + CPC^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix},$$

$$\Gamma = BD^* + APC^* = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

Since in this case all matrices are real, the unique stabilizing solution Q of the corresponding Riccati equation is real symmetric. Hence (cf., Section 12.7 in [18]) we can assume that Q is of the form

$$Q = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix},$$

and one computes that the Riccati equation (3.9) takes the form

$$\begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & q_1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -q_1 & 1-3q_1-q_2 \end{bmatrix}$$
$$\times \begin{bmatrix} 4-q_1 & 1-3q_1-q_2 \\ 1-3q_1-q_2 & 3-9q_1-6q_2-q_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -q_1 \\ 0 & 1-3q_1-q_2 \end{bmatrix}.$$

To find the stabilizing solution by hand is a problem. However we can use the standard MatLab command 'dare' from the MatLab control toolbox to compute the stabilizing solution Q for the case considered here. This yields:

$$Q = \begin{bmatrix} 0.2764 & -0.1056 \\ -0.1056 & 0.4223 \end{bmatrix}.$$

By using this Q in (3.11) we obtain

$$A_0 = \begin{bmatrix} 0.0403 & -0.1613 \\ 0.1056 & -0.4223 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0.2764 & -0.1056 \\ -0.1056 & 0.4223 \end{bmatrix}.$$

Inserting this data in the formulas of Theorem 1.1 and using MatLab to make the computations we arrive at

$$C_1 = \begin{bmatrix} 0.2764 & -0.1056 \\ -0.0652 & 0.2610 \\ 0.0403 & -0.1613 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -0.6180 \end{bmatrix},$$

$$-(I_2 - PQ)^{-1}BD_1 = \begin{bmatrix} 0 & -4.6180 \\ 0 & -2.6180 \end{bmatrix}.$$

This then shows that the least squares solution $X(z)$ is given by

$$X(z) = \begin{bmatrix} 1 & -z \\ 0 & \frac{1}{1+0.3820z} \\ 0 & \frac{-0.618}{1+0.3820z} \end{bmatrix}. \tag{5.6}$$

Remark on coprime factorization. In this final remark we use Example 2 above to show that the least squares solution (5.6) cannot be derived via the double coprime factorization approach in Chapter 4 of [30]. To see this, put

$$G_1(z) = \begin{bmatrix} 1 & z+z^2 \\ 0 & 1+z \end{bmatrix}, \; G_2(z) = \begin{bmatrix} z^2 \\ z \end{bmatrix}, \; P(z) = G_1(z)^{-1}G_2(z) = \begin{bmatrix} 0 \\ \frac{z}{1+z} \end{bmatrix}.$$

Note that $P(z) = G_1(z)^{-1}G_2(z)$ is a left coprime factorization. Using the matrices in (5.5), we see that

$$G_1(z) = I_2 + zC(I_2 - zA)^{-1}B_1, \quad G_2(z) = zC(I_2 - zA)^{-1}B_2,$$
$$\text{where } B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Furthermore,

$$P(z) = zC\left(I_2 - zA_1\right)^{-1}B_2, \quad \text{with } A_1 = A - B_1C = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now let us apply the discrete time analogue of the results of Section 4.2 in [30] to this realization of $P(z)$. Choose $K = [k_1 \quad k_2]$ such that

$$A_2 := A_1 + B_2K = \begin{bmatrix} 0 & 0 \\ k_1 & -1+k_2 \end{bmatrix} \text{ is stable.} \tag{5.7}$$

Put

$$H_1(z) = I_2 - zC(I_2 - zA_2)^{-1}B_1, \quad H_2(z) = zK(I_2 - zA_2)^{-1}B_1.$$

Then, according to the discrete time analogue of Theorem 4 in Section 4.2 of [30] (see also Section 21.5.2 in [31]), we have $G_1(z)H_1(z) + G_2(z)H_2(z) = I_2$. Hence for any choice of k_1 and k_2 in (5.7),

$$H(z) := \begin{bmatrix} H_1(z) \\ H_2(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ k_1 & k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ zk_1 & 1+z-zk_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

is a stable rational matrix function satisfying $G(z)H(z) = I_2$. Moreover, $\delta(H) \leq \delta(G)$. However, for any choice of k_1 and k_2 the value of H at zero is different for the value at zero of X given by (5.6). Thus there is no choice of k_1, k_2 such that $H = X$, and hence we cannot obtain the least-squares solution via the above coprime factorization method.

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