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Theories with the Independence Property

Abstract. A first-order theory $\mathcal{T}$ has the *Independence Property* provided $\mathcal{T} \vdash (Q)(\Phi \Rightarrow \Phi_1 \vee \cdots \vee \Phi_n)$ implies $\mathcal{T} \vdash (Q)(\Phi \Rightarrow \Phi_i)$ for some i whenever $\Phi, \Phi_1, \ldots, \Phi_n$ are formulae of a suitable type and (Q) is any quantifier sequence. Variants of this property have been noticed for some time in logic programming and in linear programming.

We show that a first order theory has the independence property for the class of basic formulae provided it can be axiomatised with Horn sentences. This condition, called *crispness*, is to some extent also necessary, but the properties are not equivalent.

The existence of so-called *free models* is a useful intermediate result. The independence property is also a tool to decide that a sentence cannot be deduced. We illustrate this with the case of the classical Carathéodory theorem for Pasch-Peano geometries.

Keywords: Complete theory, Crisp theory, Free model, Horn sentence, Independence property, Reduced product.

1. Introduction

I once noticed a student making the following kind of mistake.

> Given that $\Phi(x)$ implies $\Phi_1(x) \vee \cdots \vee \Phi_n(x)$ for all x, consider an arbitrary x satisfying Φ. Then x satisfies Φ_i for some i. Since an arbitrary x satisfying Φ satisfies this Φ_i, conclude that $\Phi(x)$ implies $\Phi_i(x)$ for all x.

This "result", and in fact, this naive use of "an arbitrary x", can sometimes be justified. We say that a first-order theory $\mathcal{T}$ has the *Independence Property* with respect to a specified class $\mathcal{F}$ of formulae provided the following holds. If $\mathcal{T} \vdash (Q)(\Phi \Rightarrow \Phi_1 \vee \cdots \vee \Phi_n)$ then $\mathcal{T} \vdash (Q)(\Phi \Rightarrow \Phi_j)$ for some j whenever $\Phi, \Phi_1, \ldots, \Phi_n$ are in $\mathcal{F}$. The expression (Q) refers to an arbitrary finite quantifier sequence. We explicitly include the case of $(Q) \bigvee_{i=1}^{n} \Phi_i$.

Such a property shows up in constraint logic programming (see [9] for a survey), theories of feature trees ([1], [2], [3], [18]), and constraint systems in real linear programming ([11]). In each of the cited examples, the class of formulae is some collection of so-called *constraints*, which is part of a standard class of "basic formulae" (comparable with *facts* in logic programming; see below).

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Occasionally, there are some differences with our viewpoint. In linear programming, for instance, there are actually two classes of formulae involved in the implications under consideration: one for "polyhedral" formulae in the antecedent and one for "affine" formulae in the consequent. Moreover, [11] takes the viewpoint of a sequent calculus. Sometimes the independence property is expressed in terms of satisfaction by a given model: If $M \models (Q)(\Phi \Rightarrow \Phi_1 \vee \cdots \vee \Phi_n)$ then $M \models (Q)(\Phi \Rightarrow \Phi_j)$ for some j. Yet another format of the property occurs, involving entailment of implications of type

$$(Q)(\Phi \Rightarrow \Phi_1 \vee \cdots \vee \Phi_n) \Rightarrow \vee_{j=1}^{n}(Q)(\Phi \Rightarrow \Phi_j).$$

Finally, in nearly all cases, only universal quantifiers are considered (existential ones in contrapositional formulations).

Most of the cited papers consider the independence property as a part of a decision procedure for solving equalities. Interestingly, it has also been used to prove that the theory at hand is complete (the theory of free equality with infinitely many operators [14]; the theory CFT of constraint feature trees [2]; the theory FT of feature trees [3]). We take the viewpoint of mathematical logic and model theory to develop some general results about the independence property. For most of the variant properties encountered, we give evidence that they amount to the same. Algorithmic features are not considered.

In section 2 we describe the class of basic formulae and a general type of "crisp" axioms which provide sufficient conditions for a theory to have the independence property for basic formulae. (Crisp sentences are logically the same as *Horn sentences* [5, p. 407].) Conversely, we show that a theory with the independence property, of which all non-crisp axioms are universal or positive, is logically equivalent with a crisp theory. We also consider a "strong" independence property, where (in the above notation) Φ is allowed to be a Horn formula. Adding crisp axioms to a theory with a "strong" independence property preserves the latter property. Finally, we found theories with the (strong) independence property which are not crisp, but such theories seem rare.

An intermediate result is the existence of so-called *free models* for crisp theories. Basic sentences, valid in a free model, can be deduced. Such models embody the famous *closed-world assumption*: if a fact can't be proved, consider it false. On a related topic, an *initial model* M of a theory has the property that for each model M' there is a unique homomorphism $M \to M'$. Such models are easily seen to be free. The existence of initial models of equational theories is known since the early days of universal algebra

([4, p. 73]); it was extended later to theories with universal definite Horn sentences. It is shown in [15, cor. 4.6] that the existence of initial models requires a theory with $\forall\exists$ axioms. In fact, such theories can be axiomatized with $\forall\exists$ Horn sentences and (up to a technical additional condition) this *characterises* theories with initial models ([15, thm. 5.9]). While initial models of necessity have a trivial automorphism group, we show that crisp theories with non-trivial models do have free models with arbitrarily large automorphism groups.

(Counter) examples and use of the independence property are discussed in section 3. A fairly detailed description of *Pasch-Peano theory* is included, with a discussion of *Carathéodory's theorem*.

The name "Independence property", occuring in several papers on feature trees, is preferred among more specific names such as "independence of negative constraints" or "independence of disequalities". The oldest reference to the property seems to be Colmerauer [6]. The most general approach so far seems to be [11], where connections between (a sequent formulation of) the independence property and Horn sentences can be found. However, example 2 in this paper is an overstatement (see the remark following lemma 2.3 below).

Do not confuse the independence property with the "disjunction property", which refers to a property in intuitionistic logic [8].

2. The main results

2.1. Preliminaries

For undefined concepts and notation in logic we refer to [5] and [16]. Let $\mathcal{L}$ be a first-order language. Given a formula Φ in $\mathcal{L}$, an array X of n distinct variables, and an array T of n terms, then $\Phi[T \leftarrow X]$ denotes the result of substituting the i^{th} variable by the i^{th} term ($i = 1, \dots, n$). If X is the array of all free variables of Φ, then $\forall\Phi$ stands for the universal closure $\forall X \Phi$ of Φ. Similarly, the existential closure $\exists X \Phi$ is abbreviated by $\exists\Phi$. For X empty, both expressions reduce to Φ.

A *sentence* is a formula without free variables whereas a *proposition* is a formula with no variables at all. An *atomic formula* is of type $P(t_1, \dots, t_n)$, where P is an $n-$ary predicate symbol of $\mathcal{L}$ and $t_1, \dots, t_n$ are terms of $\mathcal{L}$, or (in a language with equality) an equation of terms. An *elementary formula* is a conjunction of one or more atomic formulae and a *basic formula* is a formula of type $(Q)\Phi$, where (Q) represents a finite sequence of quantifiers (universal or existential) and Φ is elementary.

A theory in a first-order language is just a set of sentences (axioms) in that language. Two theories are (logically) the same if they have the same consequences. A *universal* theory has an axiom system consisting of sentences of type $\forall\Phi$, where Φ contains no quantifiers. When talking about models, an interpretation is (usually) implicitly assumed.

Part (a) of the following fact is well-known, [5]. Part (b) is a simple form of Herbrand's Theorem, which follows easily from (a) and [16, prop. 10.92].

PROPOSITION 2.1. *Let $(\mathcal{L},\mathcal{T})$ be a universal theory with $\mathcal{L}$ having at least one constant.*

(a) *For each model M of $(\mathcal{L},\mathcal{T})$ there is a submodel $M' \subseteq M$ of $(\mathcal{L},\mathcal{T})$ consisting of all interpreted variable-free terms.*

(b) *Let Φ be a formula in $\mathcal{L}$ with only one free variable x such that $(\mathcal{L},\mathcal{T}) \vdash \exists x\Phi$. Then there exist variable-free terms $t_1,\ldots,t_m$ of $\mathcal{L}$ such that*

$$(\mathcal{L},\mathcal{T}) \vdash \Phi[t_1 \leftarrow x] \vee \cdots \vee \Phi[t_m \leftarrow x].$$

2.2. Crisp theories

Below, (Q) represents a finite (possibly empty) sequence of quantifiers. We may (and always will) assume that all variables occurring in a quantifier array are distinct. A formula will be called *crisp* provided it is of type

$$(Q)(\Psi_1 \wedge \cdots \wedge \Psi_n),$$

where each formula Ψ_i has one of the following formats.

1. Φ, with Φ a basic formula.
2. $\neg\Phi$, with Φ a basic formula.
3. $\Phi_1 \Rightarrow \Phi_2$, with Φ_1 and Φ_2 basic formulae.

Quantifiers occuring in the basic formulae can be replaced with quantifiers to the right of (Q). In this prenex normal form, crisp sentences amount to the same as *Horn sentences*; cf. [5, p. 407].

A theory is called *crisp* provided it has an equivalent axiom system, consisting of crisp sentences. Some crisp theories are discussed in section §3.

We will occasionally make use of *reduced products* or *reduced powers* of models. We refer to [5, chapt. 4] for these constructions.

THEOREM 2.2. *A theory is crisp iff it is stable under reduced products.*

In this form (without the Continuum Hypothesis), the result is due to Galvin ([7, thm. 2]). As a particular and rather straightforward case, proved originally by Horn, crisp sentences are stable under (direct) products.

LEMMA 2.3. *Let $(\mathcal{L}, \mathcal{T})$ be a crisp theory in a language $\mathcal{L}$ and let P_j for $j \in J$ be atomic propositions in $\mathcal{L}$. If $\mathcal{T} \cup \{\neg P_j\}$ is consistent for each j, then $\mathcal{T} \cup \{\neg P_j : j \in J\}$ is consistent.*

PROOF. For each $j \in J$ let $M_j \models \mathcal{T} \cup \{\neg P_j\}$ and let $M := \prod_j M_j$. We may assume that each axiom of $\mathcal{T}$ is a Horn sentence. Then $M \models \mathcal{T}$. The atomic proposition P_j is not valid in M since it fails at the j^{th} factor. ∎

The conclusion of the previous lemma can be restated in this form: if P_i for $i = 1, \ldots, n$ are atomic propositions and $\mathcal{T} \vdash \vee_{i=1}^{n} P_i$, then $\mathcal{T} \vdash P_i$ for some i. This is a primitive form of the independence property, mentioned in an exercise of [17, pp. 94-95] as early as 1967. In [11, example 2], it is claimed that the above property actually holds for universally closed definite Horn clauses. No proof is given, but it is suggested that this, too, is a consequence of the stability of models under products. The following is a counterexample (without quantifiers). Let P_i and Q_i for $i = 1, 2$ be atomic propositions. Consider the theory $\mathcal{T} := \{P_i \Rightarrow Q_i : i = 1, 2\}$. Then

$$\mathcal{T} \vdash (P_1 \Rightarrow Q_2) \vee (P_2 \Rightarrow Q_1)$$

whereas $\mathcal{T} \not\vdash P_1 \Rightarrow Q_2$ and $\mathcal{T} \not\vdash P_2 \Rightarrow Q_1$.

To improve on the independence property, we have to proceed more carefully.

PROPOSITION 2.4. *A consistent and crisp universal theory $(\mathcal{L}, \mathcal{T})$ has a model M such that for each basic sentence Φ in $\mathcal{L}$,*

$$M \models \Phi \quad \textit{iff} \quad (\mathcal{L}, \mathcal{T}) \vdash \Phi.$$

PROOF. We expand $\mathcal{L}$ to a language $\mathcal{L}'$ with a countably infinite sequence of additional constants. By Lemma 2.3 and Prop. 2.1(a), there is a model M of $(\mathcal{L}', \mathcal{T})$ such that

(1) $M \models \neg P$ for each atomic proposition P of $\mathcal{L}'$ such that $\neg P$ is consistent with $\mathcal{T}$.

(2) Each member of M is the interpretation of a variable-free term of $\mathcal{L}'$.

We verify that this model is as required. Let $M \models (Q)\Phi$, where Φ is a quantifier-free elementary formula of $\mathcal{L}'$ and (Q) represents a sequence of quantifiers involving different variables. If (Q) is the empty sequence, then Φ is an elementary proposition and the result follows from (1). Proceeding by induction, suppose (Q) has length $n > 0$, let $M \models (Q)\Phi$, and suppose the result valid for quantifier sequences of length $n-1$. We have two cases to consider.

(i) $(Q) = \exists x(Q')$, where x does not occur in (Q'). Then there is $m \in M$ satisfying the interpretation of $(Q')\Phi$. By assumption (2), m is the interpretation of a variable-free term t of $\mathcal{L}'$. So $M \models (Q')\Phi[t \leftarrow x]$ and by the induction hypothesis, $(\mathcal{L}', \mathcal{T}) \vdash (Q')\Phi[t \leftarrow x]$. Therefore, $(\mathcal{L}', \mathcal{T}) \vdash \exists x(Q')\Phi$.

(ii) $(Q) = \forall x(Q')$, where x does not occur in (Q'). Let c be one of the additional constants of $\mathcal{L}'$ not occurring in Φ. We have $M \models (Q')\Phi[c \leftarrow x]$ and by the induction hypothesis, $(\mathcal{L}', \mathcal{T}) \vdash (Q')\Phi[c \leftarrow x]$. Generalization on the constant c then yields $(\mathcal{L}', \mathcal{T}) \vdash \forall x(Q')\Phi$, [16, prop. 11.17].

After completing the induction, we see that the result follows by restricting to formulae in $\mathcal{L}$. ■

By a *free model* of a theory $(\mathcal{L}, \mathcal{T})$ is meant a model M with

$$M \models \Phi \quad \text{iff} \quad (\mathcal{L}, \mathcal{T}) \vdash \Phi$$

for each basic sentence Φ in $\mathcal{L}$. Note that every model of a *complete* theory is free. In fact, it satisfies the condition for *all* sentences, not just basic ones. For crisp theories, the restriction to basic sentences is essential; the defining property may fail even for sentences of type $\neg\Phi$ or $\Phi_1 \Rightarrow \Phi_2$ with Φ, Φ_1, Φ_2 basic.

We now arrive at the first main result, linking crispness with the existence of free models and with the independence property. In fact, we can obtain a slightly stronger conclusion involving the following version of the property: A theory $(\mathcal{L}, \mathcal{T})$ has the *Strong Independence Property* provided $\mathcal{T} \vdash (Q)(\Phi \Rightarrow \Phi_1 \vee \cdots \vee \Phi_n)$ entails $\mathcal{T} \vdash (Q)(\Phi \Rightarrow \Phi_j)$ for some j whenever $\Phi_1, \ldots, \Phi_n$ are basic formulae and Φ is crisp (rather than just basic). As usual, the expression (Q) refers to an arbitrary finite quantifier sequence. Again, we explicitly include the case of $(Q) \bigvee_{j=1}^{n} \Phi_j$. (The phrase "with respect to the class of basic formulae" has been omitted above, and will mostly be omitted in the future.)

THEOREM 2.5. (1) *A crisp theory has the strong independence property.*
(2) *A consistent crisp theory has free models.*

PROOF. We establish part (1) first for a *universal* crisp theory $(\mathcal{L}, \mathcal{T})$. Let Φ_j for $j = 0, 1, \ldots, n$ be formulae in $\mathcal{L}$ of which Φ_0 is crisp, the other ones being basic, such that

$$\mathcal{T} \vdash (Q)(\Phi_0 \Rightarrow \vee_{j=1}^{n} \Phi_j).$$

We may assume that no free variables occur in the deduced formula and that Φ_0 has no quantifiers (we can live with universal quantifiers).

The aimed result is valid in case the quantifier sequence (Q) is empty. Indeed, under the given circumstances, all formulae Φ_j are sentences. If the theory $\mathcal{T} \cup \{\Phi_0\}$ is inconsistent, there is nothing left to be proved. So assume $\mathcal{T}$ is consistent with Φ_0. We have a universal crisp theory $(\mathcal{L}, \mathcal{T} \cup \{\Phi_0\})$, which has a free model M, cf. Prop. 2.4. By the Deduction Theorem,

$$(\mathcal{L}, \mathcal{T} \cup \{\Phi_0\}) \vdash \vee_{j=1}^{n} \Phi_j,$$

and hence we find that

$$M \models \vee_{j=1}^{n} \Phi_j.$$

The expression to the right is a disjunction of sentences. Hence there is $j \in \{1, \ldots, n\}$ with

$$M \models \Phi_j.$$

Now Φ_j is a basic sentence in $\mathcal{L}$. As M is a free model, we find

$$(\mathcal{L}, \mathcal{T} \cup \{\Phi_0\}) \vdash \Phi_j,$$

whence $(\mathcal{L}, \mathcal{T}) \vdash \Phi_0 \Rightarrow \Phi_j$ by the Deduction Theorem again.

We allow Φ_0 to be absent. In this case, the steps involving the deduction theorem are skipped.

Suppose next that the quantifier array (Q) has length > 0. Let $\mathcal{L}'$ be the language obtained from $\mathcal{L}$ by adding a countably infinite sequence of new constants. Starting with the sentence

$$(Q)(\Phi_0 \Rightarrow \vee_{j=1}^{n} \Phi_j)$$

(stage 0), we shall peal off the quantifiers of (Q) from left to right. Let Θ denote the formula following the sequence (Q). At each stage k, we shall obtain a statement of type

$$(\mathcal{L}', \mathcal{T}) \vdash \bigvee_{\alpha \in I_k} (Q')\Theta[T_\alpha \leftarrow X_k, C_\alpha \leftarrow Y_k].$$

where (Q') is (Q) minus the leftmost k quantifiers. The variables corresponding to the missing quantifiers fall into two arrays X_k and Y_k. Each member of X_k comes from an existential quantifier and each member of Y_k comes from a universal quantifier. The members of C_α are distinct constants of $\mathcal{L}'$ not in $\mathcal{L}$ and members of T_α are variable-free terms of $\mathcal{L}'$. Each index $\alpha \in I_k$ is a sequence of length k, consisting of non-negative numbers.

At the initial stage $k = 0$, the index collection I_0 contains only the empty sequence, the disjunction has only one member, and no substitution takes place. Suppose at stage $k \geq 0$ we have obtained the above statement with a nonempty right end (Q') and indices $\alpha \in I_k$. We have to distinguish two cases.

(I) $(Q') = \forall y(Q'')$, where y does not occur in (Q''). For each $\alpha \in I_k$, take a distinct constant c_α of $\mathcal{L}'$ not in $\mathcal{L}$ and not occurring in the formula following (Q''). Then

$$(\mathcal{L}', \mathcal{T}) \vdash \bigvee_{\alpha \in I_k} (Q'')(\Theta[T_\alpha \leftarrow X_k, C_\alpha \leftarrow Y_k][c_\alpha \leftarrow y]).$$

We introduce a new index set I_{k+1}, whose members are sequences of type $\alpha 0$ with $\alpha \in I_k$. The array T_α is renamed $T_{\alpha 0}$; X_k is renamed X_{k+1}; the array $C_{\alpha 0}$ is the old array C_α with the new constant c_α appended, and y is appended to Y_k to form the array Y_{k+1}.

(II) $(Q') = \exists x(Q'')$, where x does not occur in (Q''). We can move the quantifier $\exists x$ in front of the disjunction $\bigvee_\alpha$. The theory $\mathcal{T}$ being universal, we can apply Prop. 2.1(b). There are variable-free terms t_i of $\mathcal{L}$' for $i = 1, \ldots, n_{k+1}$, such that

$$(\mathcal{L}', \mathcal{T}) \vdash \bigvee_i \bigvee_{\alpha \in I_k} (Q'')(\Theta[T_\alpha \leftarrow X_k, C_\alpha \leftarrow Y_k][t_i \leftarrow x]).$$

We introduce a new index set I_{k+1} with indexes of type αi, where $i = 1, \ldots, n_{k+1}$. For each new index αi, the array $T_{\alpha i}$ is the old array T_α with the term t_i appended. The variable x is appended to X_k and the array $C_{\alpha i}$ is just the old array C_α.

This completes the induction, leaving us with a deduction

$$(\mathcal{L}', \mathcal{T}) \vdash \bigvee_\alpha \left(\Phi_0[T_\alpha, C_\alpha] \Rightarrow \vee_{j=1}^n \Phi_j[T_\alpha, C_\alpha] \right) \tag{1}$$

with the listed properties and an empty quantifier sequence in front of the implications. Here, and in the sequel, we are using $[T_\alpha, C_\alpha]$ as shorthand for the substitution $[T_\alpha \leftarrow X, C_\alpha \leftarrow Y]$. Elementary proposition logic yields that

$$(\mathcal{L}', \mathcal{T}) \vdash \wedge_\alpha \Phi_0[T_\alpha, C_\alpha] \Rightarrow \vee_\alpha \vee_{j=1}^n \Phi_j[T_\alpha, C_\alpha]$$

Note that $\wedge_\alpha \Phi_0[T_\alpha, C_\alpha]$ is crisp. The no-quantifier case of the independence property yields indices $\alpha*$ and $j*$ such that

$$(\mathcal{L}', \mathcal{T}) \vdash \wedge_\alpha \Phi_0[T_\alpha, C_\alpha]\big) \Rightarrow \Phi_{j*}[T_{\alpha*}, C_{\alpha*}].$$

This can be weakened to

$$(\mathcal{L}', \mathcal{T}) \vdash \wedge_\alpha \Phi_0[T_\alpha, C_\alpha] \Rightarrow \vee_\alpha \Phi_{j*}[T_\alpha, C_\alpha],$$

and again by elementary proposition logic we obtain

$$(\mathcal{L}', \mathcal{T}) \vdash \bigvee_\alpha \big(\Phi_0[T_\alpha, C_\alpha] \Rightarrow \Phi_{j*}[T_\alpha, C_\alpha]\big).$$

Now we trace back our steps through the quantifier elimination process. Suppose we are at stage $k > 0$ and that we recovered a statement of type

$$(\mathcal{L}', \mathcal{T}) \vdash \bigvee_{\alpha \in I_k} (Q')\big(\Phi_0[T_\alpha, C_\alpha] \Rightarrow \Phi_{j*}[T_\alpha, C_\alpha]\big),$$

where we restored a right end (Q') of (Q) in front of each disjunction term. If we originally reached this stage by case I, all indexes are of type $\alpha 0$ with $\alpha \in I_{k-1}$. The terms array $T_{\alpha 0}$ is just T_α. In the constants array $C_{\alpha 0}$, the last element c_α is removed, and the resulting array is C_α. By generalisation on constants, c_α may be replaced by the k-th variable y in the original quantifier arrangement (Q) and the currently restored sequence is $\forall y(Q')$ regardless of α.

If we originally reached the current stage by case II, all indexes are of type αi with $\alpha \in I_{k-1}$ and $i = 1, \ldots, n_{k+1}$. The array C_α is the same as $C_{\alpha i}$. The array T_α obtains from $T_{\alpha i}$ by deleting the last term t_i. At each position where the term t_i was introduced, we now put back the original variable x and we let each disjunction term be preceded by $\exists x$ (so-called $\exists$-introduction). Note that the disjunction terms, indexed by αi with fixed α, are now identical. Only one copy with index α is maintained.

After completing the induction we obtain

$$(\mathcal{L}', \mathcal{T}) \vdash (Q)(\Phi_0 \Rightarrow \Phi_{j*}).$$

The additional constants of $\mathcal{L}'$ are no longer involved and we can restrict ourselves to the language $\mathcal{L}$.

To obtain the result for general crisp theories, we use a method of *restricted skolemisation*. Given a sentence $(Q)\Theta$, written in prenex normal form with Θ quantifier-free, we can obtain a formula of type $\forall Y \Theta^{Sk}$ as follows. The universal quantifier sequence $\forall Y$ is what remains of (Q) after all existential quantifiers are dropped. This goes from left to right and at each stage (say, at the quantifier $\exists x$), a distinct new term is added to the language. This term depends *exactly* on all variables occurring in universal quantifiers in (Q) to the left of $\exists x$, and it will replace all occurrences of x in Θ. When all existential quantifiers are gone, we are left with the formula $\forall \Theta^{Sk}$. Note the logical implication $\forall Y \Theta^{Sk} \Rightarrow (Q)\Theta$.

We may assume that $\mathcal{T}$ is a collection of crisp sentences. Applying the above construction on each member of $\mathcal{T}$, we obtain a collection $\mathcal{T}^{Sk}$ of crisp and universal sentences in an extended language $\mathcal{L}^{Sk}$. By a routine argument, $(\mathcal{L}^{Sk}, \mathcal{T}^{Sk})$ is a conservative extension of the original theory. Suppose that

$$(\mathcal{L}, \mathcal{T}) \vdash (Q)(\Phi_0 \Rightarrow \vee_{j=1}^n \Phi_j),$$

where Φ_0 is crisp and Φ_j for $j = 1, \ldots, n$ are basic. By an observation above,

$$(\mathcal{L}^{Sk}, \mathcal{T}^{Sk}) \vdash (Q)(\Phi_0 \Rightarrow \vee_{j=1}^n \Phi_j).$$

As shown in part (i), we obtain an index $j \in \{1, \ldots, n\}$ with

$$(\mathcal{L}^{Sk}, \mathcal{T}^{Sk}) \vdash (Q)(\Phi_0 \Rightarrow \Phi_j).$$

The Skolem expansion being conservative, We conclude that

$$(\mathcal{L}, \mathcal{T}) \vdash (Q)(\Phi_0 \Rightarrow \Phi_j).$$

To see that a consistent crisp theory has free models (part (2)), we use a restricted Skolem expansion as explained in the previous paragraph. This expansion is consistent, universal, and crisp, whence it has a free model, cf. prop. 2.4. Restricting the interpretation to the original language yields a model as required. ∎

Recall ([16, p. 402]) that a formula is *positive* if it can be built exclusively with the connectives $\wedge, \vee, \forall, \exists$. Here is some additional information on free models and on the necessity of crispness for the independence property.

THEOREM 2.6. (1) *In a language with equality, a consistent crisp theory which entails $\exists x, y \neg(x \approx y)$ has free models with arbitrarily large automorphism groups.*

(2) *If a theory has the independence property and all non-crisp axioms are universal or positive, then the theory is logically equivalent to a crisp theory.*

(3) *If a theory with the strong independence property is extended with crisp axioms, then the extended theory has the strong independence property.*

PROOF. We first verify part (1). Let $(\mathcal{L}, \mathcal{T})$ be a crisp theory with $(\mathcal{L}, \mathcal{T}) \vdash \exists\neg(x \approx y)$. The elementary class of a crisp theory is stable under products by theorem 2.2. It follows that $\mathcal{T}$ (and, in fact, every consistent crisp extension of it) has infinite models.

The argument proving 2.5(2) shows that any free model of the (restricted) skolemisation of $(\mathcal{L}, \mathcal{T})$ is a free model of the original theory. Also, the skolemisation entails $\exists\neg(x \approx y)$. Therefore, without loss of generality, we may assume that $\mathcal{T}$ is universal.

Let $(L, <)$ be a linearly ordered set of any infinite cardinality and let $\mathcal{L}'$ arise from $\mathcal{L}$ by adding the members of L as constants.

Consider the collection Σ of all sentences of type

$$\neg\, a \approx b \qquad (a \neq b \in L) \tag{2}$$

$$\Phi[A \leftarrow X] \Leftrightarrow \Phi[B \leftarrow X], \tag{3}$$

where Φ is any formula in $\mathcal{L}$ with an array X of n free variables, and A, B, are $n-$tuples in L, listed in increasing order. By virtue of [5, lemma 3.3.9], the theory $(\mathcal{L}', \mathcal{T}' \cup \Sigma)$ is consistent for each consistent crisp extension $(\mathcal{L}, \mathcal{T}')$ of $(\mathcal{L}, \mathcal{T})$. In the sequel we shall consider the subset Σ' of Σ, consisting of the inequalities in (2) and all equivalences of type (3) involving atomic formulae Φ only. So, all sentences of Σ' are crisp. This leads us to a free model (M, I) of $\mathcal{T} \cup \Sigma'$ containing the set L. The theory $\mathcal{T} \cup \Sigma'$ being universal, we may replace M by the submodel, consisting of all interpreted terms $I(t)$ of $\mathcal{L}'$; cf. prop. 2.1(a). Note that this is another free model of $\mathcal{T} \cup \Sigma'$.

We first show that the restriction of M to $\mathcal{L}$ is a free model of $\mathcal{T}$. Suppose Φ is a basic sentence of $\mathcal{L}$ such that $M \models \Phi$. Then $\mathcal{T} \cup \Sigma' \vdash \Phi$. If $\mathcal{T} \not\vdash \Phi$, we have a consistent crisp extension $\mathcal{T} \cup \{\neg\Phi\}$ which (as observed above) must be consistent with Σ', a contradiction.

Let $C \subseteq M$ denote the set of interpreted constants of $\mathcal{L}'$; note that $L \subseteq C$. Modifying part of the argument in [5, thm. 3.3.11c], we achieve our goal by proving that every bijection $f : C \to C$, which restricts to an order isomorphism of L and which is the identity outside L, extends uniquely to an isomorphism of M. The prescription is as follows:

$f(I(t)(I(c_1), \ldots, I(c_n))) := I(t)(f(I(c_1)), \ldots, f(I(c_n)))$
with $c_1, \ldots, c_n$ constants of $\mathcal{L}'$.

This is mandatory by the requirements; hence uniqueness is not an issue.

To verify that we have a well-defined function, suppose that

$$I(s)(a_1,\dots,a_m) = I(t)(b_1,\dots,b_n).$$

Here, we assume that s,t are terms of $\mathcal{L}'$, $m,n \geq 0$, $a_1,\dots,a_m, b_1,\dots b_n \in L$, and $a_1 < \cdots < a_m$, $b_1 < \cdots < b_n$. Constants outside of L are not mentioned explicitly. This equation is an instance of an atomic formula $\Phi(x_1,\dots,x_p)$ in $\mathcal{L}'$, where $\max(m,n) \leq p \leq m+n$ and $c_1 < \cdots < c_p$ is an ordered listing of all a_i and b_j. By construction,

$$M \models \Phi(c_1,\dots,c_p) \Leftrightarrow \Phi(f(c_1),\dots,f(c_p)),$$

which yields

$$I(s)(f(a_1),\dots,f(a_m)) = I(t)(f(b_1),\dots,f(b_n)).$$

We next claim that the extension $f : M \to M$ is a homomorphism. This means

(i) $f(I(t)(x_1,\dots,x_n)) = I(t)(f(x_1),\dots,f(x_n))$ for each n-ary term symbol t of the language and for each $x_1,\dots,x_n \in M$.

(ii) if $I(P)(x_1,\dots,x_n))$ then $I(P)(f(x_1),\dots,f(x_n)))$ for each n-ary predicate symbol P of the language and for each $x_1,\dots,x_n \in M$.

The first statement holds by the prescription of the extension; the second statement follows with an argument as above by using suitable sentences in Σ'.

Observe that the identity function of C leads to the identity function on M. Hence, uniqueness of the induced function leads to its functoriality, from which it follows that an isomorphism of the ordered set L leads to an isomorphism of M.

To establish part (2) of the theorem, suppose $(\mathcal{L},\mathcal{T})$ is a theory with the independence property such that all non-crisp axioms are universal or positive. Let Θ be a non-crisp axiom of $\mathcal{T}$. Rewrite Θ in normal conjunctive form as

$$(Q) \wedge_{i\in I} (\vee_{p\in J_i} P^i_p \vee \vee_{n\in K_i} \neg N^i_n)$$

with a quantifier array (Q) and atomic propositions P^i_p and N^i_n of $\mathcal{L}$.

(i) Θ is a universal sentence. Rewrite Θ as

$$\wedge_{i\in I}\forall(\wedge_{n\in K_i} N^i_n \Rightarrow \vee_{p\in J_i} P^i_p).$$

Applying the independence property to each conjunct separately, we obtain
$$\mathcal{T} \vdash \forall \wedge_{i\in I} (\wedge_{n\in K_i} N^i_n \Rightarrow P^i_{p(i)})$$
for a suitable choice of $p(i) \in J_i$ for each $i \in I$. For conjuncts with no "positive" atomic propositions P, resp., with no "negative" atomic propositions N, the expression $\wedge_{n\in K_i} N^i_n \Rightarrow P^i_{p(i)}$ should be replaced by, respectively,
$$\neg(\wedge_{n\in K_i} N^i_n), \quad \text{and} \quad P^i_{p(i)}.$$
In any case, the resulting crisp sentence is a consequence of $\mathcal{T}$ and in turn implies Θ.

(ii) Θ is a positive sentence. Then Θ can be rewritten in the form
$$(Q)(\vee_{k\in K} \wedge_{l\in L_k} P^k_l).$$
Direct application of the independence property yields an index $k \in K$ such that $\mathcal{T} \vdash (Q) \wedge_{l\in L_k} P^k_l$. The resulting crisp sentence is a consequence of $\mathcal{T}$ and in turn implies Θ.

As for a proof of part (3), it suffices to consider one additional crisp axiom Ψ. Suppose
$$\mathcal{T} \cup \{\Psi\} \vdash (Q)(\Phi \Rightarrow \vee^n_{i=1} \Phi_i).$$
Applying the Deduction theorem and a logical equivalence, we obtain
$$\mathcal{T} \vdash (Q)(\Psi \wedge \Phi \Rightarrow \vee^n_{i=1} \Phi_i).$$
By the strong independence property, $\mathcal{T} \vdash (Q)(\Psi \wedge \Phi \Rightarrow \Phi_i)$ for some $i > 0$. We conclude that $\mathcal{T} \cup \{\Psi\} \vdash (Q)(\Phi \Rightarrow \Phi_i)$ for this i. ■

Part (1) confirms that free models are not the same as *initial models*, well-known from (a.o.) equational theories. By its very definition, an initial model must have a trivial automorphism group.

In the argument proving part (2), the assumed format of non-crisp axioms is such that the independence property can somehow be applied. Attempting a proof for general theories, a version of the independence property is suggested that deals with sentences of type
$$(Q) \wedge_i (\Phi^i_0 \Rightarrow \vee_j \Phi^i_j)$$
with Φ^i_0 and Φ^i_j basic for all i, j. (For some i, the part "$\Phi^i_0 \Rightarrow$" may be missing.) Note that *every* sentence can be put into the suggested format,

which would give this version of the independence property an amazingly wide range of application.

However, we do have an example of a crisp theory which does not satisfy the desired strengthening of independence.

Consider a language with two constants c, d and with five unary predicate symbols P, P', Q_1, Q_2, Q. The crisp axiom collection $\mathcal{T}$ consists of the following.

(i) $Pc \Rightarrow Q_2c$.

(ii) $Pd \Rightarrow Q_1d$.

(iii) $P'd \Rightarrow Pc$.

(iv) $P'c \Rightarrow Pd$.

(v) $\neg(Pc \wedge Pd)$.

It is easily seen that

$$\mathcal{T} \vdash \exists y((Py \Rightarrow Q_1y \vee Q_2y) \wedge (P'y \Rightarrow Qy)).$$

On the other hand, we have two models with universe $\{c, d\}$ and two interpretations of the predicates such that, respectively:

model 1	P	P'	Q_1	Q_2	Q
c	+	−	−	+	?
d	−	+	+	?	−

model 2	P	P'	Q_1	Q_2	Q
c	−	+	?	?	−
d	+	−	+	−	?

(Missing information at a question mark is irrelevant.) The first model does not satisfy $\exists y((Py \Rightarrow Q_1y) \wedge (P'y \Rightarrow Qy))$, whereas the second model does not satisfy $\exists y((Py \Rightarrow Q_2y) \wedge (P'y \Rightarrow Qy))$.

The argument proving part (3) does not extend to cover the (non-strong) independence property. However, we can adapt the argument to show that the (non-strong) independence property is preserved if axioms of type β, $\neg\beta$, $\beta_1 \Rightarrow \beta_2$, with β, β_1, β_2 basic, are added.

Our next results deal with sentences of type

$$(Q)(\Phi \Rightarrow \vee_{i=1}^{n}\Phi_i) \Rightarrow \vee_{i=1}^{n}(Q)(\Phi \Rightarrow \Phi_i),$$

$$(Q)(\vee_{i=1}^{n}\Phi_i) \Rightarrow \vee_{i=1}^{n}(Q)(\Phi_i),$$

where Φ is a basic (or crisp) formula, $\Phi_1, \ldots, \Phi_n$ are basic formulae, and (Q) is a sequence of quantifiers. We refer such sentences as "admissible", with the adjective "strong" in case the formula Φ is allowed to be crisp. As before, we allow sentences with "$\Phi \Rightarrow$" missing.

PROPOSITION 2.7. 1. *A consistent theory, which is maximal with the strong independence property, is complete.*

2. *A consistent theory with the independence property is consistent with each admissible sentence.*

3. *A consistent theory with the strong independence property is consistent with the set of all strongly admissible sentences.*

PROOF. Suppose $\mathcal{T}$ is a consistent theory which is maximal with the strong independence property. By virtue of theorem 2.6, part (3), each crisp sentence, consistent with $\mathcal{T}$, can be deduced from $\mathcal{T}$. Hence, all models of $\mathcal{T}$ satisfy exactly the same crisp sentences. By [5, thm. 6.3.18], *every* sentence in $\mathcal{L}$ is logically a Boolean combination of crisp sentences. Hence all models of $\mathcal{T}$ satisfy exactly the same sentences of $\mathcal{L}$; therefore, $\mathcal{T}$ is a *complete* theory.

As to parts (2) and (3), let $\mathcal{T}$ be a consistent theory with the (strong) independence property. We first consider the case of a single (strong) admissible sentence, with notation as above. Suppose it is inconsistent with $\mathcal{T}$. Then $\mathcal{T} \vdash (Q)(\Phi \Rightarrow \vee_{i=1}^{n}\Phi_i)$ and $\mathcal{T} \vdash \neg \vee_{i=1}^{n} (Q)(\Phi \Rightarrow \Phi_i)$. As $\mathcal{T}$ has the (strong) independence property, the first of these statements yields an index i and a proof of $(Q)(\Phi \Rightarrow \Phi_i)$. Hence there is a proof of $\vee_{i=1}^{n}(Q)(\Phi \Rightarrow \Phi_i)$, contradicting that $\mathcal{T}$ is consistent. (The alternative format is treated similarly.) This establishes (2); we continue with a proof of (3).

Note that a consistent theory with the strong independence property obviously extends to a maximal one. Hence it suffices to prove the result for a maximal consistent theory $\mathcal{T}$ with the strong independence property. Then $\mathcal{T}$ is complete by part (1). Strongly admissible sentences being individually consistent with $\mathcal{T}$, they are entailed by it. ■

So far we have been unable to extend parts (1, 3) of prop. 2.7 to cover the (non-strong) independence property.

COROLLARY 2.8. (1) *A complete theory has the (strong) independence property if and only if it entails the theory $\mathcal{A}$ ($\mathcal{SA}$) of all (strong) admissible sentences.*

(2) *A consistent theory with the strong independence property has a model M such that $M \models (Q)(\Phi \Rightarrow \vee_{i=1}^{n}\Phi_i)$ iff $M \models (Q)(\Phi \Rightarrow \Phi_i)$ for some i (Φ crisp, Φ_i basic).*

PROOF. Let $\mathcal{T}$ be a complete theory with the (strong) independence property. By proposition 2.7(2), a (strong) admissible sentence is consistent with $\mathcal{T}$. Hence it is entailed by $\mathcal{T}$. Conversely, let $\mathcal{T}$ be a complete theory that entails all (strong) admissible sentences. If $\mathcal{T} \vdash (Q)(\Phi \Rightarrow \vee_{i=1}^{n}\Phi_i)$ with Φ basic (crisp) and Φ_i for $i = 1, \ldots, n$ basic, then we can use the appropriate (strong) admissible sentence to conclude that $\mathcal{T} \vdash \vee_{i=1}^{n}(Q)(\Phi \Rightarrow \Phi_i)$. As $\mathcal{T}$ is complete, this yields an index i with $\mathcal{T} \vdash (Q)(\Phi \Rightarrow \Phi_i)$. (The alternative format is treated similarly.)

As to part (2), let $\mathcal{T}$ be a consistent theory with the strong independence property. By Zorn's Lemma and 2.7(1), $\mathcal{T}$ has a complete extension with this property. This extension entails all strong admissible sentences by part (1). Any model of it is a model of $\mathcal{T}$ as required. ■

Contrasting with the statement in part (2), certain models of a crisp theory may fail to satisfy the independence property. Here is a noteworthy example, based on an observation in [10]. Consider the theory $\mathcal{FE}$ of *free equality* in a language $\mathcal{L}$ with equality and operator symbols (among which is at least one constant). The axioms of $\mathcal{FE}$ are

1. $\forall(f(x_1, \ldots, x_n) \approx f(y_1, \ldots, y_n) \Rightarrow \wedge_{i=1}^{n} x_i \approx y_i)$, where f is an n-ary operator.
2. $\forall\neg(f(x_1, \ldots, x_{n_f}) \approx g(y_1, \ldots, y_{n_g}))$, where f, g are distinct operator symbols of arity n_f and n_g respectively.
3. $\forall x \neg(x \approx t)$, where t is a term other than x in which x occurs.

Obviously, this theory is crisp. The *Herbrand universe*, consisting of all variable-free terms of $\mathcal{L}$, is a model of $\mathcal{FE}$. If $\mathcal{L}$ has finitely many operator symbols, then the model satisfies the sentence

$$\forall x \exists Y_1 \ldots Y_k (\vee_{i=1}^{k} x \approx f_i(Y_i)),$$

where $\{f_1, \ldots, f_k\}$ are all operator symbols of $\mathcal{L}$ and Y_i is an array of variables the size of f_i's arity. All variables involved are distinct. However, if $k > 1$ then the model fails all sentences $\forall x \exists Y_i (x \approx f_i(Y_i))$ for $i = 1, \ldots, k$.

For another example, see the discussion of Pasch-Peano theory in §3).

To have *all* models of a theory $\mathcal{T}$ satisfy the strong independence property means that $\mathcal{T}$ entails the theory $\mathcal{SA}$ of all strong admissible sentences.

This is another, much stronger, format found in the literature; it is exclusively used with complete theories, in which case corollary 2.8(1) provides a justification. Alternatively, a *model complete* theory $(\mathcal{L}, \mathcal{T})$ with the strong independence property entails $\mathcal{SA}$. Indeed, by [5, thm. 3.5.1], for each model M of $\mathcal{T}$, the theory $(\mathcal{L}_M, \mathcal{T} \cup \Delta_M)$ is complete (where Δ_M is the diagram of M and $\mathcal{L}_M$ contains all elements of M as constants). It has the strong independence property by theorem 2.6(3), and hence entails $\mathcal{SA}$ by corollary 2.8(1).

In our search for examples of a non-crisp theory with the independence property, we took a closer look at the theory $\mathcal{SA}$.

THEOREM 2.9. *Let $\mathcal{L}$ be a non-trivial language. Then the theory $\mathcal{SA}$ in $\mathcal{L}$, consisting of all strong admissible sentences, is nonempty and consistent. Moreover, each subtheory of $\mathcal{SA}$ has the strong independence property. If its axioms are taken from $\mathcal{SA}$, then a subtheory is stable under reduced powers.*

PROOF. We assume that the language $\mathcal{L}$ is non-trivial just to make sure that the collection $\mathcal{SA}$ is nonempty. The theory $\mathcal{SA}$ is consistent by proposition 2.7(3).

We next prove that any subtheory $\mathcal{T}$ of $\mathcal{SA}$ has the strong independence property. Suppose

$$\mathcal{T} \vdash (Q)(\Psi \Rightarrow \vee_{i=1}^{n} \Phi_i),$$

where Ψ is crisp and each Φ_i is basic. Let Θ denote the entailed formula. Its negation is logically equivalent with

$$(Q')(\Psi \wedge \wedge_{i=1}^{n} \neg \Phi_i),$$

where (Q') is the complementary quantifier sequence. We see that $\neg\Theta$ is a crisp sentence inconsistent with $\mathcal{T}$ and hence with $\mathcal{SA}$. By proposition 2.7(2), $\neg\Theta$ must be a contradiction. Hence $\vdash (Q)(\Psi \Rightarrow \vee_{i=1}^{n} \Phi_i)$. As the "empty" theory is crisp, we conclude that $\vdash (Q)(\Psi \Rightarrow \Phi_i)$ for some i. Then certainly $\mathcal{T} \vdash (Q)(\Psi \Rightarrow \Phi_i)$.

As to the final part, note that (with our usual notation) each strong admissible sentence can be written equivalently as

$$(Q')(\Psi \wedge \wedge_{i=1}^{n} \neg \Phi_i) \vee \vee_{i=1}^{n} (Q)(\Psi \Rightarrow \Phi_i).$$

As before, (Q') is the complementary quantifier sequence of (Q). Apparently, this is a disjunction of crisp sentences. By [7, Thm. 6], a theory based on such sentences is stable under reduced powers. ■

Example 2.10. There is a non-crisp theory with the strong independence property.

Proof. We consider a language with binary predicate symbols P, P_1, P_2, P_3. For convenience, we use the following abbreviations.

$$\Psi_i := \forall x \exists y (P(x,y) \wedge \neg P_i(x,y)), \qquad i = 1,2,3;$$

$$\Psi_{i,j} := \forall x \exists y (P(x,y) \wedge \neg P_i(x,y) \wedge \neg P_j(x,y)), \qquad i \neq j \in \{1,2,3\},$$

$$\Psi_{1,2,3} := \forall x \exists y (P(x,y) \wedge \neg P_1(x,y) \wedge \neg P_2(x,y) \wedge \neg P_3(x,y)).$$

The reader may verify that each of the following two packages is consistent:

$$\{\Psi_1, \Theta_{2.3} := \exists x \forall y (P(x,y) \Rightarrow P_2(x,y) \wedge P_3(x,y))\},$$

$$\{\neg\Psi_1, \Psi_2, \Psi_3, \neg\Psi_{2,3}\}.$$

Note the conjunction in $\Theta_{2.3}$, which makes the formula different from $\neg\Psi_{2,3}$. Let M_1 and M_2 be a model of the first, resp., the second package. Both models satisfy the admissible (contrapositional) sentence

$$(*) \qquad \wedge_i \Psi_i \Rightarrow \Psi_{1,2,3}.$$

In fact, both parts of the implication fail on each model. On the other hand, if $a \in M_1$ is a value of x that must exist by $\Theta_{2.3}$, and if $b \in M_2$ is a value of x that exists by $\neg\Psi_{2,3}$, then taking $(a,b) \in M_1 \times M_2$ as x, we find that the formula

$$\exists y (P(x,y) \wedge \neg P_1(x,y) \wedge \neg P_2(x,y) \wedge \neg P_3(x,y))$$

does not hold in $M_1 \times M_2$. Yet the product does satisfy $\wedge_i \Psi_i$.

Being unstable under products of models, the theory with just the axiom (*) is not crisp by theorem 2.2. Being a subtheory of $\mathcal{SA}$, the theory (*) has the strong independence property by theorem 2.9. ■

The above example arose from a failed attempt to prove that individual admissible sentences are product-stable. However, we were able to verify the following. First, a sentence of type

$$(Q)(\Phi \Rightarrow \vee_{i=1}^n \Phi_i) \Rightarrow \vee_{i=1}^n (Q)(\Phi \Rightarrow \Phi_i),$$

with Φ crisp and Φ_i elementary for $i = 1, \ldots, n$, can be seen to be stable for products with two factors provided $n = 2$. Secondly, if $n > 2$, then the theory $\mathcal{T}$, consisting of the above sentence and all versions, involving only 2, 3, .., $n-1$ of the Φ_i, is stable for products with two factors. Hence $\mathcal{T}$

is stable under arbitrary products. By theorem 2.9, $\mathcal{T}$ is stable for reduced powers. We can now use [7, Thm. 3(f)] to conclude that the theory $\mathcal{T}$ is stable under reduced products and hence is crisp.

Being unions of such theories, both $\mathcal{SA}$ and its subtheory $\mathcal{A}$ of admissible sentences are crisp.

3. Examples and conclusions

The requirements for a theory to be crisp are quite generous. As a result, there is an abundance of theories with a crisp axiomatization. We discuss some natural (counter)examples in the fields of algebra and order, partial functions, convex geometry, and record logics.

3.1. Algebra, order, and functionality

(i) *Universal algebra* is about first order languages with equality and with terms. In the narrow sense, one considers theories consisting of (universally quantified) term equations. Such theories are crisp. Below, we shall encounter crisp theories of (abelian) *groups*, of *rings*, and of *Boolean algebras*.

Often, equational theories are extended with other non-equational axioms. In most cases, these are negations of equalities or implications between equalities. Such theories are crisp too. The theory of *torsion-free Abelian groups* has the additional axiom scheme

$$\forall x(nx \approx 0 \Rightarrow x \approx 0) \qquad (n = 2, 3, \cdots),$$

where nx stands for $n-$fold addition of x. There is a similar crisp axiom scheme describing *divisible Abelian groups*:

$$\forall x \exists y\, ny \approx x \qquad (n = 2, 3, \cdots).$$

The theory of nontrivial torsion-free divisible Abelian groups is complete ([16, Thm. 21.8, p. 351]) in addition to being crisp.

The theory of non-trivial *atomless* Boolean algebras is another example of a complete and crisp theory. Given the usual axioms of Boolean algebra, the condition of being atomless can be phrased as

$$\forall x \big(\forall y(x \cap y \approx 0 \vee x \cap y \approx x) \Rightarrow x \approx 0\big)$$

(which is logically crisp). As to the completeness of this theory, see ([16, Thm. 21.7, p. 351]). The negation of the atomless condition is a notorious example of a non-crisp sentence that is stable under products ([5, Example 6.2.3, p. 409].

Tarski's axioms for *relation algebra* (cf. [13]) involve constants $(0, 1, \mathbb{D})$ and Boolean operators (union $\cup$, intersection $\cap$, complement $-$) in addition to a binary "composition product" $x; y$ and a unary "reverse operator" $\smile x$. The axioms are those of Boolean algebra, together with the axioms of a monoid for composition (with identity element $\mathbb{D}$) and technical equivalences known as *Schröder's law*:

$$(x; y) \cap z \approx 0 \Leftrightarrow (\smile x; z) \cap y \approx 0 \Leftrightarrow (z; \smile y) \cap x \approx 0.$$

Thus, the theory of relation algebras is crisp.

There are some noteworthy examples of non-crisp theories in the realm of algebra. For instance, CUR (commutative unitary ring) theory is crisp, but no crisp extension of CUR can entail the theory of *entire rings*. Indeed, assuming the independence property, the sentence

$$\forall (x \cdot y \approx 0 \Rightarrow x \approx 0 \vee y \approx 0)$$

cannot be derived unless $0 \approx 1$ is adopted. In particular, there is no crisp theory of fields.

In all of universal algebra, fundamental use is made of so-called *simple algebras*, characterized by the property of having no homomorphic images except the obvious ones. For many common algebras, a first-order characterisation of simplicity is known. By theorem 2.6(a), a first-order theory that entails simplicity cannot be crisp. It is instructive to inspect such a first-order characterisation. For instance, *simple relation algebras* are characterised by the sentence

$$\forall x (x \approx 0 \ \vee \ 1; x; 1 \approx 1).$$

This shows there is no crisp theory of simple relation algebra unless $0 \approx 1$ is adopted.

For algebraic theories based on universal definite Horn sentences, the existence of free models with prescribed "generators" as additional constants is a well-known fact. Such models are provided by so-called *initial models*, and are usually constructed via generators and relations between them; cf. [4, p. 73].

(ii) The theory of *partial order* and the theory of *strict partial order* are axiomatized with universal definite Horn sentences and hence they are crisp.

Any coherent combination of both within one theory, however, is almost never crisp, as it somehow entails the sentence

$$\forall xy(x \leq y \Rightarrow x < y \vee x \approx y).$$

By virtue of the independence property, we would find that the partial order coincides with equality.

The theory of *dense* strict partial order and the theory of *unbounded* strict partial order are crisp too. No crisp extension of the theory of partial order can entail the theory of total order since the sentence $\forall(x \leq y \vee y \leq x)$ cannot be derived unless $\forall x \forall y\ x \approx y$ is adopted.

Suppose $\mathcal{T}$ is a crisp theory of partial order in some language $\mathcal{L}$ and Φ is a basic formula in $\mathcal{L}$ with one free variable[1]. Then the sentence

$$\exists x \Phi x \Rightarrow \exists y\big(\Phi y \wedge \forall z(\Phi z \Rightarrow y \leq z)\big)$$

is logically crisp. We may extend $\mathcal{T}$ with all sentences of this type, with Φ a basic unary formula, to obtain a crisp first-order approximation to a theory of well-order. Except in trivial cases, this imitation will never entail a total order.

Theories of partially ordered algebraic structures are another source of crisp theories: axioms linking algebraic operations with partial order are almost invariably crisp.

(iii) The theory of *partial functions* is crisp. Indeed, partial functionality of f can be described by the Horn sentence

$$\forall x y_1 y_2 (f(x, y_1) \wedge f(x, y_2) \Rightarrow y_1 \approx y_2).$$

In this way, we obtain crisp descriptions of "partial" algebras. For instance, a *groupoid* is defined as a small category where each homomorphism is an isomorphism. Alternatively, it is a partial algebra with a unary predicate $I(x)$ satisfied by the identity morphisms, with a unary reverse operator $^{\smile}x$, and with a binary partial composition operator which we treat as a ternary predicate $C(x, y, z)$. The result of composition is its third argument. Except for partial functionality of C, the axioms are the following.

(G-1) (Associativity) $\forall(C(x, y, u) \wedge C(u, z, t) \Rightarrow \exists v(C(x, v, t) \wedge C(y, z, v)))$.

(G-2) (Identity) $\forall(x \approx y \Leftrightarrow \exists w(Iw \wedge C(x, w, y)))$.

[1] For convenience, Φx denotes the formula obtained from Φ by substituting the free variable of Φ by x.

(G-3) (Left inverse) $\forall(C(x,y,z) \Rightarrow C({}^{\smile}x,z,y))$.

(G-4) (Right inverse) $\forall(C(x,y,z) \Rightarrow C(z,{}^{\smile}y,x))$.

The resulting theory is crisp and reduces to standard group theory if we (crisply) require a unique identity and a total composition. If, instead, the functionality requirement of C is dropped, we arrive at a crisp axiom system for so-called *atom structures of complete relation algebras*, [12].

3.2. The theory of Pasch-Peano spaces

Many geometries are based on a ternary predicate $Bxyz$ expressing that y is "between" x and z. Among the usual axioms are the following crisp ones, defining so-called *Pasch-Peano spaces* [19, I§4].

(PP-1) (Idempotence) $\forall xy(Bxyx \Rightarrow y \approx x)$

(PP-2) (Extensiveness) $\forall xyBxxy$

(PP-3) (Symmetry) $\forall xyz(Bxyz \Rightarrow Bzyx)$

(PP-4) (Pasch axiom) $\forall uu'vv'w(Buu'w \wedge Bvv'w \Rightarrow \exists x(Buxv' \wedge Bu'xv))$

(PP-5) (Peano axiom) $\forall uvwx\big(\exists y(Buxy \wedge Bvyw) \Rightarrow \exists y(Buyv \wedge Byxw)\big)$

A frequent additional condition is *density*,

$$\forall xy\big(x \approx y \vee \exists z(Bxzy \wedge \neg x \approx z \wedge \neg y \approx z)\big).$$

Let $\mathcal{T}_d$ denote the resulting theory of dense Pasch-Peano spaces. We can rewrite the density axiom as

$$\forall xy\exists z\big((z \approx x \wedge z \approx y) \vee (Bxzy \wedge \neg x \approx z \wedge \neg y \approx z)\big).$$

This suggests extending the language ($\mathcal{L}$, say) to a language $\mathcal{L}'$ with an additional binary term t, and to extend the theory $\mathcal{T}_d$ to a theory $\mathcal{T}'$ with two additional axioms:

(1) $\forall xy(txy \approx x \vee txy \approx y \Rightarrow x \approx y)$

(2) $\forall xyBx(txy)y$.

The extended theory is crisp because the offending density axiom can now be derived from the logically crisp axioms (1) and (2). Also, it is easy to see that $(\mathcal{L}', \mathcal{T}')$ is a conservative extension of $(\mathcal{L}, \mathcal{T}_d)$. By theorem 2.2, the former theory is stable under reduced products. Hence, as all models of $(\mathcal{L}, \mathcal{T}_d)$ obtain from models of $(\mathcal{L}', \mathcal{T}')$ by restriction, the expansion theorem

[5, thm. 4.1.8] yields that $(\mathcal{L}, \mathcal{T}_d)$ is stable under reduced products. We conclude from theorem 2.2 that dense Pasch-Peano theory is crisp.

There is an interesting class of basic formulae with a well-established geometric meaning. Given a finite array X of points $x_1, \ldots, x_n$ $(n \geq 2)$ the *convex hull* of X in a Pasch-Peano space consists of those points z that can be obtained inductively by requiring Bx_1zx_2 if $n = 2$ and by requiring $Byzx_n$, where y is in the convex hull of $x_1, \ldots, x_{n-1}$ if $n > 2$. Working this out, the statement that z is in the convex hull of $n \geq 2$ points X is seen to be expressed by a basic formula of type $\exists Y \Phi_n XYz$ with $n+1$ free variables X, z, and with an array Y of $n-2$ bounded variables. The resulting convex set is called a *polytope* with vertices X and, by virtue of the Peano axiom (PP-5), its construction does not depend on the naming order of the vertices in X. It is known [19, ch. I, §4] that the Pasch axiom (PP-4) boils down to Kakutani's separation property of disjoint polytopes by complementary half-spaces. (This statement and its proof properly belong to set theory.)

It is also customary to define *n-dimensionality* of a geometry in terms of the existence of $n+1$ "affinely independent" points. For instance, to state that a Pasch-Peano geometry is (at least) two-dimensional, we may proceed with three distinct constants a, b, c, with $a \not\approx b$ and

$$\forall z_1 z_2 \big(Baz_1b \wedge Baz_2b \wedge Bz_1z_2c \Rightarrow z_1 \approx z_2\big).$$

It is (at least) three-dimensional provided there is a fourth constant d with a, b, c as above and with d satisfying

$$(\dagger) \qquad \forall z_1 z_2 \big(\exists Y \Phi_3 abcY z_1 \wedge \exists Y \Phi_3 abcY z_2 \wedge Bz_1z_2d \Rightarrow z_1 \approx z_2\big),$$

where $\exists Y \Phi_3 abcYz$ expresses that z is in the convex hull of a, b, c and Y is an array of length one (single variable). Proceeding inductively, we can crisply state that a Pasch-Peano geometry is at least n-dimensional by adding an $(n+1)$th constant satisfying an additional formula of type $(\dagger)$ with Φ_n instead of Φ_3 and with the array of the previous n constants instead of "abc".

To formulate "exactly n-dimensional", we require that the space be the *affine hull* of the array C of $n+1$ constants, involved in the definition of "at least n-dimensional":

$$\forall x \exists vw \big(\neg v \approx w \wedge \exists Y \Phi_{n+1} CYv \wedge \exists Y \Phi_{n+1} CYw \wedge Bvwx\big).$$

Note that the description of dimension with new language constants is just a matter of convenience, not of necessity. The above array C of $n+1$ constants may be replaced by an array V of unused variables. A crisp definition of "exactly n-dimensional" then amounts to a lengthy conjunction of all previous sentences, with C replaced by V, and an existential quantifier sequence $\exists V$ in front of it.

Assume that a finite number of polytopes P, P_i are given by a crisp description of relative vertex positions. The theory of Pasch-Peano spaces (with or without density or dimension axioms) having the independence property, there can be no proof that P is included in the union of the polytopes P_i, unless P is already included in some P_i. Remarkably, many examples of Pasch-Peano spaces have an abundance of non-trivial point configurations with covering polytopes. One familiar situation occurs when a polytope is covered by its faces (a *face* is the convex hull of all but one of the vertices). The classical *Carathéodory theorem* in Euclidean n-space states that this is always the case if a polytope has $n+2$ (or more) vertices. Such a statement can never be obtained in a theory with the independence property.

Nevertheless, Carathéodory's result has been rederived in terms of Pasch-Peano spaces[2] (see [19, ch. II§1]) and, expectably, the additional assumptions involved are in conflict with the independence property. Except for density and dimension, two conditions are used:

- (*straightness*) $\forall\big(Buvw \wedge Bvwx \Rightarrow Buvx \vee v \approx w\big)$,
- (*decomposability*) $\forall\big(Bxyz \wedge Bxuz \Rightarrow Bxuy \vee Byuz\big)$.

Roughly, straightness is needed to make conditions, involved in the definition of dimension, look more symmetric. Decomposability is the main ingredient in the actual proof of Carathéodory's theorem. In fact, it is a one-dimensional variant of the theorem.

By virtue of theorem 2.5(1), the theory of Pasch-Peano spaces (with or without density or dimension axioms) has free models with arbitrarily large automorphism groups, but nothing seems to be known about them. In our proof that general free models exist, an important step is the addition of fresh constants to a language, which find their place in a free model to function as "generic elements". The following citation from the introduction of [15] illustrates our perspective.

[2] Our current treatment of "dimension" is rather biased towards a traditional view on geometry. The author's monograph [19] contains other results on Carathéodory's theorem as well, where Pasch-Peano spaces are drawn from non-traditional sources as well, e.g., distributive lattices with $Bxyz$ meaning $x \cap z \leq y \leq x \cup z$. Dimension is defined here with the *breadth* of the lattice. It is the only addition needed and *hence* it violates the independence property.

> The art of finding "generic" examples has been pushed to the extreme in Euclidean plane geometry, where we convince ourselves of many theorems by just drawing *one* picture of a non-degenerate case.

Contrasting with this observation, even in Euclidean plane, no single picture is generic for Carathéodory's theorem.

In closing this topic, we note that Euclidean n-space, seen as a model of Pasch-Peano theory with density and n dimensions, including straightness and decomposability[3], definitely does not satisfy the independence property. In particular, the "independence property" of real linear programming escapes from our approach.

3.3. Logic with records

In [20], we considered minimal axiom schemes for so-called *logics with records* and we proved a result which makes such logics interesting outside the framework of logic programming as well: Every first order theory can be faithfully interpreted into a logic with records, with provision for an assortment of additional requirements. All axiom schemes for records and all additional demands are crisp. The interpretation transforms original atomic formulae into specific basic formulae.

Hence every crisp first order theory has faithful crisp interpretations into logics with records. More generally, by virtue of theorem 2.6, part (3), every theory with the strong independence property has a faithful interpretation into a logic with records and with the strong independence property.

3.4. Problems

(1) It follows from 2.5, part (1), and 2.6, part (2), that the independence property and the strong independence property are equivalent for theories whose non-crisp axioms are universal or positive. Are the properties equivalent in other circumstances?

For instance, by virtue of corollary 2.8(1), the independence property and strong independence property are equivalent for complete theories if and only if the theories $\mathcal{A}$ (of admissible sentences) and $\mathcal{SA}$ (of strong admissible sentences) are logically the same. If either of these equivalent statements fails, there is a complete and non-crisp theory with

[3] It was a major achievement of the last century to prove that convex subspaces of Euclidean n-space can be characterised with the aid of the listed axioms ($n \geq 3$; completeness is required, too). See [19, Chap. 4 §1] for a detailed account.

the independence property. This would settle another natural question in the negative.

(2) We note that *all* non-crisp theories discussed in §3 can be recognised as such by deriving a contradiction with the independence property. This may suggest that non-crisp theories with the (strong) independence property are fairly exceptional. So far, we made little progress in finding (counter)indications for this.

3.5. Conclusion

Outside the logic or linear programming communities, the independence property of first-order theories seems to have remained largely unobserved. The importance of the independence property in logic or linear programming is, that it is part of a decision algorithm. Often, it also serves as a step in proving a theory to be complete. The property may as well serve the same purposes at other occasions, though this has not been investigated here.

The discussion of theories in §3 may illustrate a different use of the independence property as a tool to decide that certain sentences are not provable.

Finally, free models provide a flexible alternative for the traditional initial models, which require a more specialised axiom format. Some natural questions on crispness and on the (strong) independence property remain unsolved.

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