ONE-SELLER/TWO-BUYER MARKETS WITH BUYER EXTERNALITIES AND (IM)PERFECT COMPETITION

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In this paper the one-seller/two-buyer problem with buyer externalities is investigated under the assumption that the two buyers have legal opportunities to cooperate. It is shown that the Competitive equilibrium and the Core are robust with respect to negligible externalities and that the range of market prices in the Core belongs to range of Competitive equilibrium prices. However, these concepts yield no prediction for relatively severe externalities. Therefore, in order to provide a prediction the Bargaining set and the Multilateral Nash (MN) solution are also investigated. Surprisingly, in case of an empty Core the Bargaining set predicts a unique tuple of payoffs which are independent of the externalities and each pair of participants is equally likely. Markets with market imperfections are captured by the MN solution concept. The MN solution yields the paradox that the seller's price can be higher under imperfect competition than under perfect competition.

Keywords: Market game; externalities; Competitive equilibrium; Core; Bargaining set; Stable set; Multilateral Nash solution; Von Neumann–Morgenstern tuple.

JEL classification: C71, C78

1. Introduction

The simplest market situation one can think of is the situation in which one seller of an indivisible object wants to sell this object to one of two potential buyers. This problem is known as the one-seller/two-buyer problem and it has received much attention in the literature [see e.g., Hildenbrand and Kirman (1988), Shubik (1982), Osborne and Rubinstein (1991) for surveys]. Some of the well known standard results are: There always exists a (possibly degenerated) range of Competitive equilibrium prices such that demand is equal to supply; every Competitive equilibrium is efficient; the Core is non-empty; the set of Competitive equilibrium allocations coincides with the set of Core allocations; the lowest Competitive equilibrium price can be supported as the equilibrium in an auction [McAfee and McMillan (1989)].

*This paper was written while this author was visiting the Netherlands Institute for Advanced Study in the Humanities and Social Sciences. The kind hospitality and stimulating working environment of NIAS are gratefully acknowledged.

Whether or not the two buyers have the legal opportunity to enforce cooperation among them and agree not to buy the object from the seller does not matter in the results mentioned. In this paper it is assumed that the potential buyers can both commit not to buy the good and that they can agree on a monetary transfer as some sort of compensation between them. An important and (as we will see below) rather crucial assumption in this case for the above results in the standard one-seller/two-buyer problem is that each potential buyer does not suffer from external effects if the other buyer would buy and then "consume" the good. However, in many cases this assumption does not hold. For example, the UEFA (United European Football Association) holds the exclusive broadcasting rights of all soccer matches in the Champions League and both the commercial and the public broadcasting organizations would benefit from obtaining these rights but at the same time each organization has to face a serious drop in their own advertisement incomes when the other organization broadcasts Champions League matches after obtaining the rights.

In this paper externalities are modelled similar as in Jéhiel and Moldovanu (1995, 1996). Standard theory above suggests that modelling the market equilibrium mechanism as an auction would be a good start to investigate markets with externalities [e.g., Jéhiel and Moldovanu (1996)]. However, as is shown below the equilibrium of the auction is inefficient in case the magnitudes of the externalities are large. Furthermore, with large externalities the benefits of and, hence, the incentives for cooperation between the two potential buyers increase as well. If the market equilibrium mechanism is modeled as an auction, then it is *a priori* excluded that the pair of potential buyers can form a coalition. In other words, some of the strategic options available to the potential buyers are excluded. To come back to our example, in the Netherlands the Dutch commercial and the public broadcasting organizations have begun talks in order to form a coalition and avoid a harmful and fierce price competition between them.

In this paper attention is restricted to small markets, i.e., markets with two potential buyers. The motivation for this restriction is that it will be more difficult to form a (sub)coalition of potential buyers the larger this coalition becomes. We implicitly regard the effort costs of forming a coalition of potential buyers as a rapidly increasing and strict convex function in the number of buyers involved in such a coalition. To put it differently, we only expect coalitions between potential buyers if their number is very small.

In this paper the one-seller/two-buyer problem with externalities is modelled as a cooperative game known as a three-player/three-cake problem [e.g., Binmore (1986), Houba and Bennett (1997) and Houba (1994)]. This class of problems is the natural setting to study the economic market situation under consideration, because in order to obtain the maximum attainable payoff in the market it suffices to form a two-player coalition and, therefore, the economic problem can be regarded as a typical "odd man out" situation. A solution of this problem specifies which of the three two-player coalitions forms, who the excluded player is and what the payoff to each of the players is.

The main purpose of this paper is twofold. First the Competitive equilibrium concept and the Core concept will be applied to the one-seller/two-buyer problem with buyer externalities and both sets of solutions are fully characterized. It will be shown that these sets of outcomes converge to the standard results in absence of buyer externalities if we let the externalities vanish. So, the standard results are robust with respect to small values of the externalities. Moreover, the set of prices for which the object is sold corresponding to each Core allocation is always contained in the set of Competitive equilibrium prices. However, for rather high values of the externalities we find that the Competitive equilibrium as well as the Core can be equal to the empty set or that the Competitive equilibrium may exist while the Core may not exist. As in the standard case, both concepts yield an efficient allocation in case solutions to these concepts exist.

The non-existence of either of these two concepts imposes a serious problem, because for rather high values of the externalities these two theories fail to provide a satisfying answer with respect to the selection of reasonable outcomes. Clearly, this calls for some other solution concept that does not have this drawback. The second aim of this paper is to investigate whether other "classical" game theoretic solution concepts, such as the Bargaining set and the (von Neumann–Morgenstern) Stable set, can provide us with a better understanding of the economic problem. Furthermore, the relatively recent concept of the Multilateral Nash solution [e.g., Bennett (1997)] is analyzed.

It is shown that most of the classical cooperative solution concepts considered yield similar sets of solutions and, therefore, we regard the Bargaining set as the representative of these various concepts. If the Core is not empty, then the Bargaining set coincides with the Core. Otherwise, the Bargaining set admits three possible outcomes associated with the von Neumann–Morgenstern tuple. This uniquely determined tuple specifies a feasible and efficient partition of each pair's surplus with the property that for every pair each individual's payoff, if included in the pair that actually forms, is equal to this individual's foregone payoff in his alternative pair, provided that pair would have formed instead.

The Bargaining set can be regarded as a theory in which the competition among the players is perfect. However, not every market has perfect competition and it is also worthwhile to study solutions for markets with imperfect competition. At this point the concept of Multilateral Nash (MN) solution, which extends two-player axiomatic bargaining theory, becomes interesting. The MN solution is able to capture market imperfections, such as for instance lock-in effects. This is first shown in Houba and Bennett (1997) where for any three-player/three-cake problem each of the two endpoints of the (possibly degenerated) range of MN solutions corresponds to the subgame perfect equilibrium outcome of a particular non-cooperative bargaining model. These two strategic models are the market demand model [e.g., Binmore (1986) and Osborne and Rubinstein (1991), Sec. 9.3] and the proposal-making model [e.g., Binmore (1986), Chatterjee *et al.* (1993), Moldovanu (1992), Selten (1981) and Osborne and Rubinstein (1991), Sec. 9.4]. In the market demand model competition is perfect and the unique subgame perfect equilibrium outcome coincides with an MN solution and this MN solution also belongs to an outcome in the Bargaining set. In the proposal-making model competition is imperfect because the individual who is in the position to make a proposal suffers from a lock-in effect after having made his proposal. The corresponding MN solution does not lie in the Bargaining set and this particular MN solution corresponds to the situation with market imperfections. Thus, a non-cooperative underpinning of this interpretation can be given.

The remaining of the paper is as follows. In Sec. 2 we introduce the one-seller/two-buyer problem with buyer externalities and we fully characterize the set of Competitive equilibria. We also derive the necessary and sufficient condition for (non)emptiness. The standard auction is discussed and rejected as a appropriate model, because it excludes cooperation between the buyers. This section is concluded with the formulation of the economic problem as a three-player/three-cake problem. In Sec. 3 we discuss the game-theoretic solution concepts used in this paper. In Sec. 4 we translate the results of Sec. 3 into outcomes of these concepts for the one-seller/two-buyer externality game and provide the interpretation of these results. In Sec. 5 some concluding remarks are made.

2. The Economic Problem

In this section we first introduce the one-seller/two-buyers problem under buyers' externalities [e.g., Jéhiel and Moldovanu (1995, 1996)]. With buyers' externalities we mean that a buyer experiences a (negative) externality from the other buyer if the latter buyer is in possession of the object. For instance, in case of two firms competing for the ownership of some exclusive technology offered for sale by the seller, firm i, $i = 1, 2$, can make a profit w_i if he buys the technology, but makes a loss $\alpha_i \geq 0$ if his competitor $j \neq i$ buys the technology. In the remaining of the paper, let player 3 be the seller and the players 1 and 2 the two potential buyers. Moreover, let w_i be the valuation of player i, $i = 1, 2, 3$, for the item and let $\alpha_i \geq 0$ be the loss of buyer i, $i = 1, 2$, if buyer $j \neq i$ buys the commodity. Without loss of generality we assume that $w_1 > w_2 > w_3 = 0$, because the results for the non-generic cases $w_1 = w_2$ and $w_2 = w_3 = 0$ can easily be derived from Fig. 1 and Table 1.

As a benchmark we first consider the Competitive equilibrium outcome for various sets of the values of w_i and α_i, $i = 1, 2$. Recall that $w_1 > w_2 > 0$. Now, let $p \geq 0$ be the price for which the commodity is offered for sale. As long as $p < w_2$, both players want to buy the item and hence p is not an equilibrium price. So, any equilibrium price p must be at least equal to w_2. Now we have the following three cases:

$$\begin{aligned} \text{I} &: & w_1 &\geq w_2 + \alpha_2 \,, \\ \text{II} &: & w_1 + \alpha_1 &\geq w_2 + \alpha_2 > w_1 \,, \\ \text{III} &: & w_2 + \alpha_2 &\geq w_1 + \alpha_1 \geq w_1 \,. \end{aligned}$$

In the first case buyer 2 is willing to buy the item as long as $p < w_2 + \alpha_2$, because also buyer 1 is willing to pay this price. Hence, buying the item yields a payoff of $w_2 - p$, which is at least equal to $-\alpha_2$, being the payoff if buyer 1 owns the item. For $p = w_2 + \alpha_2$ buyer 2 becomes indifferent and, hence, any price p satisfying $w_2 + \alpha_2 \leq p \leq w_1$ is a Competitive equilibrium price. In such an equilibrium buyer 1 buys the item. The resulting payoffs, denoted by u_i, $i = 1, 2, 3$, are $u_1 = w_1 - p$, $u_2 = -\alpha_2$ and $u_3 = p$. Observe that the total payoff equals $w_1 - \alpha_2$. Since $w_1 \geq w_2 + \alpha_2$ we have that $w_1 - \alpha_2 \geq w_2 \geq \max\{0, w_2 - \alpha_1\}$ and, hence, the total payoff is maximized. Thus, under Case I the trade resulting from competition is efficient.

In the two other cases there does not exist a Competitive equilibrium price. Consider Case II and first suppose $p > w_2 + \alpha_2$. Then buyer 2 does not want to buy the item and, hence, buyer 1 wants to pay at most w_1. Second, suppose that $p < w_1$. Then buyer 2 wants to buy because buyer 1 is willing to pay this price. So, both buyers want to buy and so $p < w_1$ cannot be an equilibrium price. Finally, consider $w_1 \leq p \leq w_2 + \alpha_2$. In this case a buyer i, $i = 1, 2$, only wants to buy the good in order to prevent the other buyer from buying. So, i wants to buy at p if and only if $j \neq i$ wants to buy. As soon as i leaves the market also his competitor j leaves the market. Therefore, the demand is either 0 or 2 and so an equilibrium does not exist. The same reasoning holds in Case III.

Observe that in Case II it is optimal to allocate the item to player 1 if $w_1 - \alpha_2 > 0$ and to player 3 otherwise. Analogously in Case III it is optimal to allocate the item to player 2 if $w_2 - \alpha_1 > 0$ and to player 3 otherwise. Reversely this implies that there does not exist an equilibrium price if $\max[w_1 - \alpha_2, w_2 - \alpha_1] \leq 0$, i.e., if not selling the item yields an efficient allocation. Hence the main results of the above analysis are as follows:

Theorem 2.1 (Competitive Equilibrium).

(E1) *A Competitive equilibrium price exists if and only if* $\alpha_2 \leq w_1 - w_2$. *In this case the interval of Competitive equilibrium prices is* $[w_2 + \alpha_2, w_1]$ *and the good is sold to player* 1. *The equilibrium allocation is efficient*;

(E2) *if it is efficient not to sell the item, then there does not exist a Competitive equilibrium price.*

One standard result for the one-seller/two-buyer problem without externalities is that the lowest Competitive equilibrium price (and outcome) corresponds to the unique Nash equilibrium outcome of both the first-price and second-price sealed-bid auction. The unique Nash equilibrium price of this auction in case of buyer externalities is equal to $\min\{w_1 + \alpha_1, w_2 + \alpha_2\}$ and the buyer with valuation equal to $\max\{w_1 + \alpha_1, w_2 + \alpha_2\}$ obtains the good [e.g., Jéhiel and Moldovanu (1996)]. As long as $\alpha_2 \leq w_1 - w_2$, i.e., a Competitive equilibrium exists, the Nash equilibrium of the auction corresponds to the lowest Competitive equilibrium price, a result similar to the standard case without externalities. Also in case $\alpha_2 > w_1 - w_2$ the

Nash equilibrium of the auction still exists. However, in this case the outcome of the auction is not longer efficient. So, the efficiency of the outcome of the auction for the case without externalities only extends to the case with externalities for small externalities (implying robustness of the standard results). In case $\alpha_1 > w_2$ and $\alpha_2 > w_1$ the Nash equilibrium yields an inefficient allocation implying the standard efficiency result for auctions does not hold in general if the players have externalities.

In case that the magnitudes of both externalities are large it is clear that the two potential buyers could gain from forming a two-player coalition in order to prevent that one of them buys the good. Of course, this means that it must be possible for both buyers to commit themselves to not buying and that transfers between the buyers are not excluded. They can certainly commit if they have the legal opportunity to write a binding agreement or contract in which both agree not to buy the object as well as include some monetary transfer. It is clear that modelling the market as an auction excludes a priori the formation of the two-buyer coalition. In order to obtain insights into market situations in which buyers can credibly form a coalition we will model the market as a cooperative game with transferable utility.

We proceed by defining the one-seller/two-buyer problem as a transferable utility game $(\mathcal{N}, \hat{v})$, where $\mathcal{N} = \{1, 2, 3\}$ is the set of players and $\hat{v}: 2^{\mathcal{N}} \to \mathbb{R}$ the characteristic function defining for any subset S of $\mathcal{N}$ the payoff of the coalition S of players. Under the assumption that the payoff of a single player i equals the payoff this player can guarantee himself in the worse case with respect to the behaviour of the other players, the characteristic function is given by

$$\hat{v}(1) = -\alpha_1, \qquad \hat{v}(2) = -\alpha_2, \qquad \hat{v}(3) = 0, \qquad \hat{v}(1,2) = 0\,,$$
$$\hat{v}(1,3) = w_1, \qquad \hat{v}(2,3) = w_2, \quad \text{and} \quad \hat{v}(1,2,3) = \max\{0, w_1 - \alpha_2, w_2 - \alpha_1\}\,.$$

Observe that for $i = 1, 2$, the item is sold to player i if the two-player coalition $\{i, 3\}$ forms, whereas the item is not sold if the two-player coalition $\{1, 2\}$ forms. In the latter case the two buyers agree to stay out of the market. Clearly, this option maximizes the total payoff in case both $\alpha_1 > w_2$ and $\alpha_2 > w_1$, i.e., in case the profit which can be realized by one of the buyers is less than the externality of the other buyer and hence the industry is hurt by adapting the new technology. Normalizing the payoffs of the one-player coalitions equal to zero, we get the game $(\mathcal{N}, v)$ defined by the characteristic function $v: 2^{\mathcal{N}} \to \mathbb{R}$ given by

$$v(1) = v(2) = v(3) = 0\,,$$
$$v(1,2) = \alpha_1 + \alpha_2, \qquad v(1,3) = \alpha_1 + w_1, \qquad v(2,3) = \alpha_2 + w_2\,,$$

and

$$v(1,2,3) = \alpha_1 + \alpha_2 + \max\{0, w_1 - \alpha_2, w_2 - \alpha_1\}\,.$$

As noticed in Houba (1994) this bargaining problem under externalities is equivalent to a three-player/three-cake problem.[a] This can be easily seen as follows. Rewriting the last equation as $v(1,2,3) = \max\{\alpha_1 + \alpha_2, \alpha_1 + w_1, \alpha_2 + w_2\}$, we get that $v(1,2,3) = \max\{v(1,2), v(1,3), v(2,3)\}$. Therefore there always exists at least one two-player coalition $\{i,j\}$, which can divide the total payoff $v(1,2,3)$ by cooperating and excluding the third player. Hence, the grand coalition will never form and only the payoffs of the two-player coalitions matter. Therefore the game is equivalent to a three-player/three-cake problem, i.e., a game in which only pairs of players have the possibility of forming a coalition and dividing the associated cake (payoff). In fact, in a three-player/three-cake game a coalition structure $\{[i,j],[k]\}$ will form where $[i,j]$ denotes the coalition $\{i,j\}$ which forms and $[k]$ denotes the third player excluded from the coalition.

This section is concluded with the definition of the von Neumann–Morgenstern vector and tuple.[b] For a given pair $[i,j]$, let $z^{ij} = (z_i, z_j)$ be a feasible and efficient payoff vector, i.e., a pair of *non-negative* real numbers such that $z_i + z_j = v(i,j)$. That means the vector z^{ij} induces the feasible payoff vector $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$ defined by $x_i = z_i$, $x_j = z_j$ and $x_k = 0$ for $k \neq i,j$ with coalition structure $\{[i,j],[k]\}$. We now have the following definition.

Definition 2.1 (Von Neumann–Morgenstern (VNM) vector). Let $z = (z_1, z_2, z_3)^\top \in \mathbb{R}^3_+$ be such that for any pair $[i,j]$, $z^{ij} = (z_i, z_j)$ is feasible and efficient. Then the vector z is the Von Neumann–Morgenstern vector with the triple of payoff vectors $\{z^{12}, z^{13}, z^{23}\}$ as the corresponding Von Neumann–Morgenstern tuple.

By definition the VNM vector $z = (z_1, z_2, z_3)^\top$ is a solution of the system of equations

$$z_1 + z_2 = v(1,2)\,, \quad z_1 + z_3 = v(1,3)\,, \quad z_2 + z_3 = v(2,3)\,. \tag{1}$$

Although this system has a unique solution, this solution only forms a VNM vector if all of its components are non-negative. Otherwise, the VNM vector does not exist.

In case $z_i > 0$ for all $i \in \mathcal{N}$ at a VNM vector z, we have that $z_1 + z_2 + z_3 > \max\{v(1,2), v(1,3), v(2,3)\} = v(1,2,3)$. So, except for the boundary case that $z_i = 0$ for at least one i, the players cannot realize simultaneously the payoffs of the VNM vector. So, in general the VNM vector yields a non-feasible outcome. Nevertheless, the VNM vector has a nice interpretation. At such a vector any player is indifferent in choosing one of the other two players in forming a pair to divide the associated cake. Recall that at most one pair can be formed. So, player i gets

[a] This is no longer true in the more general case that any of the three players (buyers and seller) may experience externalities form the other two players [e.g., Cornet and van der Laan (1995) for a thorough discussion].

[b] By naming this vector and corresponding tuple after von Neumann and Morgenstern we follow the terminology in Binmore (1986). Below it will be shown that this tuple characterizes the Stable set in case the Core is empty.

his VNM payoff z_i if he forms a pair with either player j or k and his partner in this pair gets z_j respectively z_k. The player not in the pair that forms gets his (normalized) payoff equal to zero.

Observe that any of the three pairs may form. So, any element z^{ij} of the VNM tuple corresponds with a payoff vector x as defined above and with a coalition structure $\{[i,j],[k]\}$. Although any pair may form, the probabilities with which the pairs are formed need not to be equal. So, let p_{ij} be the probability that the pair $[i,j]$ forms. Then the expected payoff E_i of player i equals $E_i = (p_{ij} + p_{ik})z_i$, $i = 1, 2, 3$. Hence $E_1 + E_2 + E_3 = p_{ij}v(ij) + p_{ik}v(ik) + p_{jk}v(jk) \leq \max\{v(ij), v(ik), v(jk)\}$. Because also inefficient pairs (i.e., pairs not realizing the maximum payoff) may have a positive probability to form, the expected payoff is not an efficient outcome. The realized outcome is only efficient if an efficient pair forms.

3. Game Theoretic Solutions

Several game theoretic solutions for the three-player/three-cake problem have been analyzed in Houba and Bennett (1997) and more extensively in Houba (1994). In this section some of these concepts are introduced and a brief derivation of the relevant results is given. The first solution concept we want to consider is the Core in payoff configurations [e.g., Aumann and Drèze (1974), Binmore (1986) and Houba (1994)].

As is well-known, the Core is the set of all undominated payoff vectors, i.e., the set of all vectors $x \in \mathbb{R}^3$ satisfying $\sum_{i \in S} x_i \geq v(S)$ $S \subseteq \mathcal{N}$. Observe that for each element $x = (x_1, x_2, x_3)^\top$ in the Core we have that $x_k = 0$ for at least one k. To show this, let $[i,j]$ be a pair such that $v(i,j) = \max\{v(1,2), v(1,3), v(2,3)\} = v(1,2,3)$. Now, if for $k \neq i, j$, $x_k > 0$, then $x_i + x_j < v(i,j)$ and, hence, x is dominated through the coalition $\{i,j\}$. So, for each element x in the Core we have implicitly a coalition structure $\{[i,j],[k]\}$ with coalition $[i,j]$ the pair which divides its associated cake $v(i,j)$ and excludes player k from cooperation [e.g., Aumann and Drèze (1974)].

The following theorem has been proven in Binmore (1986) and Houba (1994) and shows the relationship between the Core and the VNM vector of a three-player/three-cake game.

Theorem 3.1 (Existence of Core and VNM vector). *Let* $(z_1, z_2, z_3)^\top$ *be the unique solution of* (1). *Then*

(a) *the Core consists of multiple elements if and only if* $z_i < 0$ *for at least one* $i \in \mathcal{N}$;
(b) *the VNM vector* $(z_1, z_2, z_3)^\top$ *specifies the players' payoffs corresponding to the unique element of the Core if and only if* $z_i \geq 0$ *for all* $i \in \mathcal{N}$ *and* $z_i = 0$ *for at least one* $i \in \mathcal{N}$;
(c) *the Core is empty if and only if* $z_i > 0$ *for all* $i \in \mathcal{N}$.

Theorem 3.1 states that the Core contains multiple elements if and only if the VNM vector does not exist, i.e., if the system of equations (1) does not have a non-negative vector. Reversely, the VNM vector exists and is strictly positive if and only if the Core is empty. If the system (1) has a non-negative solution with at least one of the components equal to zero, then the VNM vector is the unique element of the Core. It is important to observe the difference in the interpretation of the unique Core element and the VNM tuple. Therefore, suppose that $z_k = 0$. Then the unique Core solution says that the pair $[i,j]$, $i \neq k$ and $j \neq k$, is the only pair that forms and partitions its cake $v(i,j)$ according to $z^{ij} = (z_i, z_j)$. According to the VNM tuple any pair $[h,k]$ may form with the element z^{hk} of the VNM tuple as the partition of its cake $v(h,k)$. Observe that only if the pair $[i,j]$ forms the payoffs obtained from the VNM vector are equal to the payoffs obtained from the Core solution.

In Houba (1994) it is also shown that both the Bargaining set and the Stable set are also fully characterized by Theorem 3.1. These results are summarized in the next theorem. Therefore, let $(z_1, z_2, z_3)^\top$ be again the solution of (1) and let k be the index such that $z_k = \min_j z_j$. Observe that if $z_k < 0$, then $z_i > 0$, $z_j > 0$ and $v(i,j) = \max\{v(1,2), v(1,3), v(2,3)\}$ for $i,\ j \neq k$.

Theorem 3.2 (Characterization of Bargaining set and Stable set). *Let* $(z_1, z_2, z_3)^\top$ *be the solution of* (1) *and let* k *be the index such that* $z_k = \min_j z_j$. *Then we have the following cases*:

(a) $z_k < 0$: *then the Bargaining set is equal to the Core and the Stable set is given by the set*

$$\{x \in \mathbb{R}^3_+ | x_i + x_j = v(i,j), i, j \neq k\,, \quad \textit{and} \quad x_k = 0\}\,.$$

At any outcome of this set the pair $[i,j]$, $i,\ j \neq k$, *forms a coalition and divides its associated cake*;

(b) $z_k \geq 0$: *then both the Bargaining set and the Stable set are given by the collection of the three vectors* $\{(z_1, z_2, 0)^\top, (z_1, 0, z_3)^\top, (0, z_2, z_3)^\top\}$, *Any pair may form. Player* i *gets payoff* z_i *if he is in the pair that forms. Otherwise the player gets a payoff equal to* 0.

Combining Theorems 3.1 and 3.2 we see that the Bargaining set coincides with the Core if the VNM vector does not exist (Case (a) of Theorem 3.1). In this case the Stable set is given by the set of payoff vectors satisfying that the two players which can realize the highest payoff divide this payoff among themselves. In case the VNM vector exists (Cases (b) and (c) of Theorem 3.1) the Bargaining set is equal to the Stable set and consists of the three payoff vectors induced by the VNM tuple. So, the Bargaining set and the Stable set immediately follow from the Core and the VNM vector. Therefore we do not further discuss the Bargaining set and Stable set in the remaining of the paper.

We now want to consider the concept of the Multilateral Nash solution. Suppose that for every pair $[i,j]$ all three players in $\mathcal{N}$ conjecture that $x^{ij} \in \mathbb{R}^2_+$ represents the partition of the cake $v(i,j)$ for this pair, with the convention that the pair $[i,j]$

will not form if x^{ij} is not feasible. Then the tuple $\underline{x} = \{x^{12}, x^{13}, x^{23}\}$ represents the conjectured partitions for every cake. Given these conjectured partitions $\underline{x}$ each member of the pair $[i,j]$ has a conjectured outside option, namely abandon the pair $[i,j]$ and go to the third player and form a coalition with this player. It is assumed that all three players in $\mathcal{N}$ conjecture that the value of player i's outside option $o_i^{ij}(\underline{x}) = \max\{0, v(i,k) - x_k^{ik}\}$, where x_k^{ik} denotes player k's conjectured payoff in x^{ik}. The reason for this is that if x^{ik} is not feasible, then player i can only execute his outside option if he gives in player k's demand x_k^{ik} in order to form the pair $[i,k]$. This gives us a tuple $\underline{o}(\underline{x}) = \{o^{12}(\underline{x}), o^{13}(\underline{x}), o^{23}(\underline{x})\}$ of conjectured outside options for each pair. Now, in case of an infeasible pair of outside options $o^{ij}(\underline{x})$ it is assumed that the members of the pair $[i,j]$ agree not to form this pair and to execute their outside options $o^{ij}(\underline{x})$. In case of a feasible pair of outside options $o^{ij}(\underline{x})$ the members of the pair $[i,j]$ have the possibility to form a coalition and to negotiate about the division of the cake. It is assumed that the resulting agreement of these negotiations is the constrained Nash bargaining solution $\arg\max_{(x_i,x_j)} x_i x_j$, s.t. $(x_i, x_j) \geq o^{ij}(\underline{x})$ [e.g., Sutton (1986)]. In other words, it is assumed that the negotiated agreement $N^{ij}(\underline{o}(\underline{x}))$ within the pair $[i,j]$ is given by

$$N^{ij}(\underline{o}(\underline{x})) = \begin{cases} \displaystyle\arg\max_{(x_i,x_j)\geq o^{ij}(\underline{x})} x_i x_j, & \text{if } o^{ij}(\underline{x}) \text{ is feasible,} \\ o^{ij}(\underline{x}) & \text{otherwise.} \end{cases}$$

Finally, consistency requirements impose the condition that for each pair $[i,j]$ the conjectured agreement x^{ij} is equal to $N^{ij}(\underline{o}(\underline{x}))$, i.e., a fixed point argument. All these considerations lead to the following definition.

Definition 3.1 (Multilateral Nash (MN) solution). A tuple $\underline{x} = \{x^{12}, x^{13}, x^{23}\}$ is a multilateral Nash solution if

$$x^{ij} = N^{ij}(\underline{o}(\underline{x}))\,, \quad \text{for all } [i,j] \in \{[1,2],[1,3],[2,3]\}\,.$$

For every MN solution $\underline{x} = \{x^{12}, x^{13}, x^{23}\}$ it holds that there exists a vector $y = (y_1, y_2, y_3) \in \mathbb{R}_+^3$ such that $x_i^{ij} = x_i^{ik} = y_i$, for all i, j, $k \in \mathcal{N}$ and $i \neq j$, $i \neq k$ [e.g., Bennett (1997)]. Analogously to the VNM vector we summarize an MN solution $\underline{x} = \{x^{12}, x^{13}, x^{23}\}$ by its associated vector of demands $y = (y_1, y_2, y_3)$. The i-th component can be thought of as an endogenous reservation value for player i and player i does not participate in any pair if he does not get at least a payoff of y_i. For details we refer to Bennett (1997) or Houba and Bennett (1997).

In the remaining of the paper, let the three players i, j, k be ordered such that

$$v(i,j) \geq v(i,k) \geq v(j,k)\,.$$

The following definitions will prove to be useful in characterizing the set of MN solutions.

Definition 3.2. If $v(i,j) \geq v(i,k) \geq v(j,k)$, then we say that

(a) Player i is the *dominant* player, player k is the *dominated* player and player j is the *modal* player.
(b) If $v(i,j) > 2v(i,k)$, then players i and j are the Nash-dominant pair.

We now consider the two cases, either the pair $[i,j]$ is Nash-dominant, or not. In the first case the pair $[i,j]$ is called the Nash dominant pair, because the standard Nash bargaining solution with disagreement point $(0,0)$ of this pair's cake gives $\frac{1}{2}v(i,j) > v(i,k) = \max\{v(i,k), v(j,k)\}$ to each player in this pair and, hence, none of these two players can improve by leaving the pair $[i,j]$ and form a pair with the third player k. Note that for any tuple $\underline{x}$ it holds that $o_i^{ik}(\underline{x}) \leq v(i,k)$ and $o_j^{jk}(\underline{x}) \leq v(j,k)$. So, for any tuple $\underline{x}$ it immediately follows from the definitions of $N^{ij}(\underline{o}(\underline{x}))$ and the Nash-dominant pair that $y_i = y_j = \frac{1}{2}v(i,j)$. Then $y_k = 0$ follows also. Thus, if $[1,2]$ is the Nash-dominant pair, then the vector $y = (\frac{1}{2}v(1,2), \frac{1}{2}v(1,2), 0)^\top$ is the unique MN vector of demands. The Nash-dominant pair $[i,j]$ is the only pair that can form, because x^{ik} and x^{jk} are not feasible for the pairs $[i,k]$ respectively $[j,k]$. In this case for each of these players the payoff is at least equal to the maximum payoff they can obtain if forming a pair with the third player. Therefore, forming a pair with the third player k is not a relevant outside option for the players in the Nash-dominant pair. Observe that this unique MN solution lies in the Core.

If the pair $[i,j]$ is not Nash-dominant, then the MN solution implies that the dominant player is always in the pair that forms. The reasoning behind this is that the dominant player i has always the possibility to offer the modal player j a payoff of at least $v(j,k)$, which is the maximum payoff that player j can obtain in forming a pair with the dominated player k. So, player i has the possibility to object against the pair $[j,k]$. In characterizing the set of MN solutions for this case it is convenient to write down the following system of (in)equalities:

$$y_i + y_j = v(i,j)\,, \tag{2}$$

$$y_i + y_k = v(i,k)\,, \tag{3}$$

$$y_i \geq v(i,j)/2\,, \tag{4}$$

$$y_j + y_k \geq v(j,k)\,, \tag{5}$$

$$y_k \geq 0\,, \tag{6}$$

The next lemma follows immediately.

Lemma 3.1.

(a) *The system of* (*in*)*equalities* (2)–(6) *has a* (*possibly degenerated*) *line piece of non-negative solutions* $y \in \mathbb{R}^3_+$ *if and only if the pair* $[i,j]$ *is not Nash-dominant.*

(b) *If the VNM vector does not exists, then the system of* (*in*)*equalities* (2)–(6) *has the non-empty set of solutions*

$$Y = \left\{ y \in \mathbb{R}^3_+ | y_j = v(i,j) - y_i\,, \quad y_k = v(i,k) - y_i\,, \right.$$
$$\left. y_i \in \left[\frac{1}{2} v(i,j), v(i,j) - v(j,k) \right] \right\},$$

which reduces to the unique point $y = (\frac{1}{2}v(1,2), \frac{1}{2}v(1,2), 0)^\top$ *when* $v(i,k) = \frac{1}{2}v(i,j)$.

If the VNM vector z *exists, then the system of* (*in*)*equalities* (2)–(6) *has the non-empty set of solutions*

$$Y = \left\{ y \in \mathbb{R}^3_+ | y_j = v(i,j) - y_i\,, \quad y_k = v(i,k) - y_i\,, \quad y_i \in \left[\frac{1}{2} v(i,j), z_i \right] \right\},$$

which includes the VNM vector z. *When* $v(i,j) = v(i,k) = v(j,k)$, *then* $Y = \{z\}$.

If the pair $[i,j]$ is not Nash-dominant, then the dominant player i will form a pair with either the modal player j or the dominated player k. So, the solution must satisfy Eqs. (2) and (3). As in the VNM vector, he is indifferent between these two alternatives. Moreover, player i can always claim at least an amount of at least $\frac{1}{2}v(i,j)$, being the payoff obtained from the Nash solution when he forms a pair with the modal player j. This is reflected by inequality (4). The upper bound of the payoff of the dominant player i follows from the condition that $z_j + z_k$ must be at least equal to $v(j,k)$ (inequality (5)), preventing that the modal and the dominated player can improve from forming a pair together, and from the condition that all payoffs must be non-negative, which is guaranteed by $z_k \geq 0$ (inequality (6)). Finally, observe that only the pairs $[i,j]$ and $[i,k]$ can form. So, player i can play off both j and k, because i can realize z_i in either $[i,j]$ or $[i,k]$, whereas the players j and k can only form a pair with i. However, this is only reasonable for any MN solution with $z_j + z_k > v(j,k)$. At the MN solution z of the system (2)–(6) satisfying $z_j + z_k = v(j,k)$ we have that z also solves the system (1) determining the VNM vector. According to this latter solution all pairs may form. For values of z_i above the VNM value it becomes beneficial for the other players to form the pair $[j,k]$. This threat puts the VNM outcome as an upper bound on how far player i can go in playing off the players j and k. So, player i reaches its maximum payoff at the VNM vector. Moreover, as this solution the threat of the other players to form a pair becomes credible.

We summarize the above results for the MN solutions by stating the following theorem, which also follows directly by applying the results in Houba and Bennett (1997).

Theorem 3.3 (Multilateral Nash solution).

(a) *If the pair* $[i,j]$ *is Nash-dominant, then the MN solution is unique and characterized by the unique demand vector* $y = (y_1, y_2, y_3)^\top$ *given by* $y_i = y_j = \frac{1}{2}v(i,j)$ *and* $y_k = 0$*. The resulting coalition structure is given by* $\{[i,j],[k]\}$*.*
(b) *If no pair is Nash dominant, then the MN solution is (generic) non-unique, the set of associated demand vectors is equal to the set* Y *and either* $\{[i,j],[k]\}$ *or* $\{[i,k],[j]\}$ *results as a coalition structure. If the VNM vector* z *exists, then for* $y = z$ *(and only if* $y = z$*) the coalition structure* $\{[j,k],[i]\}$ *can also results.*

In Houba and Bennett (1997) it is shown that the endpoints of the curve of MN solutions correspond to two non-cooperative bargaining models, namely the market demand model and the proposal-making model. The first one describes a negotiation situation in which competition among the three players is perfect. The outcome of this bargaining model corresponds with either a Core solution or the VNM vector. In the second model competition among the three players is imperfect because each player faces a lock-in effect. If the pair $[i,j]$ is Nash dominant, then this lock-in effect plays no role for the players i and j, which is reflected in the fact that the same outcome results as in the market bargaining model. However, otherwise the dominant player i suffers from the lock-in effect, (because his payoff is less than in the market bargaining model) and receives a payoff of at least $\frac{1}{2}v(i,j)$ as if he is in a two-player negotiation situation together with player j bargaining over the cake $v(i,j)$ while player k is not present at all. As is argued in Houba and Bennett (1997) the intermediate MN solutions can also be regarded to reflect imperfect competition among the three players. This interpretation is nicely illustrated by the unique equilibrium in the wage bargaining model in Shaked and Sutton (1984), because this equilibrium corresponds also to a MN solution and the equilibrium changes due to the lock-in effect.

4. Solutions to The Market Situation

In this section we translate the game theoretic solutions of the previous section into the original market problem. First, the VNM vector z that solves the system (1) as a function of the values w_i and α_i, $i = 1,2$, is given by

$$z = \left(\alpha_1 + \frac{w_1 - w_2}{2}, \alpha_2 + \frac{w_2 - w_1}{2}, \frac{w_1 + w_2}{2}\right)^\top .$$

Since by assumption $w_1 > w_2 > 0$ and $\alpha_i \le 0$, $i = 1,2$ these values are non-negative and hence the VNM vector exists if $\alpha_2 \ge \frac{w_1 - w_2}{2}$, i.e., if the externality of buyer 2 is large enough. Observe that both the VNM vector and the Competitive equilibrium exist if $\frac{w_1 - w_2}{2} \le \alpha_2 \le w_1 - w_2$. In the remaining of the paper we only consider the payoffs of the underlying one-seller/two-buyers problem and we denote these payoffs by $u = (u_1, u_2, u_3)$. Recall that any solution $z = (z_1, z_2, z_3)$ of the normalized game yields payoffs $u_i = z_i - \alpha_i$, $i = 1, 2$ and $u_3 = z_3$ to the two buyers and the seller. So, the payoffs at the VNM vector become

$$u = \left(\frac{w_1 - w_2}{2}, \frac{w_2 - w_1}{2}, \frac{w_1 + w_2}{2}\right)^\top ,$$

which are independent of the externalities. So, at the VNM vector player 1 gets a payoff $u_1 = \frac{w_1 - w_2}{2}$ if he forms a pair with either player 2 or player 3. In the first case the two buyers agree not to buy and buyer 2 pays a compensation $\frac{w_1 - w_2}{2}$ to buyer 1. The seller stays outside the coalition and realizes a payoff of zero. In the second case buyer 1 buys the item against price $\frac{w_1 + w_2}{2}$. Now, buyer 2 stays outside the coalition and gets a payoff of $-\alpha_2$. If the pair $[2, 3]$ forms then buyer 2 buys the item against price $\frac{w_1 + w_2}{2}$ and gets buyer 1 a payoff of $-\alpha_1$. So, if a player is in the pair that forms, his payoff is not only independent of the externalities, but also of his partner. However, only the two players in the coalition realize their VNM payoffs. The player outside the coalition can not realize his VNM payoff. If $\alpha_2 = \frac{w_1 - w_2}{2}$ then the VNM payoff of player 2 is $u_2 = \frac{w_2 - w_1}{2} = -\alpha_2$ and does not depend on being in or out the pair. In this case $z_2 = 0$ and hence it follows from Theorem 3.1, case (b) that the VNM vector is equal to the unique element of the Core. However, the Core payoffs are only realized if the pair $[1, 3]$ forms. The resulting outcome is not efficient and also not in the Core if according to the VNM vector one of the other pairs forms. According to Theorem 3.1, case (c) we have that the Core is empty if the VNM vector is strictly positive, i.e., if $\alpha_2 > \frac{w_1 - w_2}{2}$. In this case an efficient outcome is only achieved if the (efficient) pair $[i, j]$ forms. For $\alpha_2 \leq w_1 - w_2$ (Case I of the Competitive solution) this is the pair $[1, 3]$. On the other hand it follows from the cases (a) and (b) that the Core exists if system (1) does not have a strictly positive solution, i.e., if $\alpha_2 \leq \frac{w_1 - w_2}{2}$. So, if the Core is not empty, there exists a competitive price. It follows from straightforward calculations that the set of Core outcomes is given by $C = \{u \in \mathbb{R}^3 | u_1 = w_1 - u_3,\ u_2 = -\alpha_2,\ w_2 + \alpha_2 \leq u_3 \leq w_1 - \alpha_2\}$. Summarizing these results we have the following theorem.

Theorem 4.1 (Core solution and VNM vector).

(C1) *The Core is given* $C = \{u \in \mathbb{R}^3 | u_1 = w_1 - u_3,\ u_2 = -\alpha_2,\ w_2 + \alpha_2 \leq u_3 \leq w_1 - \alpha_2\}$ *and hence is non-empty if and only if* $\alpha_2 \leq \frac{w_1 - w_2}{2}$;

(C2) *if the Core is not empty, then there exists a Competitive equilibrium price and the interval* $[w_2 + \alpha_2, w_1 - \alpha_2]$ *of prices corresponding to the Core belongs to the interval* $[w_2 + \alpha_2, w_1]$ *of Competitive equilibrium prices*;

(C3) *the VNM vector exists if and only if* $\alpha_2 \geq \frac{w_1 - w_2}{2}$ *and the corresponding payoff vector is given by* $u = (\frac{w_1 - w_2}{2}, \frac{w_2 - w_1}{2}, \frac{w_1 + w_2}{2})^\top$.

Comparing the Theorems 2.1 and 4.1 it follows immediately from the properties E1, C1 and C2 that the Core does not coincide with the set of Competitive equilibria if the externalities are strictly positive. In particular we have that the Core is empty and the set of Competitive equilibria is not empty if $\frac{w_1 - w_2}{2} < \alpha_2 \leq w_1 - w_2$.

Observe that the set of Core prices is a subset of the set of Competitive equilibrium outcomes, namely the outcomes corresponding to the Competitive prices $w_2+\alpha_2 \leq p \leq w_1 - \alpha_2$. For prices $w_1 - \alpha_2 < p \leq w_1$, the Competitive equilibrium outcome is outside the Core, because in this case the two buyers can do better by agreeing not to buy. Moreover we see that the Core shrinks if α_2 increases and reduces to just one point for the earlier found value $\alpha_2 = \frac{w_1-w_2}{2}$. In this case the unique Core point coincides with the VNM vector and the outcome corresponds to the minimum value $w_2 + \alpha_2$ of the Competitive equilibrium price. Furthermore, an increase in the value of externality of buyer 2 increases the minimum Core payoff of buyer 1 and decreases the maximum Core payoff of buyer 1 with the same amount. With respect to the Core outcomes the externality of buyer 2 effects the outcomes of buyer 1 and the seller equally. Finally, observe that the Core outcomes do not depend on the externality of buyer 1. Obviously, this is because at any Core solution the pair $[1, 3]$ forms.

We now consider the set of MN solutions. First of all, it follows from straightforward calculations that only the pair $[1, 3]$ can be a Nash-dominant pair. To be so, we must have that $\frac{w_1+\alpha_1}{2} > \max\{w_2+\alpha_2, \alpha_1+\alpha_2\}$, which implies that $\alpha_2 < \frac{w_1-w_2}{2}$. Hence, if $[1, 3]$ is indeed a Nash-dominant pair, then the corresponding unique MN solution lies in the Core and hence the Core is not empty. The only other case that there exists a unique MN solution is when $w_1 + \alpha_1 = w_2 + \alpha_2 = \alpha_1 + \alpha_2$. Then this unique solution is equal to the VNM vector. In all other cases there exists a set of MN solutions, depending on the values of the parameters. According to the six permutations (i, j, k) over the set of players, there are six different regimes. Recall that i is the dominant player, j is the modal player and k is the dominated player. For given values of w_1 and w_2 these six regimes can be drawn in the (α_1, α_2) space and are determined by the three equations $v(1, 2) = v(1, 3)$, $v(1, 2) = v(2, 3)$ and $v(1, 3) = v(2, 3)$. The first equation gives the line

$$\alpha_2 = w_1 - w_2 + \alpha_1 \, ,$$

the second equation the line $\alpha_1 = w_2$ and the last equation the line $\alpha_2 = w_1$. These lines are drawn heavily in Fig. 1 for given values w_1 and w_2 such that $\frac{1}{2}w_1 > w_2$. Moreover in this figure the equation

$$v(1, 3) = 2 \cdot \max\{v(1, 2), v(2, 3)\}$$

is represented by the curve DNE. The region "NASH" below this curve is the region of values for which the pair $[1, 3]$ is Nash-dominant.

Below the horizontal lines $\alpha_2 = \frac{w_1-w_2}{2}$ and $\alpha_2 = w_1 - w_2$ the Core is not empty, respectively there exists a Competitive equilibrium. Figure 1 does not change much if $w_2 > w_1/2$. In this case the point D moves to the horizontal α_1-axes, implying that then the origin does not belong the NASH-dominant region. We will explain the dashed lines in the figure later.

We now consider the six different regions. In region A1 we have that $i = 3$, $j = 1$ and $k = 2$. According to the existence of the Core and/or the Competitive

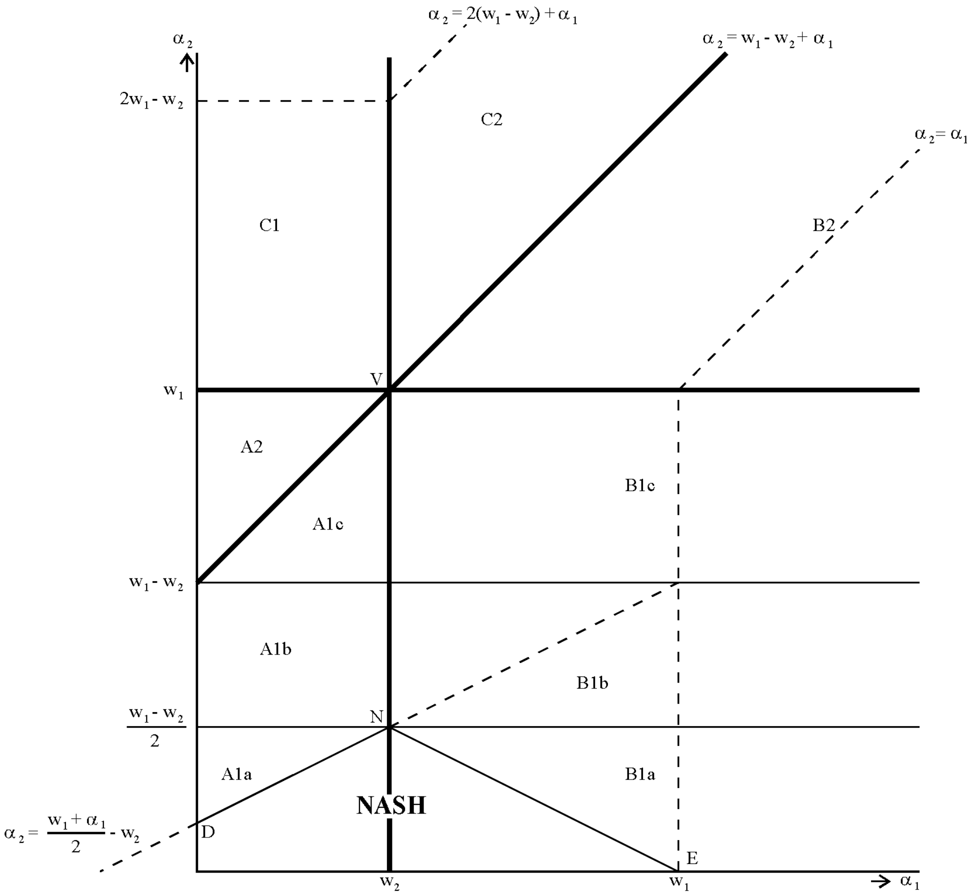

Fig. 1. The partition of the (α_1, α_2) space into regions of dominance.

equilibrium, the regions A1 and B1 (in which $i = 1$, $j = 3$ and $k = 2$) are partitioned up into three subregions a, b and c. For each (sub)region all the data are given in Table 1. This table characterizes each region by the ordering (i, j, k) of the players with respect to domination, respectively the Nash dominant pair, and gives the payoff vectors of the set of the MN solutions, and, if exist, the VNM vector, the set of Core solutions and the set of Competitive outcomes. Outside the Nash region the set of MN solutions follows from solving the system (2)–(6). The upper bound of the payoff u_i of the dominant player i follows from (6) in A1a and B1a, and from (5) in all other regions. The lower bound follows from (4). Recall that in case of the VNM vector and the MN-solutions a player only realizes the reported payoff if he is in the pair that forms. If not, he stays at his initial value. Moreover, with respect the MN solutions outside the NASH region only the pairs (i, j) and (i, k) can form. Finally observe that at the point V in Fig. 1 we have that $\alpha_1 = w_2$ and $\alpha_2 = w_1$. So, at this point all two-player coalition have the same value in the normalized

Table 1. The Competitive equilibria and game theoretic solutions for various values of the externalities. The payoff u_3 is equal to the price the seller would get if he is included in the pair that forms.

Region and ordering		The players' payoffs		
		u_1	u_2	u_3
NASH	MN	$w_1 - u_3$	$-\alpha_2$	$\frac{1}{2}(w_1 + \alpha_1)$
pair $[1, 3]$	Core	$w_1 - u_3$	$-\alpha_2$	$[w_2 + \alpha_2, w_1 - \alpha_2]$,
is dominant	Comp	$w_1 - u_3$	$-\alpha_2$	$[w_2 + \alpha_2, w_1]$,
A1a	MN	$w_1 - u_3$	$w_2 - u_3$	$\left[\frac{1}{2}(w_1 + \alpha_1), w_2 + \alpha_2\right]$,
$(3, 1, 2)$	Core	$w_1 - u_3$	$-\alpha_2$	$[w_2 + \alpha_2, w_1 - \alpha_2]$
	Comp	$w_1 - u_3$	$-\alpha_2$	$[w_2 + \alpha_2, w_1]$
A1b	MN	$w_1 - u_3$	$w_2 - u_3$	$\left[\frac{1}{2}(w_1 + \alpha_1), \frac{1}{2}(w_1 + w_2)\right]$,
$(3, 1, 2)$	VNM	$\frac{1}{2}(w_1 - w_2)$	$\frac{1}{2}(w_2 - w_1)$	$\frac{1}{2}(w_1 + w_2)$
	Comp	$w_1 - u_3$	$-\alpha_2$	$[w_2 + \alpha_2, w_1]$
A1c	MN	$w_1 - u_3$	$w_2 - u_3$	$\left[\frac{w_1 + \alpha_1}{2}, \frac{w_1 + w_2}{2}\right]$,
$(3, 1, 2)$	VNM	$\frac{1}{2}(w_1 - w_2)$	$\frac{1}{2}(w_2 - w_1)$	$\frac{1}{2}(w_1 + w_2)$
A2	MN	$w_1 - u_3$	$w_2 - u_3$	$\left[\frac{w_2 + \alpha_2}{2}, \frac{w_1 + w_2}{2}\right]$,
$(3, 2, 1)$	VNM	$\frac{1}{2}(w_1 - w_2)$	$\frac{1}{2}(w_2 - w_1)$	$\frac{1}{2}(w_1 + w_2)$
B1a	MN	$\left[\frac{1}{2}(w_1 - \alpha_1), \alpha_2\right]$	$-u_1$	$w_1 - u_1$
$(1, 3, 2)$	Core	$[\alpha_2, w_1 - w_2 - \alpha_2]$	$-\alpha_2$	$w_1 - u_1$
	Comp	$[0, w_1 - w_2 - \alpha_2]$	$-\alpha_2$	$w_1 - u_1$

Table 1. (*Continued*).

Region and ordering		The players' payoffs		
		u_1	u_2	u_3
B1b	MN	$\left[\frac{1}{2}(w_1 - \alpha_1), \frac{1}{2}(w_1 - w_2)\right]$	$-u_1$	$w_1 - u_1$
$(1, 3, 2)$	VNM	$\frac{1}{2}(w_1 - w_2)$	$\frac{1}{2}(w_2 - w_1)$	$\frac{1}{2}(w_1 + w_2)$
	Comp	$[0, w_1 - w_2 - \alpha_2]$	$-\alpha_2$	$w_1 - u_1$
B1c	MN	$\left[\frac{1}{2}(w_1 - \alpha_1), \frac{1}{2}(w_1 - w_2)\right]$	$-u_1$	$w_1 - u_1$
$(1, 3, 2)$	VNM	$\frac{1}{2}(w_1 - w_2)$	$\frac{1}{2}(w_2 - w_1)$	$\frac{1}{2}(w_1 + w_2)$
B2	MN	$\left[\frac{1}{2}(\alpha_2 - \alpha_1), \frac{1}{2}(w_1 - w_2)\right]$	$-u_1$	$w_1 - u_1$
$(1, 2, 3)$	VNM	$\frac{1}{2}(w_1 - w_2)$	$\frac{1}{2}(w_2 - w_1)$	$\frac{1}{2}(w_1 + w_2)$
C1	MN	$-u_2$	$\left[\frac{1}{2}(w_2 - \alpha_2), \frac{1}{2}(w_2 - w_1)\right]$	$w_2 - u_2$
$(2, 3, 1)$	VNM	$\frac{1}{2}(w_1 - w_2)$	$\frac{1}{2}(w_2 - w_1)$	$\frac{1}{2}(w_1 + w_2)$
C2	MN	$-u_2$	$\left[\frac{1}{2}(\alpha_1 - \alpha_2), \frac{1}{2}(w_2 - w_1)\right]$	$w_2 - u_2$
$(2, 1, 3)$	VNM	$\frac{1}{2}(w_1 - w_2)$	$\frac{1}{2}(w_2 - w_1)$	$\frac{1}{2}(w_1 + w_2)$

game and therefore the MN solution is unique and equal to the VNM vector. At the point N we have that the VNM vector is equal to both the unique MN solution and the unique Core outcome.

Inspection of Table 1 leads to the following observations. First of all, depending on the values of the externalities, any dominance ordering of the players can occur. Obviously, in the one-seller/two-buyers game without externalities either the pair $[1,3]$ is Nash-dominant (if $\frac{w_1}{2} \geq w_2$) or the seller is the dominant player. In case of externalities this is only true for low values of the externalities. High values give the buyers an incentive to stick together and to agree not to buy. So, the occurrence of externalities has two opposite effects on the position of the seller. If he is in the pair that forms, he obtains a higher price. On the other hand, if he is not in the pair, then he is left with a payoff of zero. Observe that the value of u_3 equals the price against the item is sold in case the seller is in the pair that forms. So, Table 1 also provides the price of the item. Observation of the table therefore immediately shows that the price induced by the VNM vector equals the maximum of the set of prices induced by the set of MN solutions if the seller is the dominant player (case A). In case one of the buyers is the dominant player (Cases B and C) we have that the price induced by the VNM vector equals the minimum of the set of prices induced by the set of MN solutions. We now consider the different regions in more detail.

In region A1a (the dominant seller forms a pair with either the modal buyer 1 or the dominated buyer 2) the maximum payoff of the seller on the set of MN solutions is equal to the minimum payoff of the seller on the set of Core solutions and on the set of Competitive outcomes. So, in this region we have that at any MN solution the dominant seller sells the item at a price which is at most equal to the price at any Core (Competitive) outcome. This is because the MN solution allows that the (inefficient) pair $[2,3]$ forms and player 2 is not willing to pay a price above $w_2 + \alpha_2$.

In region A1b the payoff of the seller on the set of MN solutions is at most equal to his payoff in the VNM vector, i.e., at any MN solution the item is sold against a price at most equal to the price at the VNM tuple. However, remember that at any MN solution the seller is sure to be in the pair that forms, whereas at the VNM vector also the pair of buyers can form. Moreover, since $\alpha_2 > \frac{w_1 - w_2}{2}$ we have that the minimal Competitive price $w_2 + \alpha_2$ is higher than the maximal price $\frac{w_1+w_2}{2}$ on the set of MN solutions. So, again we observe that the dominant player suffers from the lock-in effect. Once a pair is formed with one of the buyers, the two buyers can not be played off against each other any longer.

In the region A1c the value of the externality of player 2 becomes so large that competition does not lead to a solution. Now, the only possibility for the seller to sell the item is to start negotiations with one of the two buyers. As soon as he has formed a pair with one of these buyers, he suffers again from the lock-in effect.

Finally, observe that in the region A1 the minimum payoff of the seller on the set of MN solutions increases in the value of α_1. In fact, an increase of α_1

brings the seller in a better position. An increase in α_2 has a positive effect on the Competitive price, but also makes it more attractive for the buyers to avoid competitive behavior. For α_2 high enough the seller can not take advantage any longer from the competition between the buyers.

In the region A2 buyer 2 becomes the modal player and buyer 1 the dominated player. Now the minimum payoff of the seller on the set of MN solutions increases in α_2 instead of α_1.

In region B buyer 1 is the dominant player. In region B1 (the seller is the modal player) the minimum payoff of buyer 1 on the set of MN solutions is equal to $\frac{w_1-\alpha_1}{2}$. So, at the right side of the (dashed) line $\alpha_1 = w_1$ there are solutions in which player 1 is willing to pay a price above w_1 if he forms a pair with the seller or to pay a compensation if he forms a pair with buyer 2. So, in this case we have the striking result that there are solutions in which the dominated buyer 2 gets a positive and the dominant player is willing to accept a loss.

In region B1a the maximum payoff of buyer 1 on the set of MN solutions is equal to $u_1 = \alpha_2$. This is equal to the minimum payoff of buyer 1 on the set of Core solutions. So, in this region we have again that at any MN solution the payoff of the dominant player is at most equal to his payoff at any Core solution. Observe that this payoff is increasing in α_2 and hence the corresponding payoff of the seller is decreasing in α_2. So, the maximum price according to the Core solution and the minimum price according to the MN solution are decreasing in the externality of buyer 2. An increase in the externality of this buyer puts his competitor in a better bargaining position with respect to the seller. This is because the willingness of buyer 2 to pay a compensation for an agreement not to buy becomes larger. Since $\alpha_2 \leq \frac{w_1-w_2}{2}$ we also have that for any MN solution there is a Competitive outcome which gives the dominant buyer 1 a higher payoff. Moreover, the Competitive outcomes guarantees buyer 1 a payoff of at least zero.

In region B1b the Core is empty. Analogous to region A, in this case we have again that the payoff of the dominant player on the set of MN solutions is at most equal to his payoff in the VNM vector. However, remember again that at any MN solution buyer 1 is sure to be in the pair that forms, whereas at the VNM vector also the pair $[2,3]$ can form. The minimum payoff of the dominant player on the set of MN solutions is equal to $\frac{w_1-\alpha_1}{2}$. Above the dashed line $\alpha_2 = \frac{w_1+\alpha_1}{2} - w_2$ this payoff is higher than the maximum payoff on the Competitive outcome. Below the dashed line the minimum payoff of buyer 1 on the set of MN solutions is below his maximum payoff of the Competitive outcomes and for high values of α_1 there even exist MN solutions with a negative payoff for buyer 1, whereas all Competitive outcomes yield a non-negative payoff. So, for high values of α_1 buyer 1 prefers competition, for values of α_1 above the dashed line buyer 1 prefers an MN solution. The reverse is true for the seller, provided that he is in the pair that forms. The dominated buyer 2 never prefers competition, because he always looses the competition and gets a payoff of $-\alpha_2$. Any MN solution gives buyer 2 at least the same payoff and a higher payoff if he is in the pair that forms.

In region B1c the maximum payoff of the dominant player 1 on the set of MN solutions is equal to the VNM payoff, whereas his minimum payoff decreases in α_1.

In region B2 buyer 2 becomes the modal player. In this case the minimum payoff of the dominant player equals $\frac{\alpha_2-\alpha_1}{2}$ and hence is negative below the dashed line $\alpha_2 = \alpha_1$ in Fig. 1. Below this line the dominant buyer is willing to pay a price above his reservation value w_1 or to compensate buyer 2 for an agreement not to buy. We also see that the minimum payoff of buyer 1 increases in α_2. So, when forming a pair with the seller this means that the maximum price decreases in α_2 and hence an increase in the externality of the modal buyer 2 weakens the position of the dominated seller in bargaining with buyer 1. This is because an increase in the externality of buyer 2 makes him willing to increase his compensation to player 1 for getting an agreement not to buy.

Region C2 is analogous to region B2 with a change of roles between the two buyers. Now buyer 2 is the dominant buyer. His minimum payoff on the set of MN solutions is $\frac{\alpha_1-\alpha_2}{2}$ and hence is increasing in α_1, So, the maximum price is decreasing in α_1. Above the dashed line $\alpha_2 = \alpha_1 + 2(w_1 - w_2)$ we have that the minimum payoff of buyer 2 is less than $w_2 - w_1$, i.e., buyer 2 is willing to pay more than the reservation value w_1 of buyer 1. In region C1 the maximum payoff of the dominant buyer 2 is equal to his payoff at the VNM vector, while his minimum payoff is decreasing in α_2. So, in this case the price does not depend on α_1 and is increasing in α_2. Above the dashed line $\alpha_2 = 2w_1 - w_2$ buyer 2 is willing to pay a price above w_1.

One final remark is in place. As mentioned before the MN solution captures the situation of imperfect competition among the three participants and in case no pair is Nash-dominant it is the dominant player who suffers from this imperfect competition. In region A, which includes the standard case without externalities, the seller is the dominant player. Hence, imperfect competition implies that the seller receives a price that is either the price corresponding to the VNM vector or at most the lowest price corresponding to the Core. Note that the seller, as the dominant player, is always included in the pair that forms (except if the MN vector equals the VNM vector). In regions B and C the seller is not the dominant player and one of the buyers suffers from imperfect competition. This is reflected in the fact that every MN price is at least the price corresponding to the VNM vector. This leads to the paradox that negative buyer externalities and imperfect competition raises the price paid for the good. This “higher” price is however not without costs for the seller, because the seller is not automatically included in the pair that forms. So, the seller only benefits from imperfect competition if he is included in the pair that forms.

5. Concluding Remarks

In this paper we have analyzed the one-seller/two-buyer problem of bargaining in case of externalities between the buyers and the presence of legal opportunities for cooperation between the two buyers, i.e., they can commit not buying the object and paying a monetary transfer among them. If the externalities are large compared to valuations and if buyers can cooperate, then one should expect that coalitions between buyers form in small markets. In the existing literature [e.g., Jéhiel and Moldovanu (1995, 1996)] cooperation between the two buyers is not taken into account. For the standard market problem without externalities there is no difference in results between the two cases, but our analysis shows that it does matter in the presence of buyer externalities.

In order to study these markets the problem is modeled as a three-player/three-cake problem and the cooperative solution concepts of the Core, the Bargaining set, the Stable set and the Multilateral Nash solution are applied and their outcomes are compared with the Competitive outcome, as far as the latter exists. We have shown that the Competitive outcome exists as long as the sum of the externality value α_2 and the reservation value w_2 of the weakest buyer is smaller than the reservation value w_1 of the strongest buyer. Moreover, the set of Core outcomes is a strict subset of the set of Competitive outcomes, which is quite different result than for the standard case with zero externalities. Finally, all standard results known for the one-seller/two buyer problem without externalities can be obtained as the limit result by letting the externalities vanish and, hence, the standard results are robust with respect to small externalities.

The non-existence of the Competitive equilibrium as well as the Core is a serious drawback of these concepts, because this implies that these concepts fail to provide a satisfying answer to the economic problem. The Bargaining set is able to produce an answer for the whole class of one-seller/two-buyer problems with buyer externalities. A remarkable result is that, if the Core is empty, all three pairs may form and that the payoffs of the participants in each pair do not depend on the externalities but only on the buyer valuations w_1 and w_2 of the buyers. If the seller is included in the pair that forms and, thus, the object is sold, then the seller obtains the average of the buyer valuations w_1 and w_2. This price is always included in the set of Core prices, provided the Core is not empty. If instead the pair of the two buyers forms and, thus, the good is not sold, then it is always the buyer with the lowest valuation who makes a monetary transfer equal to half of (the absolute value of the) difference between the buyers' valuations to the other buyer. This transfer is equal to the foregone consumer surplus of the buyer with the highest valuation that would have been obtained if this buyer buys the object. Thus, similar as in the standard case without externalities the asymmetry between the two buyers is based upon the asymmetry in valuations w_1 and w_2.

Finally, the Bargaining set represents a solution for markets with perfect competition. Markets with imperfect competition, such as lock-in effects for one of the participants, exists as well. We have implicitly investigated markets with imperfect competition by looking at the MN solution. We defined the notion of a dominant player and saw that this player is always included in the pair that forms (neglecting the MN solution that coincides with the VNM vector). Imperfect competition is bad for the dominant player, because this player is not able to fully play off the other two players as would have been the case with perfect competition. For small buyer externalities the seller is the dominant player (just as in the case without externalities) and, hence, the seller is always included in the pair that forms and the seller suffers from the imperfect competition because the resulting price is lower than the price that would have been obtained under perfect competition. However, for larger buyer externalities the seller is no longer the dominant player. Since one of the buyers is the dominant player this buyer will suffer from imperfect competition and, hence, this yields the paradox that the price of the object is higher under imperfect competition than under perfect competition. Since the seller is not automatically included in the pair that forms it is clear that the seller only benefits from imperfect competition if he is included in the pair that forms.

Acknowledgment

We thank Maarten Cornet for useful comments on an earlier draft of this paper. We also want to thank Xander Tieman for providing the figure. This research is part of the Research-program "Competition and Cooperation", which is conducted at the Free university.

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