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Measure-Valued Differentiation for the Cycle Cost Performance in the G/G/1 Queue in the Presence of Heavy-Tailed Distributions

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Abstract. We consider the accumulate costs over a cycle of a phase-homogeneous random walk. For this model we establish sufficient conditions for the existence of the derivative of the cycle cost and we establish an unbiased gradient estimator. The main stability condition for our analysis is that the expected cycle costs are finite. We thereby improve the results known in the literature so far, where usually finiteness of higher moments of the cycle length is assumed in order to establish unbiasedness of a particular gradient estimator.

Keywords: measure-valued differentiation, perturbation analysis, Markov chains

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1. Introduction

Let $X_\theta(n)$ be the waiting time of the nth customer in an G/G/1 queue, depending on a (vector-valued) parameter $\theta \in \Theta$ with $X_\theta(0) \in \alpha$, for some measurable set α. We denote the drift of $X_\theta(n)$ by $\xi_\theta(n)$, in formula

$$X_\theta(n+1) \quad = \quad \max(\, X_\theta(n) + \xi_\theta(n)\, ,\, 0\,) \quad n \geq 1\,. \tag{1.1}$$

We assume that the drift sequence $\xi_\theta(n)$ is i.i.d. and that the system is stable, i.e., we assume that $\mathbb{E}[\xi_\theta(1)] < \infty$. A broad class of problems can be modeled by

$$\mathbb{E}\Big[\sum_{n=0}^{\tau_{\alpha,\theta}(s)-1} g(X_\theta(n))\Big], \tag{1.2}$$

where $\tau_{\alpha,\theta}$ denotes the first entrance time after $n = 0$ of X_θ into α and g is some cost function.

In optimization and sensitivity analysis, one is interested in computing/ estimating the derivative of the overall performance in (1.2) with respect to θ. It has already been observed in the literature that there are many situations where the derivative of (1.2) can be obtained from observing the process up to time $\tau_{\alpha,\theta}$. However, for establishing unbiasedness of the estimator one usually requires that the second or third moment of $\tau_{\alpha,\theta}$ has finite expected value. See, for example, [1, 3, 8].

This moment condition on the cycle-length can be harmful in the presence of heavy-tailed distributions. For example, if g in (1.2) is bounded by a polynomial of degree p, then the derivative of the cycle performance exists if the $(p+1)$st moment of the drift is finite. For example, for g bounded, we require finiteness of the expected value of $\tau_{\alpha,\theta}$. Since finiteness of the expected value of $\tau_{\alpha,\theta}$ is already necessary for the cycle performance in (1.2) to exist, our analysis provides a set of minimal conditions for unbiasedness of a gradient estimator for the derivative of the cycle cost. To summarize, we show that for waiting times in the G/G/1 queue existence of the cycle performance already implies unbiasedness of the gradient estimator for bounded cost functions.

The paper is organized as follows. In Section 2 preliminary results are presented. In particular, a brief introduction to the theory of measure-valued differentiation is provided. The main result of the paper is presented in Section 3 and applications are discussed in Section 4. The technical analysis is provided in Section 5 and Section 6.

The paper has two main contributions. The first contribution is that differentiability of cycle costs of the G/G/1 queue is established that extends the results in [6] to systems satisfying much weaker stability conditions. Specifically, heavy tailed distributions can be treated with the framework provided in this paper, whereas this type of distribution is out ruled by the conditions required for the analysis in [6]. The second contribution is that conditions for unbiasedness of gradient estimators provided in this paper are minimal and it is the first result of this type.

2. Preliminaries

2.1. Taboo kernels and the potential kernel

Let $(S, \mathcal{T})$ be a Polish measurable space. Let $\mathcal{M}(S, \mathcal{T})$ denote the set of finite (signed) measures on $(S, \mathcal{T})$ and $\mathcal{M}_1(S, \mathcal{T})$ that of probability measures on $(S,\mathcal{T})$. The mapping $P : S \times \mathcal{T} \to [0,1]$ is called a (homogeneous) *transition kernel* on $(S,\mathcal{T})$ if (i) $P(s;\cdot) \in \mathcal{M}(S,\mathcal{T})$ for all $s \in S$; and (ii) $P(\cdot;B)$ is $\mathcal{T}$ measurable for all $B \in \mathcal{T}$. If, in condition (i), $\mathcal{M}(S,\mathcal{T})$ can be replaced by $\mathcal{M}_1(S,\mathcal{T})$, then P is called a *Markov kernel* on $(S,\mathcal{T})$. Denote the set of transition kernels on $(S,\mathcal{T})$ by $\mathcal{K}(S,\mathcal{T})$ and the set of Markov kernels on $(S,\mathcal{T})$ by $\mathcal{K}_1(S,\mathcal{T})$. A transition kernel $P \in \mathcal{K}(S,\mathcal{T})$ with $0 < P(s;S) < 1$ for at least one $s \in S$ is called a *defective* Markov kernel, and terms "transition kernel" and "defective Markov kernel" are synonyms.

Consider a family of Markov kernels $(P_\theta : \theta \in \Theta)$ on $(S,\mathcal{T})$, with $\Theta \subset \mathbb{R}$, and let $\mathcal{L}^1(P_\theta;\Theta) \subset \mathbb{R}^S$ denote the set of measurable mappings $g : S \to \mathbb{R}$, such that $\int_S P_\theta(s;du)\,|g(u)|$ is finite for all $\theta \in \Theta$ and $s \in S$. A kernel P_θ is called $\mathcal{D}$*-preserving*, with $\mathcal{D} \subset \mathcal{L}^1(P_\theta;\Theta)$, if $g \in \mathcal{D}$ implies $\int_S P_\theta(\cdot;du)g(u) \in \mathcal{D}$. To simplify the notation, we set

$$(P_\theta g)(s) \triangleq \int\limits_S P_\theta(s;du)g(u)$$

for $g \in \mathcal{L}^1(P_\theta;\Theta)$ and $s \in S$.

For $P_\theta \in \mathcal{K}_1(S,\mathcal{T})$ and $V \in \mathcal{T}$, the taboo operator associated with P_θ for some taboo set V is defined as

$$\forall g \in \mathcal{L}^1(P_\theta;\Theta) : \quad ({}_V P_\theta g)(s) \triangleq \int\limits_{u \notin V} P_\theta(s;du)g(u)\,,$$

for $s \in S$. Note that if $P_\theta(s,V) > 0$ for some $s \in S$, then ${}_V P_\theta$ is defective. Taking $\alpha = V$, the expression in (1.2) reads

$$\mathbb{E}\Big[\sum_{n=0}^{\tau_{\alpha,\theta}-1} g(X_\theta(n))\Big] \;=\; \sum_{n=0}^{\infty} {}_\alpha P_\theta^n g\,,$$

provided that it exists. The operator

$$\mathrm{H}_\theta \triangleq \sum_{n=0}^{\infty} {}_\alpha P_\theta^n$$

is called the *potential of* ${}_\alpha P_\theta$. Note that the potential of ${}_\alpha P_\theta$ yields the distribution of a cycle of X_θ, in formula,

$$(\mathrm{H}_\theta g)(s) = \mathbb{E}\Big[\sum_{n=0}^{\tau_{\alpha,\theta}-1} g(X_\theta(n)) \mid X_\theta(0) = s\Big],$$

for any $s \in S$ and for any g for which (1.2) exists. Denoting by e the mapping that maps any $s \in S$ onto 1, gives for any $s \in S$:

$$\mathbb{P}_\theta(\tau_{\alpha,\theta} > n \,|\, X_\theta(0) = s) \;=\; ({}_\alpha P_\theta^n e)(s) \tag{2.1}$$

and

$$\mathbb{E}[\tau_{\alpha,\theta} \mid X_\theta(0) = s] \;=\; \sum_{n=0}^{\infty} ({}_\alpha P_\theta^n e)(s) \;=\; (\mathrm{H}_\theta e)(s)\,.$$

2.2. Measure-valued differentiation

In what follows, we let Θ be an open neighborhood of θ_0 and assume that $\mathcal{D} \subset \mathcal{L}^1(P_\theta;\Theta)$.

Definition 2.1. We call $P_\theta \in \mathcal{K}(S,\mathcal{T})$ differentiable at θ with respect to $\mathcal{D}$, or $\mathcal{D}$-differentiable for short, if $P'_\theta \in \mathcal{K}(S,\mathcal{T})$ exists such that for any $g \in \mathcal{D}$ and any $s \in S$:

$$\frac{d}{d\theta}\int_S P_\theta(s;du)\,g(u) \;=\; \int_S P'_\theta(s;du)\,g(u). \tag{2.2}$$

If the left-hand side of equation (2.2) equals zero for all $g \in \mathcal{D}$, then we say that P'_θ is not significant.

We denote the set of bounded continuous mappings from S to $\mathbb{R}$ by $C^b(S)$ and assume, unless stated otherwise, that $C^b(S) \subset \mathcal{D}$. This implies that P'_θ in (2.2) is uniquely defined provided that P_θ is $\mathcal{D}$-differentiable. For more details on measure-valued differentiation (MVD), we refer to [5, 10, 12, 13].

Definition 2.2. Let $P_\theta \in \mathcal{K}(S,\mathcal{T})$ be $\mathcal{D}$-differentiable at θ. Any triple $(c_{P_\theta}(\cdot), P_\theta^+, P_\theta^-)$, with $P_\theta^\pm \in \mathcal{K}_1(S,\mathcal{T})$ and c_{P_θ} a measurable mapping from S to $\mathbb{R}$ such that

$$\forall g \in \mathcal{D}\;:\quad \int_S P'_\theta(s;du)\,g(u) = c_{P_\theta}(s)\left(\int_S P_\theta^+(s;du)\,g(u) \;-\; \int_S P_\theta^-(s;du)\,g(u)\right)$$

is called a $\mathcal{D}$-derivative of P_θ.

Remark 2.1. If P_θ is $\mathcal{D}$-differentiable, so is ${}_V P_\theta$ provided that V is independent of θ. Moreover, if $(c_{P_\theta}(\cdot), P_\theta^+, P_\theta^-)$ is an instance of a $\mathcal{D}$-derivative for P_θ, then an instance of a $\mathcal{D}$-derivative of ${}_V P_\theta$ is given by

$$\left(c_{{}_V P_\theta}, {}_V P_\theta^+, {}_V P_\theta^-\right),$$

with $c_{{}_V P_\theta}(s) = c_{P_\theta}(s)$ for $s \in S$.

Let $v : S \to \mathbb{R}$ be a measurable mapping such that

$$\inf_{s\in S} v(s) \geq 1. \tag{2.3}$$

The set of mappings from S to $\mathbb{R}$ can be equipped with the so-called functional *v-norm*, where

$$\|f\|_v = \sup_{s \in \mathcal{S}} \frac{|f(s)|}{|v(s)|}.$$

For μ a (signed) measure the associated measure norm is

$$\|\mu\|_v = \sup_{\|f\|_v \leq 1} |\mu f|$$

and for a kernel P the associated operator norm reads

$$\|P\|_v = \sup_{s \in S} \sup_{\|f\|_v \leq 1} \frac{\left|\int f(z)\, P(s; dz)\right|}{|v(s)|} .$$

If g has finite v-norm, then $|g(s)| \leq c\, v(s)$ for any $s \in S$ and some finite constant c. Let $\mathcal{H}$ be an arbitrary set of measurable mappings and let $v \in \mathcal{L}^1(P_\theta; \Theta)$. We denote the subset of $\mathcal{H}$ constituted out of the v-dominated functions by $(\mathcal{H}, v)$; in formula:

$$(\mathcal{H}, v) \triangleq \{g \in \mathcal{H} \,:\, ||g||_v < \infty\}.$$

Let $v_p(s) = \sum_{k=0}^{p} d_k |s|^k$ for finite constants $d_k \geq 0$, for $p \geq k > 0$ and $d_0 > 0$, then $(\mathcal{H}, p) \triangleq (\mathcal{H}, v_p)$ denotes the set of mappings $g \in \mathcal{H}$ that are bounded by a polynomial of degree p, that is, $g \in (\mathcal{H}, p)$ implies that

$$|g(s)| \leq \sum_{k=0}^{p} c_k |s|^k$$

for some finite constants $c_k \geq 0$, $k = 0, \ldots, p$.

We call $(\mathcal{H}, v)$ (resp. $(\mathcal{H}, p)$) *Banach* if $(\mathcal{H}, v)$ (resp. $(\mathcal{H}, p)$) is a Banach space with respect to the v-norm.

Let P_θ be $(\mathcal{H}, v)$-differentiable at $\theta \in \Theta$ with $(\mathcal{H}, v)$ Banach. Then, for any neighborhood $U = [\theta - \Delta, \theta + \Delta] \subset \Theta$ of $\theta \in \Theta$ a finite constant M exists such that

$$\forall |h| \leq \Delta : \quad \|P_{\theta+h} - P_\theta\|_v \leq |h|\, M, \tag{2.4}$$

see [7]. In words, P_θ is locally v-norm Lipschitz. For a signed measure μ on $(S, \mathcal{S})$ we denote its positive part by $[\mu]^+$ and its negative part by $[\mu]^-$. The absolute value of μ, in symbols $|\mu|$, is defined by $|\mu| = [\mu^+] + [\mu^-]$ and it holds that

$$\forall |h| \leq \Delta : \quad \int g(u) |P_{\theta+h} - P_\theta|(du) \leq ||P_{\theta+h} - P_\theta||_v v(s) \tag{2.5}$$

for all g such that $||g||_v \leq 1$, see [7] for details.

For our analysis we require a set $\mathcal{D}$ of performance measures that satisfies the following conditions:

(i) There exists $v \in \mathcal{L}^1(P_\theta;\Theta)$ such that $\mathcal{D}$ endowed with the $||\cdot||_v$-norm becomes a Banach space.

(ii) P_θ is $\mathcal{D}$-differentiable.

In the following we discuss typical examples for $\mathcal{D}$, where $C(S)$ denotes the set of all continuous mappings belonging to $\mathcal{L}^1(P_\theta;\Theta)$.

- Let $\mathcal{H} = C(S)$ and let $v \in \mathcal{L}^1(P_\theta;\Theta)$ such that v satisfies (2.3). Then, $\mathcal{D} = (\mathcal{H}, v)$ is the set of all continuous mappings bounded by v up to multiplicative constant and $\mathcal{D}$ equipped with the $||\cdot||_v$-norm becomes the Banach space of continuous mappings with finite v-norm[1]. In particular, for $v \equiv 1$, $\mathcal{D}$ becomes the set of bounded continuous mappings, denoted by $C^b(S)$.

- Let $\mathcal{H} = \mathcal{L}^1(P_\theta;\Theta)$ and let $v \in \mathcal{L}^1(P_\theta;\Theta)$ such that v satisfies (2.3). Then, $\mathcal{D} = (\mathcal{H}, v)$ is the set of all measurable mappings bounded by v up to a multiplicative constant and $\mathcal{D}$ equipped with the v-norm becomes the Banach space of measurable mappings with finite v-norm.

The question whether (ii) is satisfied for $\mathcal{D}$ depends on P_θ. It may happen that P_θ is only $\mathcal{D}$-differentiable for a particular choice of $\mathcal{D}$. Roughly speaking, $\mathcal{D}$-differentiability with respect to $\mathcal{D} = (\mathcal{L}^1(P_\theta;\Theta), v)$ is the most restrictive condition, since it requires that indicator functions are differentiable. On the other hand, $\mathcal{D} = C^b(S)$, that is, $\mathcal{D} = (C(S), v \equiv 1)$ is the least restrictive choice for $\mathcal{D}$, however, excluding the analysis of possibly unbounded cost functions. See the discussion in [13] for details.

2.3. Random walks

For our analysis, we model the waiting times $X_\theta(n)$ in an G/G/1 queue as a collection of Markov chains on the positive half-line with jump variables $\xi_\theta(x)$ in state x. More specifically, let S_θ be a sample of the service time and let A_θ be an independent sample of the interarrival time. Then, the drift of $X_\theta(n)$ is denoted by $\xi_\theta = S_\theta - A_\theta$. For $x \in \mathbb{R}^+ \triangleq \{x \in \mathbb{R} : x \geq 0\}$ let

$$\xi_\theta(x) = \max(\xi_\theta + x, 0) = x + \max(\xi_\theta, -x), \tag{2.6}$$

with $\mathbb{E}[\xi_\theta]$ finite for any $\theta \in \Theta$. Hence, the $(n+1)$st waiting time is obtained from $X_\theta(n+1) = \xi_\theta(x) = \max(\xi_\theta + x, 0) = \max(S_\theta - A_\theta + x, 0)$, where $X_\theta(n) = x$. Observe that the very definition of $\xi_\theta(x)$ implies that

$$\text{(S1)} \qquad x \leq y \quad \Rightarrow \quad \xi_\theta(x) \leq \xi_\theta(y).$$

[1] One may slightly deviate from the requirement of continuity. For example, let g be bounded by v and denote the set of discontinuities of g by D_g. Provided that $P_\theta(s, D_g) = 0$ for any $\theta \in \Theta$ and $s \in S$, we may consider $\mathcal{D} \cup \{g\}$ for our analysis.

Let $\mathcal{M}$ be the class of nonnegative non-decreasing functions on $\mathbb{R}^+$. Standard are the following notions for stochastic comparison; see, for example, [11]. For stochastic variables X_1, X_2

$$X_1 \leq_{st} X_2 \Longleftrightarrow \mathbb{E}[f(X_1)] \leq \mathbb{E}[f(X_2)] \quad \text{for } f \in \mathcal{M}.$$

As shown in [11], $X_1 \leq_{st} X_2$ is equivalent to $\tilde{X}_1 \leq \tilde{X}_2$ a.s. for suitably chosen versions $\tilde{X}_1$ and $\tilde{X}_2$. A possibly defective transition kernel Q is monotone if

$$Qf \in \mathcal{M} \quad \text{for } f \in \mathcal{M}.$$

Let Θ be a neighborhood of θ, and assume that

$$\xi_\theta \leq_{st} \xi_{\theta'} \quad \text{for } \theta \leq \theta'.$$

Identifying the random variables with an appropriate version that translates $\leq_{st}$-ordering to almost sure ordering, we assume that

$$\text{(S2)} \qquad\qquad \xi_\theta \leq \xi_{\theta'} \quad \text{for } \theta \leq \theta'$$

with probability one. Hence, in the following we will work with random mappings $\xi_\theta(x)$ that are a.s. monotone in both arguments, where (S1) is guaranteed by definition and (S2) is an assumption that has to be verified in applications.

By (S2) it then holds with probability one that

$$\sup_{\theta \in \Theta_0} \xi_\theta = \xi_{\theta_r} \triangleq \xi \tag{2.7}$$

and, since we have assumed that the expected value of the drift is finite, it follows that

$$\mathbb{E}\big[\sup_{\theta \in \Theta_0} \xi_\theta\big] = \mathbb{E}[\xi] \tag{2.8}$$

is finite.

On $\mathbb{R}_+$ we define a Markov kernel by

$$P_\theta(x, B) \triangleq \mathbb{P}(\xi_\theta(x) \in B) = \mathbb{E}[1_B(\max(\xi_\theta + x, 0))], \tag{2.9}$$

where $x \in \mathbb{R}_+$ and B a Borel-set. For given initial state x_0, the above Markov kernel defines a random walk on the positive half-line. The increment variable ξ_θ in (2.6) represents the drift of the random walk. Note that one typically assumes for stability that $\mathbb{E}[\xi] < 0$.

The Markov kernel P_θ defined in (2.9) enjoys the following properties.

Lemma 2.1. *The kernel is monotone, that is*

$$P_\theta f \in \mathcal{M} \quad \text{for } f \in \mathcal{M}.$$

If (S2) *holds, then the kernel is monotone w.r.t.* θ, *that is*

$$P_\theta f \leq P_{\theta'} f \quad \text{for } \theta \leq \theta',$$

for any monotone integrable mapping f.

Proof. The definition of the kernel in (2.9) yields $(P_\theta f)(s) = \mathbb{E}[f(\xi_\theta(s))]$ and the first part of the lemma is a direct consequence of (S1), whereas the second part of the lemma follows directly from (S2). □

Lemma 2.2. *Suppose that* (S2) *holds. Then*

(i) *for* $\theta' \geq \theta$
$$\xi_{\theta'}(x) - \xi_\theta(x) \leq \xi_{\theta'} - \xi_\theta\,, \quad x \in S;$$

(ii) *for* $f \in \mathcal{M}$, *it holds for* $\theta' \geq \theta$,
$$f(\xi_{\theta'}(x)) - f(\xi_\theta(x)) \leq f(x + \xi_{\theta'}) - f(x + \xi_\theta)\,, \quad x \in S.$$

Proof. For the proof note that $\phi(x) = \max(x, 0)$ is a non-decreasing contraction. Hence, it holds that $\phi(x) \leq \phi(y)$ for $x \leq y$, which proves (i). The second part of the lemma follows from $\phi(f(y)) - \phi(f(x)) \leq \max(f(y) - f(x), f(y) - f(0), 0)$, which stems from the fact that $\phi(y) = 0$ implies $\phi(x) = 0$. □

Lemma 2.3. *Let* $\mathcal{H} \subset \mathcal{L}^1(P_\theta; \Theta)$. *Provided that* $\mathbb{E}[|\xi|^k]$ *is finite, for* $1 \leq k \leq p$, *it holds that* P_θ^n *is* $\mathcal{D}$*-preserving on* Θ *for all* n, *where* $\mathcal{D} = (\mathcal{H}, p)$.

Proof. Note that for $g \in \mathcal{D}$ it holds for all $\theta \in \Theta_0$ that
$$\begin{aligned}
|(P_\theta g)(x)| &\leq c_0 + \sum_{k=1}^{p} c_k \mathbb{E}\big[(\max(\xi_\theta + x, 0))^k\big] \\
&\leq c_0 + \sum_{k=1}^{p} c_k \mathbb{E}\big[|\xi_\theta + x|^k\big] \\
&\leq c_0 + \sum_{k=1}^{p} c_k \sum_{l=0}^{k} \tbinom{k}{l} x^l \mathbb{E}\big[|\xi_\theta|^{k-l}\big] \\
&\leq c_0 + \sum_{k=1}^{p} c_k \sum_{l=0}^{k} \tbinom{k}{l} x^l \mathbb{E}\big[|\xi|^{k-l}\big].
\end{aligned}$$

Hence, provided that $\mathbb{E}[|\xi|^k]$ is finite for $1 \leq k \leq p$, it follows that $P_\theta g \in \mathcal{D}$ for $g \in \mathcal{D}$. The claim then follows from finite induction. □

Lemma 2.4. *Let* $\mathcal{D} = (\mathcal{H}, p)$, *for* $\mathcal{H} \subset \mathcal{L}^1(P_\theta; \Theta)$. *If* ξ_θ *has* $\mathcal{D}$*-derivative* $(c_\theta, \xi_\theta^+, \xi_\theta^-)$, *then* P_θ *is* $\mathcal{D}$*-differentiable and* P'_θ *is* $\mathcal{D}$*-preserving.*

Proof. Let μ_θ denote the distribution of ξ_θ and let $\mu_\theta^\pm$ denote the distribution of $\xi_\theta^\pm$. The assumption that ξ_θ has a $\mathcal{D}$-derivative implies that
$$\forall g \in \mathcal{D}: \quad \frac{d}{d\theta} \int g(s) \mu_\theta(ds) = \int g(s) \mu'_\theta(ds),$$

with $\mu'_\theta = c_\theta(\mu_\theta^+ - \mu_\theta^-)$. Note that $g \in \mathcal{D} = (\mathcal{H}, p)$ implies that $g(\max(\xi_\theta + x, 0))$ as a function of ξ_θ lies in $\mathcal{D}$ as well. Hence, for $g \in \mathcal{D}$ it holds that

$$\begin{aligned} \frac{d}{d\theta} \int P_\theta(s; du) g(u) &= \frac{d}{d\theta} \int g(\max(u+s, 0)) \mu_\theta(du) \\ &= \int g(\max(u+s, 0)) \mu'_\theta(du). \end{aligned}$$

For $A \in \mathcal{T}$, set

$$P'_\theta(s; A) \triangleq \int 1_A(\max(u+s, 0))\, \mu'_\theta(du),$$

then

$$\forall g \in \mathcal{D}: \quad \frac{d}{d\theta} \int P_\theta(s; du) g(u) = \int g(u) P'_\theta(s; du),$$

which establishes $\mathcal{D}$-differentiability of P_θ.

We now show that P'_θ is $\mathcal{D}$-preserving. If ξ_θ has $\mathcal{D}$-derivative $(c_\theta, \xi_\theta^+, \xi_\theta^-)$, then $\mathbb{E}[|\xi_\theta^+|^k]$ and $\mathbb{E}[|\xi_\theta^-|^k]$ are finite for $1 \leq k \leq p$. Since $g \in \mathcal{D}$ implies that $g(\max(\xi_\theta + x, 0))$ as a function of ξ_θ lies in $\mathcal{D}$ as well, $\mathcal{D}$-differentiability of ξ_θ yields

$$\begin{aligned} |(P'_\theta g)(x)| &= c_\theta \big| \mathbb{E}\big[g(\max(\xi_\theta^+ + x, 0))\big] - \mathbb{E}\big[g(\max(\xi_\theta^- + x, 0))\big] \big| \\ &\leq c_\theta \sum_{k=0}^{p} d_k \mathbb{E}\big[|\xi_\theta^+ + x|^k\big] + c_\theta \sum_{k=0}^{p} d_k \mathbb{E}\big[|\xi_\theta^- + x|^k\big] \\ &\leq c_\theta \sum_{k=0}^{p} \sum_{l=0}^{k} \binom{k}{l} x^l \big(\mathbb{E}\big[|\xi_\theta^+|^{k-l}\big] + \mathbb{E}\big[|\xi_\theta^-|^{k-l}\big] \big). \end{aligned}$$

Since $\mathbb{E}[|\xi_\theta^\pm|^k]$ are finite for $1 \leq k \leq p$, it follows that $P'_\theta g \in \mathcal{D}$. □

3. Main result

In this section, we present the main result of this paper. The technical analysis is postponed to Section 5. To simplify the presentation of results, we summarize the stability conditions required for our analysis in the following definition.

Definition 3.1. We say that a *Lyapunov condition* holds for p, with $p \geq 0$, if

- condition (S2) holds,
- it holds that $\mathbb{E}[\xi] < 0$ and

$$\mathbb{E}\big[|\xi|^{p+1}\big] < \infty,$$

- for each $x_0 \in \mathbb{R}^+$ it holds that $\sup_{s \in S, \theta \in \Theta} \mathbb{E}_\theta[N_\alpha(x_0, s)]$ is finite, where $N_\alpha(x_0, s)$ denotes the number of visits to $[0, x_0]$ with $X_\theta(0) = s$ and without hitting α.

As we will show in Section 5.2, if the Lyapunov condition holds for p, the $\mathrm{H}_\theta g(x)$ is bounded as a function in x in the following way: if g is bounded by a polynomial of order p, then $\mathrm{H}_\theta g(x)$ is bounded by a polynomial of order $p+1$. Surprisingly enough, as our analysis put forward in Section 5.3 shows, provided that g is monotone, multiplying the expected cycle cost by the weak derivative of the kernel reduces the order of the bound and $P'_\theta H_\theta g$ is bounded by a constant, i.e., a polynomial of degree 0. Hence, finiteness of $H_\theta g$ for any monotone cost function bounded by a polynomial of degree p implies that $H_\theta P'_\theta H_\theta g$ exists. The precise technical conditions are put forward in the following theorem. The proof of the theorem will be postponed to Section 6.

Theorem 3.1. *Let $(\mathcal{H}, p+1)$ be Banach, for $p \geq 0$, and $\mathcal{H} \subset \mathcal{L}^1(P_\theta; \Theta)$. Let $\Theta_1 \subset \Theta$ be a neighborhood of θ. Suppose that*

(i) *the Lyapunov condition holds for p,*

(ii) *ξ_θ is $(\mathcal{H}, p+1)$-differentiable on Θ_1 and for $l = 1, \ldots, p+1$*

$$\sup_{\theta' \in \Theta_1} \left| \frac{d}{d\theta}\Big|_{\theta=\theta'} \mathbb{E}\left[(\xi_\theta)^l\right] \right| < \infty,$$

(iii) *for $g \in (\mathcal{H}, p)$ it holds that*

$$\sup_x \sup_{\hat{\theta} \in \Theta_1} \left| P'_\theta \mathrm{H}_\theta g(x) - P'_{\hat{\theta}} \mathrm{H}_\theta g(x) \right| < \infty.$$

Then it holds for any nonnegative and monotone $g \in (\mathcal{H}, p)$ that

$$\frac{d}{d\theta} \sum_{k=1}^{\infty} P_\theta^k g = \sum_{k=1}^{\infty} P_\theta^k P'_\theta \sum_{l=1}^{\infty} P_\theta^l g \,\in (\mathcal{H}, 1).$$

4. Applications

In this section, we apply our results to performance characteristics of the G/G/1 queue. In the first example, we study the dependence of the overflow probability of a certain level in a busy cycle with θ a parameter of the service time distribution. In the second example, we consider the same performance measure but this time for the M/G/1 queue where θ is the intensity of the arrival stream. We will base the analysis of the second example on the thinning model of Poisson processes.

4.1. Service time dependence on θ

Let $W_\theta(n)$ be the waiting time of the nth customer in a G/G/1 queue. Let $\{A(n)\}$ be the i.i.d. sequence of interarrival times with finite expected value and let $\{S_\theta(n)\}$ the i.i.d. sequence of service times, respectively. We assume that the system is stable, i.e., $\sup_{\theta\in\Theta}\mathbb{E}[S_\theta(1)] < \mathbb{E}[A(1)]$. Set

$$\xi_\theta(n) \triangleq S_\theta(n) - A(n)$$

and

$$\xi_\theta(n, w) = \max(w + \xi_\theta(n), 0),$$

for $n \geq 1$. Lindley's recursion yields:

$$W_\theta(n+1) = \max(\, W_\theta(n) + \xi_\theta(n)\,,\, 0\,) = \xi_\theta(n, W_\theta(n))\,, \quad n \geq 1\,,$$

and $W_\theta(1) = 0$. Let $\alpha = \{0\}$ denote the event that the waiting times regenerate.

We assume that $S_\theta(n)$ follows a Pareto $(\theta,2)$ distribution, i.e.,

$$\mathbb{P}(S_\theta(n) > x) = \frac{\theta^2}{(\theta + x)^2}.$$

Then $\mathbb{E}[S_\theta(n)] = \theta$, for any n, and the variance of $S_\theta(n)$ fails to exist. For U uniformly distributed on $[0,1]$, a sample of $S_\theta(n)$ can be obtained by the inverse probability function through $\theta((1-U)^{(-1/2)} - 1)$. From this construction it follows that $S_\theta(n)$ is monotone with respect to θ which in turn implies (S2).

Let $f_{\theta,k}$ with

$$f_{\theta,k}(x) = k\frac{\theta^k}{(\theta + x)^{k+1}}$$

denote the density of the Pareto (θ, k) distribution. Take as $\mathcal{D} = (\mathcal{L}^1(P_\theta;\Theta), 1)$ the set of integrable measurable mappings. Then, the Pareto distribution is $\mathcal{D}$-differentiable and for $k = 2$ and $g \in \mathcal{D}$, differentiating with respect to θ yields,

$$\begin{aligned} \frac{d}{d\theta}\int g(x)f_{\theta,2}(x)\,dx &= 4\int g(x)\frac{\theta}{(\theta+x)^3}\,dx - 6\int g(x)\frac{\theta^2}{(\theta+x)^4}\,dx \\ &= \frac{2}{\theta}\bigg(\int g(x)f_{\theta,2}(x)\,dx - \int g(x)f_{\theta,3}(x)\,dx\bigg), \end{aligned} \tag{4.1}$$

for $g \in \mathcal{D}$. Hence, the Pareto $(\theta,2)$ distribution has $\mathcal{D}$-derivative

$$(2/\theta, \mathrm{Pareto}(\theta,2), \mathrm{Pareto}(\theta,3)).$$

Note that the Pareto $(\theta,3)$ distribution has finite first and second moment. Moreover, the positive part of the $\mathcal{D}$-derivative of the Pareto $(\theta,2)$ distribution is the Pareto $(\theta,2)$ distribution itself.

We now apply Theorem 3.1 to $\mathcal{D} = (\mathcal{L}^1(P_\theta; \Theta), 0)$ (the set of bounded performance measures). First we show that the Lyapunov condition holds for $p = 0$. Note that $\mathbb{E}[|\xi|]$ is finite and $\mathbb{E}[\xi] < 0$, since $S_\theta(n)$ and $A(n)$ have finite expected values and $\sup_{\theta \in \Theta} \mathbb{E}[S_\theta(1)] < \mathbb{E}[A(1)]$, by assumption. Moreover, $\mathbb{P}(A(n) > x) > 0$ for any $x \geq 0$ implies the third condition in Definition 3.1, see Lemma 5.4.

Condition (ii) of Theorem 3.1 follows from (4.1). It remains to be shown that for $g \in (\mathcal{H}, 0)$ it holds that

$$\sup_x \sup_{\hat\theta \in \Theta_1} \left| P'_\theta \mathrm{H}_\theta g(x) - P'_{\hat\theta} \mathrm{H}_\theta g(x) \right| < \infty .$$

To see this note that $P^+_\theta = P_\theta$ and that P^-_θ is the transition kernel with a Pareto (θ,3) distributed service time. Hence,

$$|P'_\theta \mathrm{H}_\theta g(x) - P'_{\hat\theta} \mathrm{H}_\theta g|(x) = \frac{2}{\theta} |P_\theta \mathrm{H}_\theta g(x) - P_{\hat\theta} \mathrm{H}_\theta g(x)| + \frac{2}{\theta} |(P^-_\theta \mathrm{H}_\theta g(x) - P^-_{\hat\theta} \mathrm{H}_\theta g(x)| .$$

A direct application of Lemma 5.6 yields

$$\sup_x \sup_{\hat\theta \in \Theta_1} |P_\theta \mathrm{H}_\theta g(x) - P_{\hat\theta} \mathrm{H}_\theta g(x)| \leq C .$$

Moreover, applying Lemma 5.6 to the version of the kernel with a Pareto (θ,3) distributed service time yields

$$\sup_x \sup_{\hat\theta \in \Theta_1} |P^-_\theta \mathrm{H}_\theta g(x) - P^-_{\hat\theta} \mathrm{H}_\theta g(x)| \leq \hat{C} ,$$

which yields condition (iii) of Theorem 3.1.

Hence, for a suitable neighborhood of θ, Theorem 3.1 applies and we obtain for any nonnegative and monotone cost function g out of $\mathcal{D}'$:

$$\frac{d}{d\theta} \mathbb{E}\Big[\sum_{n=1}^{\tau_{\alpha,\theta}-1} g(W_\theta(n)) \Big] = \sum_{k=0}^{\infty} P^k_\theta P'_\theta \sum_{l=0}^{\infty} P^l_\theta g ,$$

where P_θ denotes the taboo kernel of the waiting times with taboo set $\alpha = \{0\}$. In order to write the expression on the right-hand side in terms of random variables, we introduce the following variant of the waiting time sequence. For $j \in \mathbb{N}$, set

$$W^-_\theta(j; n+1) \quad = \quad \max(\, W^-_\theta(j; n) + S_\theta(n) - A(n) \,,\, 0 \,) \,, \quad n \neq j \,,$$

with $W^-_\theta(j; 1) = 0$, and for $j = n$, let

$$W^-_\theta(j; n+1) \quad = \quad \max(\, W^-_\theta(j; n) + S^-_\theta(n) - A(n) \,,\, 0 \,) \,, \quad n \geq 1 \,,$$

where $S_\theta^-(n)$ follows a Pareto (θ,3) distribution. We denote the first time that $\{W_\theta(n)\}$ and $\{W_\theta^-(j;n)\}$ simultaneously hit α by $\tau^{\pm}_{\alpha,\theta}(j)$. Then it holds that

$$\frac{d}{d\theta}\mathbb{E}\left[\sum_{n=1}^{\tau_{\alpha,\theta}-1} g(W_\theta(n))\right] = \frac{2}{\theta}\mathbb{E}\left[\sum_{j=1}^{\tau_{\alpha,\theta}-1}\sum_{n=j+1}^{\tau^{\pm}_{\alpha,\theta}(j)-1}\big(g(W_\theta(n)) - g(W_\theta^-(j;n))\big)\right],$$

for details see [6]. For example, letting, for some $b > 0$, $g_b(x) = 1$ for $x \geq b$ and zero otherwise, $\mathrm{H}_\theta g_b$ is the expected number of overflows of level b in a busy cycle. Note that, on the one hand, $S_\theta(n)$ fails to have a finite second moment which implies that $\tau_{\alpha,\theta}$ fails to have a finite second moment too. On the other hand, $\tau_{\alpha,\theta}$ needs to have a finite first moment for the cycle cost to exists. Hence, the key condition for applying Theorem 3.1 is the existence of the cycle cost.

4.2. Thinning of a Poisson process

In this section we consider the waiting time of the nth customer in a G/G/1 queue in a slightly different setting. Let $\{A(n)\}$ be an i.i.d. sequence exponential distributed random variables with rate λ constituting the interarrival times and let $\{S(n)\}$ be the i.i.d. sequence of service times, respectively. We introduce an i.i.d. sequence of $\{0,1\}$ random variables $\{\eta_\theta(n)\}$ with distribution

$$\mathbb{P}(\eta_\theta = 1) = \theta = 1 - \mathbb{P}(\eta_\theta = 0),$$

for $\theta \in \Theta = [0,1]$. Set

$$\xi_\theta(n) \triangleq \eta_\theta(n)S(n) - A(n) \quad \text{ and } \quad \xi_\theta(n,w) = \max(w + \xi_\theta(n), 0),$$

for $n \geq 1$. Note that for $\theta \in [0,1]$ it holds

$$\xi_\theta(n) \leq \xi_1(n) \triangleq S(n) - A(n), \tag{4.2}$$

for $n \geq 1$, and that (S2) holds. Lindley's recursion yields:

$$W_\theta(n+1) = \max(\, W_\theta(n) + \eta_\theta(n)S(n) - A(n)\,,\, 0\,) = \xi_\theta(n, W_\theta(n))\,, \quad n \geq 1\,,$$

and $W_\theta(1) = 0$. We assume that the system is stable for any $\theta \in [0,1]$, i.e., $\mathbb{E}[S_\theta(1)] < \mathbb{E}[A(1)]$.

The above model has the following interpretation. Customers arrive according to a Poisson-λ-process at the queue. An arriving customer is admitted to the queue with probability $1-\theta$. The total number of admitted customers out of the first n arriving customers after the initial one is

$$m(n) \triangleq \sum_{k=1}^{n} \eta_\theta(k).$$

From this construction it follows that $W_\theta(n+1)$ is the waiting time of the $(m(n)+1)$st customer in a single-server queue with Poisson-λ-arrival stream. The above thinning of Poisson process yields again a Poisson process but with intensity $\lambda\theta$. Hence, $W_\theta(n+1)$ can also be interpreted as the waiting time of the $(m(n)+1)$st customer in single-server queue with Poisson-$\lambda\theta$-arrival stream.

Let $\alpha = \{0\}$ denote the event that the waiting times regenerate and take $\mathcal{D} = (\mathcal{L}^1(P_\theta;\Theta), p)$. Provided that $S(n)$ and $A(n)$ have finite pth moments it holds for any $g \in \mathcal{D}$

$$\begin{aligned}\frac{d}{d\theta}\mathbb{E}[g(\xi_\theta(n))] &= \frac{d}{d\theta}\big(\mathbb{E}[g(\xi_\theta(n)) \mid \eta_\theta(n)=1]\theta + \mathbb{E}[g(\xi_\theta(n)) \mid \eta_\theta(n)=0](1-\theta)\big)\\ &= \frac{d}{d\theta}\big(\mathbb{E}[g(S(n)-A(n))]\theta + \mathbb{E}[g(-A(n))](1-\theta)\big)\\ &= \mathbb{E}[g(S(n)-A(n))] - \mathbb{E}[g(-A(n))].\end{aligned}$$

Since the right-hand side of the above equation is independent of θ and finite, it holds that

$$\sup_{\theta\in[0,1]}\left|\frac{d}{d\theta}\mathbb{E}[g(\xi_\theta(n))]\right| < \infty.$$

It remains to be shown that for $g \in \mathcal{D}$ it holds that

$$\sup_x \sup_{\hat\theta\in\Theta_1}\left|P'_\theta \mathrm{H}_\theta g(x) - P'_{\hat\theta}\mathrm{H}_\theta g(x)\right| < \infty.$$

To see this note that $P'_\theta = (P_1 - P_0)$ is independent of θ, which gives for all $x \in \mathbb{R}_+$ and all $\theta, \hat\theta \in [0,1]$ that

$$\left|P'_\theta \mathrm{H}_\theta g(x) - P'_{\hat\theta}\mathrm{H}_\theta g(x)\right| = 0.$$

From the above it is straightforward to see that Theorem 3.1 applies to any $g \in \mathcal{D}$ provided that $S(n)$ and $A(n)$ have finite $(p+1)$st moments. Specifically, for $j \in \mathbb{N}$, set

$$W^+_\theta(j;n+1) \quad = \quad \max(\,W^+_\theta(j;,n) + \eta_\theta S(n) - A(n)\,,\,0\,)\,, \quad n \neq j\,,$$

with $W^+_\theta(j;1) = 0$, and for $n = j$

$$W^+_\theta(j;n+1) \quad = \quad \max(\,W^+_\theta(j;,n) + S(n) - A(n)\,,\,0\,)\,,$$

and define the '$-$' version by

$$W^-_\theta(j;n+1) \quad = \quad \max(\,W^-_\theta(j;n) + \eta_\theta S(n) - A(n)\,,\,0\,)\,, \quad n \geq 1\,,$$

with $W^-_\theta(j;1) = 0$ and for $j = n$

$$W^-_\theta(j;n+1) \quad = \quad \max(\,W^-_\theta(j;n) - A(n)\,,\,0\,)\,.$$

We denote the first time that $\{W_\theta^+(j;n)\}$ and $\{W_\theta^-(j;n)\}$ simultaneously enter α by $\tau^{\pm}_{\alpha,\theta}(j)$, with $\alpha = \{0\}$. Then, it holds that

$$\frac{d}{d\theta}\mathbb{E}\left[\sum_{n=1}^{\tau_{\alpha,\theta}-1} g(W_\theta(n))\right] = \mathbb{E}\left[\sum_{j=1}^{\tau_{\alpha,\theta}-1}\sum_{n=j+1}^{\tau^{\pm}_{\alpha,\theta}(j)-1}\big(g(W_\theta^+(j;n)) - g(W_\theta^-(j;n))\big)\right],$$

for any $g \in \mathcal{D}^p$, for details see [6].

For example, taking $p = 0$, it is sufficient that $S(n)$ and $A(n)$ have finite first moment. Again, our result applies in the case that $S(n)$ has no finite second moment which in turn implies that $\tau_{\alpha,\theta}$ has no finite second moment. In [8] finiteness of the second moment of $\tau_{\alpha,\theta}$ is required which is an improvement on [1] where even the third moment has to be finite. For a first study of this problem we refer to [4].

For example, letting, for some $b > 0$, $g_b(x) = 1$ for $x \geq b$ and zero otherwise, $\mathbb{E}\big[\sum_{n=1}^{\tau_{\alpha,\theta}-1} g_b(W_\theta(n))\big]$ is the expected number of overflows of level b in a busy cycle.

5. Technical analysis

Section 5.1 establishes sufficient conditions for differentiability of the potential. In Section 5.2, we establish a bound for cost accumulated over a cycle. This result will be used in Section 5.3 to establish bounds on the effect of a perturbation of θ on the cost accumulated over a cycle. With the preliminary results established in this section, the proof of Theorem 3.1 will then be given in Section 6.

5.1. Differentiating the potential

Let $(P_\theta : \theta \in \Theta)$ be a collection of (possibly defective) Markov kernels on $(S, \mathcal{T})$, i.e. $P_\theta \in \mathcal{K}(S, \mathcal{T})$. For example, P_θ may be obtained through ${}_VP_\theta$ for $P_\theta \in \mathcal{K}_1(S, \mathcal{T})$ and $V \in \mathcal{T}$, as explained in the previous section. In this section, we will compute the derivative of the potential of P_θ:

$$\frac{d}{d\theta}\sum_{k=1}^{\infty} P_\theta^k g \,,$$

where $g \in \mathcal{D}$, for appropriately defined set of cost functions $\mathcal{D}$. Note that since g is independent of θ is holds that $dg/d\theta = 0$, which implies that

$$\frac{d}{d\theta}\sum_{k=0}^{\infty} P_\theta^k g = \frac{d}{d\theta}\sum_{k=1}^{\infty} P_\theta^k g.$$

To simplify the notation, we write H_θ for the potential of P_θ even though P_θ does not necessarily have to be a taboo kernel.

We will show that, under appropriate conditions, $\mathrm{H}_\theta g$ is differentiable with derivative

$$\mathrm{H}_\theta P'_\theta \mathrm{H}_\theta g\,. \tag{5.1}$$

Starting point is a $\mathcal{D}$-differentiable Markov kernel P_θ. Let $\Theta_0 \triangleq (\theta_l, \theta_r) \subset \Theta$ be a neighborhood of θ such that $(\theta_l, \theta_r] \subset \Theta$. The following theorem presents minimal conditions for (5.1) to hold.

Theorem 5.1. *For $p \geq 0$, let $\mathcal{H} \subset \mathcal{L}^1(P_\theta; \Theta)$. Let P_θ be $(\mathcal{H}, p{+}1)$-differentiable, and assume that for $\hat\theta$ out of an open neighborhood of θ it holds that*

(i) *$P^n_{\hat\theta}$ is $(\mathcal{H}, p)$-preserving,*

(ii) *for all $g \in (\mathcal{H}, p)$ it holds that $\mathrm{H}_{\hat\theta} g \in (\mathcal{H}, p+1)$,*

(iii) *for all $g \in (\mathcal{H}, p+1)$ it holds that $P'_\theta \mathrm{H}_\theta g \in (\mathcal{H}, 0)$,*

(iv) *for all $g \in (\mathcal{H}, p+1)$ it holds that $(P_{\hat\theta} - P_\theta)\mathrm{H}_{\hat\theta} g \in (\mathcal{H}, 0)$.*

A sufficient condition for

$$(\mathrm{H}_\theta g)' = \mathrm{H}_\theta P'_\theta \mathrm{H}_\theta g\,,$$

or, more explicitly,

$$\frac{d}{d\theta} \sum_{k=1}^{\infty} P^k_\theta g = \sum_{k=1}^{\infty} P^k_\theta P'_\theta \sum_{l=1}^{\infty} P^l_\theta g$$

is given in the following:

$$\text{(B1)} \qquad \lim_{\Delta \to 0} \frac{1}{\Delta} \left| \mathrm{H}_\theta (P_{\theta+\Delta} - P_\theta) \mathrm{H}_\theta g - \Delta \mathrm{H}_\theta P'_\theta \mathrm{H}_\theta g \right| = 0,$$

$$\text{(B2)} \qquad \lim_{\Delta \to 0} \frac{1}{\Delta} \left| \mathrm{H}_\theta (P_{\theta+\Delta} - P_\theta)(\mathrm{H}_{\theta+\Delta} - \mathrm{H}_\theta) g \right| = 0.$$

Proof. Note that by the conditions put forward in the theorem the expressions on the right-hand side in the statement of the theorem are well-defined. By calculation,

$$\begin{aligned} &\frac{1}{\Delta}\Big| \sum_{n=1}^{\infty} P^n_{\theta+\Delta} g - \sum_{n=1}^{\infty} P^n_\theta g - \Delta \sum_{k=1}^{\infty} P^k_\theta P'_\theta \sum_{l=1}^{\infty} P^l_\theta g \Big| \qquad (5.2) \\ &= \frac{1}{\Delta}\Big| \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} P^k_\theta (P_{\theta+\Delta} - P_\theta) P^{n-1-k}_{\theta+\Delta} g - \Delta \sum_{k=1}^{\infty} P^k_\theta P'_\theta \sum_{l=1}^{\infty} P^l_\theta g \Big| \\ &= \frac{1}{\Delta}\Big| \sum_{k=0}^{\infty} P^k_\theta (P_{\theta+\Delta} - P_\theta) \sum_{l=0}^{\infty} P^l_{\theta+\Delta} g - \Delta \sum_{k=1}^{\infty} P^k_\theta P'_\theta \sum_{l=1}^{\infty} P^l_\theta g \Big| \end{aligned}$$

$$\leq \frac{1}{\Delta}\Big|\sum_{k=0}^{\infty} P_\theta^k (P_{\theta+\Delta} - P_\theta) \sum_{l=0}^{\infty} P_\theta^l g - \Delta \sum_{k=1}^{\infty} P_\theta^k P_\theta' \sum_{l=1}^{\infty} P_\theta^l g\Big|$$
$$+ \frac{1}{\Delta}\Big|\sum_{k=0}^{\infty} P_\theta^k (P_{\theta+\Delta} - P_\theta) \sum_{l=0}^{\infty} (P_{\theta+\Delta}^l - P_\theta^l) g\Big|.$$

Hence, we arrive at

$$\frac{1}{\Delta}\Big|\sum_{n=1}^{\infty} P_{\theta+\Delta}^n g - \sum_{n=1}^{\infty} P_\theta^n g - \Delta \sum_{k=1}^{\infty} P_\theta^k P_\theta' \sum_{l=1}^{\infty} P_\theta^l g\Big|$$
$$\leq \frac{1}{\Delta}\big|\mathrm{H}_\theta (P_{\theta+\Delta} - P_\theta)\mathrm{H}_\theta g - \Delta \mathrm{H}_\theta P_\theta' - \mathrm{H}_\theta g\big|$$
$$+ \frac{1}{\Delta}\big|\mathrm{H}_\theta (P_{\theta+\Delta} - P_\theta)(\mathrm{H}_{\theta+\Delta} - \mathrm{H}_\theta) g\big| = 0$$

and by assumptions (B1) and (B2) the last terms tend to 0 and thus the expression in (5.2) tends to 0. □

5.2. Lyapunov conditions

Lemma 5.1. *Let $V \in \mathcal{T}$ and $g \in \mathcal{L}^1(P_\theta; \Theta)$, with $g(s) \geq 0$ for all s. Suppose that there exists a Lyapunov function $g^\lambda \in \mathcal{L}^1(P_\theta; \Theta)$ such that*

$$g + ({}_V P_\theta g^\lambda) \leq g^\lambda, \tag{5.3}$$

and moreover, suppose that for some c

$$\sup_{s \in V} g^\lambda(s) \leq c.$$

Then, for $N_\theta(s)$ the number of visits to the set V provided that $X_\theta(0) = s$, it holds

$$\mathbb{E}\Big[\sum_{t=1}^{\infty} g(X_\theta(t)) \mid |X_\theta(0) = s\Big] \leq g^\lambda(s) + c\mathbb{E}[N_\theta(s)].$$

Proof. Let $\tau_1 < \tau_2 < \tau_3 < \ldots$ be the successive recurrence times to the set V, and let $\tau_0 = 0$. Then rewriting the expected costs with direct cost function g over the infinite horizon in blocks over the periods between the successive recurrence times to the set V, we get

$$\mathbb{E}\Big[\sum_{t=1}^{\infty} g(X_\theta(t)) \mid X_\theta(0) = s\Big] = \mathbb{E}\Big[\sum_{k=1}^{\infty} \sum_{t=\tau_{k-1}}^{\tau_k - 1} g(X_\theta(t)) \mid X_\theta(0) = s\Big]. \tag{5.4}$$

In the following we show that the expected costs over the first cycle are bounded by g^λ. Multiplying the Lyapunov inequality (5.3) from the left by ${}_V P_\theta$ gives

$${}_V P_\theta g + ({}_V P_\theta^2 g^\lambda) \leq {}_V P_\theta g^\lambda.$$

Adding g on both sides of the inequality and using the Lyapunov inequality for the expression on the right-hand side of the inequality yields

$$g + {}_V P_\theta g + {}_V P_\theta^2 g^\lambda \leq g^\lambda \,.$$

Repeating this argument n-times gives

$$\sum_{l=0}^{n} {}_V P_\theta^l g + {}_V P_\theta^{n+1} g^\lambda \leq g^\lambda \,.$$

Since $g^\lambda \geq g \geq 0$, we find by taking the limit as n tends to infinity that

$$\sum_{l=0}^{\infty} {}_V P_\theta^l g \leq g^\lambda . \tag{5.5}$$

Note that the l-th term in this sum is operator notation for the expected costs at time l on the event that the first recurrence time $\tau_1 > l$, i.e.

$$({}_V P_\theta^l\, g)(s) = \mathbb{E}[g(X_\theta(l))\, \mathbf{1}(\tau_1 > l) \mid X_\theta(0) = s],$$

where $\mathbf{1}(\tau_1 > l)$ is the indicator function of the event that the first recurrence time is larger than l. Hence,

$$\begin{aligned} \sum_{l=0}^{\infty} ({}_V P_\theta^l\, g)(s) &= \sum_{l=0}^{\infty} \mathbb{E}[g(X_\theta(l))\, \mathbf{1}(\tau_1 > l) \mid X_\theta(0) = s] \\ &= \mathbb{E}\Big[\sum_{t=1}^{\tau_1 - 1} g(X_\theta(t)) \mid X_\theta(0) = s\Big] \\ &\leq g^\lambda(s), \end{aligned} \tag{5.6}$$

where the last inequality follows from (5.5).

In a similar way, an upper bound for the expected costs over the kth cycle can be obtained. Indeed, by using the Markov property we find with $N_\theta(s)$ the number of recurrences to the set V (note that we will allow that $N_\theta(s)$ is equal to infinity with positive probability, in this case the assertion is obvious true), for $k = 2, 3, \ldots$

$$\mathbb{E}\Big[\sum_{t=\tau_{k-1}}^{\tau_k - 1} g(X_\theta(t)) \mid N_\theta(s) > k-1, X_\theta(\tau_{k-1}) = s\Big] = \sum_{l=0}^{\infty} ({}_V P_\theta^l\, g)(s) \leq g^\lambda(s),$$

where the inequality follows from (5.5). Since τ_{k-1} is a recurrence time to the set V, we have $X_\theta(\tau_{k-1}) \in V$ and, moreover, from the assumption

$$g^\lambda(X_\theta(\tau_{k-1})) \leq c$$

we obtain for any $u \in V$:

$$\mathbb{E}\Big[\sum_{t=\tau_{k-1}}^{\tau_k-1} g(X_\theta(t)) \mid N_\theta(s) > k-1, X_\theta(\tau_{k-1}) = u\Big] < c.$$

By the strong Markov property this yields

$$\mathbb{E}\Big[\sum_{t=\tau_{k-1}}^{\tau_k-1} g(X_\theta(t)) \mid N_\theta(s) > k-1, X_\theta(\tau_{k-1}) = u\,, X_\theta(0) = s\Big] < c\,, \quad \text{for all } u \in V.$$

The above bound holds uniformly on V and since $X_\theta(\tau_{k-1}) \in V$, we can disregard the condition $X_\theta(\tau_{k-1}) = u$ in the above bound which yields

$$\mathbb{E}\Big[\sum_{t=\tau_{k-1}}^{\tau_k-1} g(X_\theta(t)) \mid N_\theta(s) > k-1, X_\theta(0) = s\Big] < c$$

and we finally arrive at

$$\mathbb{E}\Big[\sum_{t=\tau_{k-1}}^{\tau_k-1} g(X_\theta(t)) \mid X_\theta(0) = s\Big] \leq c\mathbb{P}(\, N_\theta(s) > k-1 \mid X_\theta(0) = s\,). \tag{5.7}$$

We now combine the above results in order to establish an upper bound for the overall expected costs as given on the right-hand side of (5.4):

$$\begin{aligned}
\mathbb{E}\Big[\sum_{t=1}^{\infty} g(X_\theta(t)) \mid X_\theta(0) = s\Big] &= \mathbb{E}\Big[\sum_{k=1}^{\infty}\sum_{t=\tau_{k-1}}^{\tau_k-1} g(X_\theta(t)) \mid X_\theta(0) = s\Big] \\
&\overset{(5.6)}{\leq} g^\lambda(s) + \mathbb{E}\Big[\sum_{k=2}^{\infty}\sum_{t=\tau_{k-1}}^{\tau_k-1} g(X_\theta(t)) \mid X_\theta(0) = s\Big] \\
&\overset{(5.7)}{\leq} g^\lambda(s) + c\sum_{k=2}^{\infty} \mathbb{P}(\, N_\theta(s) > k-1 \mid X_\theta(0) = s) \\
&\leq g^\lambda(s) + c\mathbb{E}[\, N_\theta(s) \,|\, X_\theta(0) = s\,],
\end{aligned}$$

which completes the proof. □

Recall that we are interested in the cumulative cost until the Markov chains hits a predefined set α. We will apply the above lemma to the particular defective Markov transition kernel ${}_\alpha P_\theta$.

Lemma 5.2. *Let $\alpha \subset V$ and for $X_\theta(0) = s$ denote by $N_{\alpha,\theta}(s)$ the number of visits to V without hitting α. Suppose that for $x \notin V$*

$$g + ({}_V P_\theta g^1) \leq g^1, \tag{5.8}$$

and

$$c \triangleq \sup_{x \in V} (g(x) + ({}_V P_\theta g^1(x)) < \infty. \tag{5.9}$$

Then

$$g^\lambda(x) \triangleq \begin{cases} g^1(x) & \text{for } x \notin V, \\ c & \text{for } x \in V, \end{cases}$$

satisfies the conditions in Lemma 5.1 *and it holds that*

$$\mathbb{E}\Big[\sum_{t=1}^{\tau_\alpha - 1} g(X_\theta(t)) \mid X_\theta(0) = s \Big] \leq g^\lambda(s) + c\mathbb{E}[N_{\alpha,\theta}(s)] , \quad s \in S.$$

Proof. The proof follows from Lemma 5.1 by replacing P_θ with ${}_\alpha P_\theta$. □

Remark 5.1. For verifying inequality (5.3) for g^λ, it is sufficient to check it for g^1 for $x \notin V$ and to verify (5.9). In our applications we will verify (5.8) and (5.9). Note that if g^1 is bounded in absolute value by a polynomial of degree p then so is g^λ.

5.3. Bounds on the effect of a finite perturbation

The Lyapunov condition allows to bound $\mathrm{H}_\theta g(x)$ as a function in x. The precise statement is given in the following lemma.

Lemma 5.3. *Suppose that the Lyapunov condition holds for p, with $p \geq 0$. Let $\mathcal{H} \subset \mathcal{L}^1(P_\theta; \Theta)$, then for each $g \in (\mathcal{H}, p)$ a function $f \in (\mathcal{H}, p+1)$ exists such that* $\sup_{\theta \in \Theta_0} \mathrm{H}_\theta g \leq f$.

Proof. Suppose that $g(x) \leq \sum_{k=0}^p c_k x^k$, then

$$\mathrm{H}_\theta g \leq \sum_{k=0}^p c_k \mathrm{H}_\theta g_k$$

with $g_k(x) = x^k$. Hence, it suffices to show the assertion for g_k. We will show it for $g(x) = x^p$ for $x \geq 0$, the proof for the other terms goes similarly. We try to satisfy the condition of Lemma 5.2 with the function $g^1(x) = cx^{p+1}$ and so we try to find $V \subset [0, \infty)$ such that

$$x^p + \int_{y \notin V} g^1(y) \mathbb{P}(\xi_\theta(x) \in dy) \leq cx^{p+1}.$$

Set $\hat{\xi}_\theta(x) \triangleq \max(\xi_\theta, -x)$, which gives $\hat{\xi}_\theta(x) + x = \xi_\theta(x)$. By computation,

$$x^p + \int_{y \notin V} c\, y^{p+1} \mathbb{P}(\xi_\theta(x) \in dy) \quad \leq \quad x^p + \int_{y \geq 0} c\, y^{p+1} \mathbb{P}(\xi_\theta(x) \in dy)$$

$$\leq \quad x^p + \int\limits_{y\geq 0} c(x+y)^{p+1}\mathbb{P}(\hat{\xi}_\theta(x) \in dy)$$
$$\leq \quad x^p + c\sum_{k=0}^{p+1}\binom{p+1}{k}x^{p+1-k}\mathbb{E}\big[\hat{\xi}_\theta(x)^k\big].$$

To simplify the notation, set

$$h(x) \triangleq \sum_{k=2}^{p+1}\binom{p+1}{k}x^{p+1-k}\big|\mathbb{E}\big[\hat{\xi}_\theta(x)^k\big]\big|.$$

With this notation, we obtain

$$x^p + \int\limits_{y\notin V} c\, y^{p+1}\mathbb{P}(\xi_\theta(x) \in dy)$$
$$\leq x^p + c\big(x^{p+1} + (p+1)x^p\mathbb{E}\big[\hat{\xi}_\theta(x)\big] + h(x)\big). \qquad (5.10)$$

Recall that $\hat{\xi}_\theta(x) = \max(\xi_\theta, -x)$, which implies $\hat{\xi}_\theta(x) \approx \xi_\theta$ for x large. This motivates the following line of argument. Choose $\varepsilon > 0$ small enough such that

$$\mathbb{E}[\xi] + \varepsilon \triangleq \gamma < 0$$

and x_0' large enough such that $\mathbb{E}[|\xi|\, 1_{\xi<-x_0'}] < \varepsilon$. Then it holds for all $x \geq x_0'$ that

$$\begin{aligned} \forall\theta: \quad \mathbb{E}[\hat{\xi}_\theta(x)] &= \mathbb{E}[\max(\xi_\theta, -x)] \\ &\leq \mathbb{E}[\max(\xi, -x)] \\ &\leq \gamma\,. \end{aligned}$$

Inserting this into (5.10) yields

$$x^p + \int\limits_{y\notin V} c\, y^{p+1}\mathbb{P}(\xi_\theta(x) \in dy) \leq x^p + c\left(x^{p+1} + (p+1)x^p\gamma + h(x)\right)$$
$$= cx^{p+1} + (1 + c(p+1)\gamma)x^p + ch(x). \qquad (5.11)$$

We now take $c > 0$ such that

$$1 + c(p+1)\gamma < 0$$

and $x_0 \geq x_0'$ so large that for $x \geq x_0$

$$(1 + c(p+1)\gamma)x^p + ch(x) \leq 0. \qquad (5.12)$$

Inserting (5.12) into (5.11) yields

$$x^p + \int_{y \notin V} c\, y^{p+1} \mathbb{P}(\xi_\theta(x) \in dy) \le c x^{p+1}.$$

This establishes for $V = \{x \,|\, x \le x_0\}$ the inequality

$$\forall \theta : \quad x^p + \int_{y \notin V} g^1(y) \mathbb{P}(\xi_\theta(x) \in dy) \le c x^{p+1}.$$

By Lemma 5.2 we find a Lyapunov function of the type

$$g^\lambda(x) = \begin{cases} c x^{p+1} & \text{for } x \notin V, \\ c_0 & \text{for } x \in V, \end{cases}$$

where c_0 is defined as in (5.9). Hence, by Lemma 5.2 and our assumption that $\sup_{s \in S,\, \theta \in \Theta} \mathbb{E}_\theta[N_\alpha(x_0, s)]$ is finite, it follows that $\sup_{\theta \in \Theta} H_\theta g$ is bounded by a polynomial of degree $p + 1$. □

The following lemma provides a sufficient condition for $\sup_{\theta \in \Theta} \mathbb{E}_\theta[N_\alpha(x_0, s)]$ to be finite for any $s \in S$.

Lemma 5.4. *Let $\alpha = \{0\}$. Suppose that for each $x_0 > 0$*

$$\mathbb{P}(\xi \le -x_0) \triangleq p(x_0) > 0.$$

Then $\sup_{s \in S,\, \theta \in \Theta} \mathbb{E}_\theta[N_\alpha(x_0, s)]$ is finite.

Proof. For $x_0 > 0$ set $V = \{x \mid x \ge x_0\}$. Note that for each $x_0 > 0$ there is $p(x_0) > 0$ such that

$$\begin{aligned} &\inf_{x \in V} \inf_{\theta \in \Theta} \mathbb{P}(X_\theta(t+1) = 0 \mid X_\theta(t) = x, X_\theta = s) \\ &\quad = \inf_{x \in V} \inf_{\theta \in \Theta} \mathbb{P}(\xi_\theta \le -x) \\ &\quad \ge \mathbb{P}(\xi \le -x_0) = p(x_0) > 0, \end{aligned}$$

for any $s \in S$. In words, the probability that the process jumps from a state in V immediately to $\alpha = \{0\}$ is at least $p(x_0)$. A simple geometrical trial argument (with probability of success $p(x_0)$) then shows that

$$\sup_{s \in S,\, \theta \in \Theta} \mathbb{E}_\theta[N_\alpha(x_0, s)] \le \frac{p(x_0)}{1 - p(x_0)}, \quad s \in S,$$

which completes the proof. □

Lemma 5.5. *Let $\alpha = \{0\}$. Let $g \in \mathcal{L}^1(P_\theta;\Theta) \cap \mathcal{M}$ and assume that*

(i) $\mathrm{H}_\theta g$ *is bounded by a function f in a neighborhood Θ_1 of θ:*

$$\forall \theta \in \Theta_1 : \quad h_\theta(x) \triangleq \mathrm{H}_\theta g(x) \leq f(x);$$

(ii) (S2) *holds.*

Let $\theta, \theta' \in \Theta_1$, then for $\theta \leq \theta'$

$$|(P_{\theta'} h_\theta)(x) - (P_\theta h_\theta)(x)| \leq \mathbb{E}\left[f\big(\xi_{\theta'} - \xi_\theta\big)\right] \tag{5.13}$$

and for $\theta \geq \theta'$

$$|(P_{\theta'} h_\theta)(x) - (P_\theta h_\theta)(x)| \leq \mathbb{E}\left[f\big(\xi_\theta - \xi_{\theta'}\big)\right] \; ; \tag{5.14}$$

moreover,

$$|h_{\theta'}(x) - h_\theta(x)| \leq \mathbb{E}\left[\big|f\big(\xi_{\theta'} - \xi_\theta\big)\big|\right] \sum_{n=0}^{\infty} P_\theta^n e. \tag{5.15}$$

Proof. Since g is monotone, it follows from Lemma 2.1 by induction that

$$h_\theta = \sum_{k=0}^{\infty} P_\theta^k g \in \mathcal{M}.$$

The coupling of two processes with the same transition operator but different starting states, say x and $x+y$ with $y \geq 0$ gives that

$$h_\theta(x+y) \leq h_\theta(x) + h_\theta(y),$$

since before absorbing in $\alpha = \{0\}$ the state of the process with starting state $x+y$ is always larger than that with starting state x and at absorption it is at most y. With the monotonicity of $h_\theta(x)$ in x we have the inequalities

$$h_\theta(x) \leq h_\theta(x+y) \leq h_\theta(x) + h_\theta(y),$$

and consequently

$$h_\theta(x+y) - h_\theta(x) \leq h_\theta(y).$$

Condition (S2) implies $\xi_{\theta'}(x) - \xi_\theta(x) \geq 0$ a.s. Substituting $x + \xi_\theta(x)$ for x and $(\xi_{\theta'}(x) - \xi_\theta(x))$ for y in the above inequality yields

$$\begin{aligned} h_\theta(x+\xi_{\theta'}(x)) - h_\theta(x+\xi_\theta(x)) &\leq h_\theta(\xi_{\theta'}(x) - \xi_\theta(x)) \\ &\leq h_\theta(\xi_{\theta'} - \xi_\theta), \end{aligned} \tag{5.16}$$

where the last inequality follows from Lemma 2.2 (i). For $\theta \leq \theta'$ we have $\xi_\theta(x) \leq_{st} \xi_{\theta'}(x)$ and, since $h_\theta \in \mathcal{M}$, we obtain

$$\begin{aligned} |P_{\theta'} h_\theta - P_\theta h_\theta|(x) &= |\mathbb{E}\left[h_\theta(x+\xi_{\theta'}(x)) - h_\theta(x+\xi_\theta(x))\right]| \\ &= \mathbb{E}\big[h_\theta(x+\xi_{\theta'}(x)) - h_\theta(x+\xi_\theta(x))\big] \\ &\leq \mathbb{E}\big[h_\theta(\xi_{\theta'} - \xi_\theta)\big], \end{aligned}$$

where the last inequality follows from (5.16). Since $h_\theta(x) \leq f(x)$, we obtain

$$|P_{\theta'} h_\theta - P_\theta h_\theta|(x) \leq \mathbb{E}\big[f(\xi_{\theta'} - \xi_\theta)\big]. \tag{5.17}$$

The proof for $\theta \geq \theta'$ is similar.

The identity

$$P_{\theta'}^n - P_\theta^n = \sum_{k=0}^{n-1} P_{\theta'}^k (P_{\theta'} - P_\theta) P_\theta^{n-k-1}$$

can be proved by induction. This implies

$$\begin{aligned} \mathrm{H}_{\theta'} g - \mathrm{H}_\theta g &= \sum_{n=0}^{\infty} (P_{\theta'}^n - P_\theta^n) g \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} P_{\theta'}^k (P_{\theta'} - P_\theta) P_\theta^{n-k-1} g \\ &= \sum_{n=0}^{\infty} P_{\theta'}^n (P_{\theta'} - P_\theta) \sum_{k=0}^{\infty} P_\theta^k g \\ &= \sum_{n=0}^{\infty} P_{\theta'}^n (P_{\theta'} - P_\theta) h_\theta . \end{aligned}$$

Hence, together with (5.17) we find,

$$|\mathrm{H}_{\theta'} g - \mathrm{H}_\theta g| \leq \mathbb{E}\big[\,|f(\xi_{\theta'} - \xi_\theta)|\,\big] \sum_{n=0}^{\infty} P_{\theta'}^n e\,. \tag{5.18}$$

Reversing the roles of θ' and θ proves the last assertion. □

Lemma 5.6. *Suppose that the Lyapunov condition holds for p, with $p \geq 0$. Let $\Theta_1 \subset \Theta$ such that, for $l = 0, \ldots, p+1$,*

$$\sup_{\theta' \in \Theta_1} \left| \frac{d}{d\theta}\Big|_{\theta=\theta'} \mathbb{E}\big[(\xi_\theta)^l\big] \right| \triangleq a_l < \infty.$$

Then for $g \in (\mathcal{H}, p) \cap \mathcal{M}$ it holds for all $\theta, \theta' \in \Theta_1$

$$|(P_{\theta'} \mathrm{H}_\theta g)(x) - (P_\theta \mathrm{H}_\theta g)(x)| \leq c_0 + |\theta' - \theta|\, C_1, \tag{5.19}$$

where

$$C_1 \triangleq \sum_{k=1}^{p} c_k a_k.$$

Proof. Let $\theta' > \theta$. By Lemma 5.3 together with Lemma 5.5, it holds that

$$|(P_{\theta'} \mathrm{H}_\theta g)(x) - (P_\theta \mathrm{H}_\theta g)(x)| \leq \mathbb{E}[f(\xi_{\theta'} - \xi_\theta)]$$

with $f(x) = \sum_{k=0}^{p+1} c_k x^k$. By calculation,

$$\mathbb{E}\big[|f(\xi_{\theta'} - \xi_\theta)|\big] \quad \leq \quad c_0 + \sum_{k=1}^{p+1} c_k \mathbb{E}\left[|\xi_{\theta'} - \xi_\theta|^k\right].$$

For $\theta' > \theta$, it holds

$$\begin{aligned} \mathbb{E}\left[|\xi_{\theta'} - \xi_\theta|^k\right] &= \mathbb{E}\left[(\xi_{\theta'} - \xi_\theta)^k\right] \\ &\leq \mathbb{E}\left[\xi_{\theta'}^k\right] - \mathbb{E}\left[\xi_\theta^k\right], \end{aligned}$$

where we use the fact that $\xi_{\theta'} \geq \xi_\theta$ a.s., and, for $\theta' < \theta$, we obtain

$$\begin{aligned} \mathbb{E}\left[|\xi_{\theta'} - \xi_\theta|^k\right] &= \mathbb{E}\left[(\xi_\theta - \xi_{\theta'})^k\right] \\ &\leq \mathbb{E}\left[\xi_\theta^k\right] - \mathbb{E}\left[\xi_{\theta'}^k\right], \end{aligned}$$

where we use the fact that $\xi_\theta \geq \xi_{\theta'}$ a.s. Combining the above inequalities we arrive at

$$\mathbb{E}\left[|\xi_{\theta'} - \xi_\theta|^k\right] \leq \left|\mathbb{E}\left[\xi_{\theta'}^k\right] - \mathbb{E}\left[\xi_\theta^k\right]\right|,$$

and using the fact that a_k is a Lipschitz-constant for $\mathbb{E}[\xi_\theta^k]$ proves the claim for $\theta' > \theta$. The proof for the case $\theta' < \theta$ follows from the same reasoning. □

Lemma 5.7. *Let the Lyapunov condition be satisfied for p, with $p \geq 0$, and let $\Theta_1 \subset \Theta$ be a neighborhood of θ. If*

(i) *ξ_θ is $(\mathcal{H}, p+1)$-differentiable on Θ_1,*

(ii) *for $g \in (\mathcal{H}, p)$ it holds that*

$$\sup_x \sup_{\hat\theta \in \Theta_1} \left| P'_\theta \mathrm{H}_\theta g(x) - P'_{\hat\theta} \mathrm{H}_\theta g(x)\right| < \infty,$$

then it holds for $g \in (\mathcal{H}, p) \cap \mathcal{M}$ and $\theta', \theta \in \Theta_1$ that

$$|(P_{\theta'} - P_\theta)\mathrm{H}_\theta g(x)| \leq |\theta' - \theta| f_0(x)$$

for $f_1 \in (\mathcal{H}, 0)$, and

$$|(\mathrm{H}_{\theta'} g(x) - \mathrm{H}_\theta g(x)| \leq |\theta' - \theta| f_1(x)$$

for $f_1 \in (\mathcal{H}, 1)$.

Proof. Let $\mathrm{H}_\theta g = h_\theta$. We have assumed that $g \in (\mathcal{H}, p)$ and Lemma 5.3 yields $h_\theta \in (\mathcal{H}, p+1)$. The second part of the lemma is a direct consequence of the first part. To see this, note that

$$h_{\theta'} - h_\theta = \sum_{n=0}^{\infty} P_{\theta'}^n (P_{\theta'} - P_\theta) h_\theta,$$

see the proof of Lemma 5.5 for details. The first part of the lemma implies

$$|h_{\theta'} - h_\theta| \leq |\theta' - \theta| \sum_{n=0}^{\infty} P_{\theta'}^n f_0 = |\theta' - \theta| \mathrm{H}_{\theta'} f_0,$$

for $f_0 \in (\mathcal{H}, 0)$. By Lemma 5.3, $\mathrm{H}_{\theta'} f_0 \in (\mathcal{H}, 1)$, which concludes the proof of the second part of the lemma.

We now turn to the proof of the first part of the lemma. By (i), P_θ is $(\mathcal{H}, p+1)$-differentiable, see Lemma 2.4. The Mean Value Theorem implies that

$$(P_{\theta'} - P_\theta) h_\theta(x) = (\theta' - \theta) P'_{\theta' + \delta(x)} h_\theta(x), \tag{5.20}$$

for $|\delta(x)| \leq |\theta' - \theta|$ for $|\theta' - \theta|$ sufficiently small.

Suppose that

$$\sup_x |P'_\theta h_\theta(x)| = \infty.$$

Then, a sequence (x_k, Δ_k) exists such that

$$\lim_{k \to \infty} \Delta_k = 0 \quad \text{and} \quad \lim_{k \to \infty} \Delta_k |P'_\theta h_\theta(x_k)| = \infty. \tag{5.21}$$

Inserting this sequence into (5.20), we obtain

$$\begin{aligned} (P_{\theta + \Delta_k} - P_\theta) h_\theta(x_k) &= \Delta_k P'_{\theta + \delta(x_k)} h_\theta(x_k), \\ &= \Delta_k P'_\theta h_\theta(x_k) + \Delta_k (P'_{\theta + \delta(x_k)} h_\theta(x_k) - P'_\theta h_\theta(x_k)). \end{aligned}$$

Note that for k sufficiently large it holds that

$$|P'_{\theta + \delta(x_k)} h_\theta(x_k) - P'_\theta h_\theta(x_k)| \leq \sup_x \sup_{\theta' \in \Theta_1} |P'_{\theta'} h_\theta(x) - P'_\theta h_\theta(x)|.$$

Condition (ii) yields that

$$\sup_x \sup_{\theta' \in \Theta_1} |P'_{\theta'} h_\theta(x) - P'_\theta h_\theta(x)| < \infty,$$

which implies

$$\lim_{k \to \infty} \Delta_k (P'_{\theta + \delta(x_k)} h_\theta(x_k) - P'_\theta h_\theta(x_k)) = 0$$

and we obtain from (5.21) that

$$\limsup_{k\to\infty}(P_{\theta+\Delta} - P_\theta)h_\theta(x_k) \in \{\infty, -\infty\},$$

which contradicts equation (5.19) in Lemma 5.6. Hence,

$$\sup_x |P'_\theta h_\theta(x)| < \infty,$$

and from condition (v) it thus follows that

$$\sup_x |P'_{\hat\theta} h_\theta(x)| < \infty$$

for any $\hat\theta$ such that $|\theta - \hat\theta| \leq \delta$ for δ sufficiently small. We have thus shown that

$$C \stackrel{\text{def}}{=} \sup_{\{\hat\theta:|\theta-\hat\theta|\leq\delta\}} \sup_x |P'_{\hat\theta} h_\theta(x)| < \infty.$$

Applying the Mean Value Theorem now gives

$$|(P_{\theta'} - P_\theta)h_\theta(x)| \leq |\theta' - \theta| C$$

for $|\theta' - \theta| \leq \delta$, which proves the claim. □

6. Proof of Theorem 3.1

For the proof we will apply Theorem 5.1. To this end note that under the assumptions of Theorem 3.1, P_θ is $(\mathcal{H}, p+1)$-differentiable according to Lemma 2.4 and condition (i) in Theorem 5.1 follows from Lemma 2.3, condition (ii) in Theorem 5.1 follows from Lemma 5.3, and conditions (ii) and (iii) in Theorem 5.1 follow from Lemma 5.7. For the proof of Theorem 3.1 it therefore remains to show that conditions (B1) and (B2) hold.

We now show (B1) for $g \in (\mathcal{H}, p) \cap \mathcal{M}$. Let $h_\theta \triangleq H_\theta g$. By Lemma 5.7, $f_0 \in (\mathcal{H}, 0)$ exists such that

$$\sup_{\{\Delta:\theta+\Delta\in\Theta_1\}} \frac{1}{|\Delta|} |(P_{\theta+\Delta} - P_\theta)h_\theta(x)| \leq f_0(x). \tag{6.1}$$

We have already established that $h_\theta \in (\mathcal{H}, p+1)$ and that P_θ is $(\mathcal{H}, p+1)$-differentiable, which yields

$$\lim_{\Delta\to 0} \frac{1}{\Delta}(P_{\theta+\Delta} - P_\theta)h_\theta = P'_\theta h_\theta.$$

Applying the dominated convergence theorem then yields

$$\lim_{\Delta\to 0} \frac{1}{\Delta} H_\theta(P_{\theta+\Delta} - P_\theta)h_\theta = \mathrm{H}_\theta P'_\theta h_\theta, \tag{6.2}$$

which establishes (B1). Moreover, by (6.1) together with (6.2) it follows from Lemma 5.3 that $\mathrm{H}_\theta P'_\theta h_\theta \in (\mathcal{H}, 1)$.

We now turn to condition (B2) for $g \in (\mathcal{H}, p) \cap \mathcal{M}$. By Lemma 5.7,

$$|h_{\theta+\Delta} - h_\theta| \leq |\Delta| f_1,$$

for Δ sufficiently small and $f_1 \in (\mathcal{H}, 1)$. Recall that the positive part (of the Hahn−Jordan decomposition) of a signed measure μ is denoted by $[\mu]^+$ and the negative part by $[\mu]^-$. Note that

$$\begin{aligned}
&\Big| \frac{1}{\Delta} (P_{\theta+\Delta} - P_\theta)(h_{\theta+\Delta} - h_\theta) \Big| \\
&\quad \leq \frac{1}{|\Delta|} \big[(P_{\theta+\Delta} - P_\theta) \big]^+ |h_{\theta+\Delta} - h_\theta| + \frac{1}{|\Delta|} \big[(P_{\theta+\Delta} - P_\theta) \big]^- |h_{\theta+\Delta} - h_\theta| \\
&\quad \leq [(P_{\theta+\Delta} - P_\theta)]^+ f_1 + [(P_{\theta+\Delta} - P_\theta)]^- f_1.
\end{aligned}$$

It holds that

$$[(P_{\theta+\Delta} - P_\theta)]^\pm f_1 \leq ||P_{\theta+\Delta} - P_\theta||_{f_1} f_1 \leq |\Delta| M f_1,$$

for some finite number M, see (2.4) together with (2.5). Hence,

$$\Big| \frac{1}{\Delta} (P_{\theta+\Delta} - P_\theta)(h_{\theta+\Delta} - h_\theta) \Big| \leq 2 |\Delta| \, M \, f_1$$

and

$$\lim_{\Delta \to 0} \Big| \frac{1}{\Delta} (P_{\theta+\Delta} - P_\theta)(h_{\theta+\Delta} - h_\theta) \Big| \quad = \quad 0. \tag{6.3}$$

Note that

$$\Big| \frac{1}{\Delta} (P_{\theta+\Delta} - P_\theta)(h_{\theta+\Delta} - h_\theta) \Big| \leq \frac{1}{|\Delta|} |(P_{\theta+\Delta} - P_\theta) h_{\theta+\Delta}| + \frac{1}{|\Delta|} |(P_{\theta+\Delta} - P_\theta) h_\theta|.$$

For $|\Delta|$ sufficiently small, Lemma 5.7 yields

$$\Big| \frac{1}{\Delta} (P_{\theta+\Delta} - P_\theta)(h_{\theta+\Delta} - h_\theta) \Big| \leq \tilde{f}_0,$$

for $\tilde{f}_0 \in (\mathcal{H}, 0)$, which implies

$$\mathrm{H}_\theta \Big(\sup_{|\Delta|} \frac{1}{\Delta} (P_{\theta+\Delta} - P_\theta)(h_{\theta+\Delta} - h_\theta) \Big) < \infty. \tag{6.4}$$

By (6.3) together with (6.4), condition (B2) follows from the Dominated Convergence Theorem, which completes the proof.

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