Economic Theory 14, 691–704 (1999)

Economic Theory

General equilibrium and resource economics*

Jan H. van Geldrop[1] **and Cees A. Withagen**[2]

[1] Department of Mathematics Computing Science, Eindhoven University of Technology, P.O. Box 513, NL-5600 MB Eindhoven, THE NETHERLANDS

[2] Department of Economics and Tinbergen Institute, Free University Amsterdam, De Boelelaan 1105, NL-1081 HV Amsterdam, THE NETHERLANDS (e-mail: cwithagen@econ.vu.nl)

Received: May 2, 1996; revised version: May 13, 1998

Summary. For a number of reasons a large class of general equilibrium models from the field of resource economics does not allow for an equilibrium analysis along the lines of the theory of infinite dimensional commodity spaces. The reasons concern the choice of the commodity space and the applicability of properness assumptions with respect to preferences and the technology. This paper illustrates the difficulties and shows for a prototype model how the problems can successfully be tackled by the use of a limit argument on equilibria in the truncated economies.

Keywords and Phrases: General equilibrium, Resource economies, Truncation.

JEL Classification Numbers: D50, Q30.

1 Introduction

The present paper deals with a general equilibrium analysis of a prototype model with exhaustible resources. The modelling of the economy entails an infinite dimensional commodity space, about which there is now an abundant and rich literature [see Mas-Colell and Zame (1991) for an excellent survey and Aliprantis, Brown and Burkinshaw (1989) for a comprehensive text book]. Unfortunately, it turns out however that the existing literature is hard to apply to the kind of economy we have in mind. To see this we briefly sketch some of the basic constituent elements of the main stream equilibrium theory in infinite dimensional spaces. This description is based on Mas-Colell and Zame (o.c.).

First the commodity space is identified, endowed with a topology and order structure, such that the positive orthant of the commodity space is a

* We would like to thank Michiel Keyzer for helpful comments.

Correspondence to: C.A. Withagen

closed convex cone. Here arises a first problem. Section 2 of this paper will illustrate that it is rather difficult to make an appropriate choice for the commodity space a priori. There are examples where L_1 is clearly the only natural candidate but a slight modification of the model may have a proper subset of L_1 or L_∞ as the commodity space. The next step is to introduce consumers, the consumption sets, preferences and initial endowments. In the context of exchange economies Mas-Colell and Zame (o.c. p. 1849) put forward that there are three main difficulties: compactness of attainable sets, the continuity of wealth and the supportability of preferred sets by prices. It is well-known that in the case of L_p $(p < \infty)$ spaces the first two problems are not very serious (in fact compactness poses then usually no problem at all) but, on the other hand, the interior of the positive orthant of all L_p spaces is empty which may cause serious difficulties. These are generally tackled by making an assumption with respect to the (uniform) properness of the preference relations, introduced by Mas-Colell (1986). There are several notions of properness (see also Yannelis and Zame, 1987) but they all entail a priori bounds on the marginal rates of substitution in consumption [see e.g. Zame (o.c. p. 1087) for a nice interpretation]. Now, in resource economics it is quite customary to employ instantaneous utility functions with marginal utility going to infinity as the rate of consumption goes to zero. This assumption is of course made to have positive consumption along optimal paths. Therefore, if we want to adhere to the traditional resource economics models, and if it is known beforehand that the positive cone of commodity space has empty interior, or if we do not know beforehand what the appropriate commodity space is, there is an important obstacle to apply the infinite dimensional equilibrium theory. Matters get only worse when production is added to the model. For commodity spaces with a non-empty positive cone assumptions like the possibility of truncation (see Prescott and Lucas, 1972) or exclusion (Bewley, 1972) will help to solve the supportability issue. However, it is well-known that additional assumptions need to be made if the positive cone has an empty interior. Zame (o.c.) introduces another properness assumption which in some sense bounds the marginal efficiency of production. Again, the standard modelling practice in resource economics is at variance with this assumption: it is quite common to assume that the marginal product of the raw material from the exhaustible resources goes to infinity as the input goes to zero.

We conclude that the existing theory on infinite dimensional commodity spaces cannot be used to tackle the existence of general equilibria in a large class of models with exhaustible resources. In this paper we present an alternative approach.

In Section 2 we sketch a prototype neoclassical economy with exhaustible resources and study the question of the "appropriate" commodity space. The issue we address in Section 3 is existence. The question is whether or not the existence of a general equilibrium of the infinite horizon economy can be ascertained from the existence of equilibria in every truncated economy. The answer is in the affirmative if, among other things, the equilibrium prices in the truncated economies have bounded growth rates, which will be the case in a large class of models. In Section 4 we state some conclusions resulting from our approach.

2 A prototype model

In the economy under consideration there are m (≥ 1) stocks of exhaustible resources. There is one other commodity that will be called the composite commodity. It serves as a capital stock in production and as a flow of a consumer good. So, a typical commodity bundle consists of $m+2$ elements. The first element is the *stock* of the composite commodity. Elements 2 up to $m+1$ are the *stocks* of the m exhaustible resources and the final element is a mapping from $[0,\infty)$ to the positive reals denoting the *flow* of the composite commodity.

There are l (≥ 1) *consumers* or dynasties, infinitely living. Consumer h ($h = 1, 2, \dots, l$) has initial holding $\omega_h := (\bar{k}_h, \bar{s}_{h1}, \dots, \bar{s}_{hm}, 0) = (\bar{k}_h, \bar{s}_h, 0)$, where $\bar{k}_h$ denotes the initial holding of capital, $\bar{s}_{hj}$ the initial holding of resource stock of type j ($j = 1, 2, \dots, m$) and "0" says that the flow of the consumer good accruing to the consumer as initial endowment is zero. We assume that $\bar{k}_h > 0$ (all h) and $\bar{s}_{hj} > 0$ for at least one h (all j). The aggregate initial stocks are denoted by $k_0 := \sum_h \bar{k}_h$ and $s_0 := (s_{01}, s_{02}, \dots, s_{0m}) = \left(\sum_h \bar{s}_{h1}, \sum_h \bar{s}_{h2}, \dots, \sum_h \bar{s}_{hm}\right)$.

Consumer h is entitled to a given constant proportion ϑ_h in the profits of the firm to be described below. The consumption set of consumer h is denoted by X_h. It consists of the set of tuples of the form $(0, 0, c)$ with $c : [0,\infty) \to \mathbb{R}_+$ a measurable function. Consumers only derive utility from the consumption of the composite commodity. Preferences of consumer h are represented by a utility functional of the following type:

$$U_h(x_h) = U_h(0, 0, c_h) = \int_0^\infty e^{-\rho_h s} u_h(c_h(s))\, ds. \tag{2.1}$$

Here ρ_h is a given positive constant denoting the rate of time preference and u_h is the instantaneous utility function. This function satisfies

Assumption 1. $u_h : \mathbb{R}_+ \to \mathbb{R}$ is strictly monotonic and strictly concave. $u_h'(0) = \infty$; $u_h'(\infty) = 0$. $\eta_h(c) := u_h''(c)c/u_h'(c) \leq \eta$ for all $c > 0$ and some $\eta < 0$.

This assumption is quite common in the literature on resource economics. In view of the unboundedness of marginal utility, preferences in this economy are not proper.

There are two types of *firms*. One type is engaged in the exploitation of the exhaustible resources. These firms buy resource stocks and deplete them with the aid of capital as a factor of production. Each deposit of a resource requires its own extraction technology: $G_j(v_j)$ of capital is needed to extract the amount v_j from resource j. The material once above the ground is homogeneous. The second type firms convert it, together with capital, into a flow of the composite commodity according to a production function F.

About F and G_j $(j = 1, 2, \dots, m)$ the following assumptions are made.

Assumption 2. $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is continuous, increasing, and concave on $\mathbb{R}_+^2$ and continuously differentiable on $\mathbb{R}_{++}^2$.

Assumption 3. $G_j : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, convex and continuously differentiable. Moreover $G_j(0) = 0$.

Since we assume convexity of the individual production sets we can aggregate over the firms and do as if there were only one single firm. Without loss of generality it will be assumed that the firm buys all of the initial endowments (k_0, s_0). If a commodity is not used, its price will be zero and we can do as if it is bought at a zero price. The economy's aggregate technology Y is now given by the following

Definition 2.1. $y = (-k_0, -s_0, z)$ belongs to the aggregate production set Y if and only if there are absolutely continuous mappings k and $s = (s_1, \dots, s_m)$ and mappings $v = (v_1, \dots, v_m)$, belonging to the space of Lebesgue integrable functions, such that (omitting t when there is no danger of confusion)

$$\dot{k} = F\left(k - \sum_j G_j(v_j), \sum_j v_j\right) - \mu k - z, \; k(0) = k_0 \tag{2.2}$$

$$\dot{s}_j = -v_j, \; s_j(0) = s_{j0}, \; s_j \geq 0 \quad (j = 1, 2, \dots, m) \tag{2.3}$$

$$v_j \geq 0 \quad (j = 1, 2, \dots, m) \tag{2.4}$$

$$k - \sum_j G_j(v_j) \geq 0 \tag{2.5}$$

Here μ (≥ 0) is the rate of decay of capital. Differential equation (2.2) is to be interpreted as follows. Let the capital stock at some instant of time be k. If the firm wants to use an amount Σv_j of the raw material in the production of the composite commodity, $\Sigma G_j(v_j)$ of the capital stock is needed to extract that amount. Therefore, $k - \Sigma G_j(v_j)$ is available as capital in the production process of the composite commodity. Total net production of this commodity is then $F - \mu k$, which is used for net investments $(\dot{k})$ and consumption (z).

The spirit of the model stems from the field of resource economics. For the case of a single consumer, no extraction costs, no depreciation and a CES specification of F it coincides with the model extensively studied by Dasgupta and Heal (1974). Stiglitz (1974) uses a Cobb-Douglas F and allows for population growth and technological process. Chiarella (1980) deals with two consumers and Cobb-Douglas functional forms for F and the u_h's. Elbers and Withagen (1984) also study the case of two consumers but they do not include production; on the other hand they introduce extraction costs. Heal (1976) and Kay and Mirrlees (1975) assume a linear extraction technology. The model also closely relates to van Geldrop et al. (1991) who employ a discrete time setting. We should also mention the work of Mitra (1978) and (1980) and Toman (1986) who study efficiency in a one consumer world.

We next address the remarks made in the Introduction. They are illustrated by means of a number of examples.

a) The simplest example is the classical cake-eating problem. The production function F is identically zero. Any feasible allocation pattern in the economy should satisfy $\dot{k} = -z - \mu k$.

   If there is no depreciation ($\mu = 0$) then obviously L_1 is the appropriate commodity space. If depreciation is positive, then

$$\int_0^\infty e^{\mu t} z(t)\, dt < \infty$$

   so that any feasible consumption pattern lies in L_1. However, not all elements of L_1 can be consumed, even with an arbitrarily large initial capital stock. Since marginal utility is unbounded at zero consumption, consumption wil be positive at all instants of time. It is then easily seen that equilibrium prices will not lie in L_1. As expected the equilibrium price will grow at the rate μ. If preferences would display properness then zero consumption would be possible in equilibrium and prices would lie in the dual of L_1. Zame (o.c.) shows this in a discrete-time setting, where ℓ_1 is the commodity space. Note that in the Hotelling problem equilibrium prices are in L_∞ in spite of the violation of the properness assumption.

b) The second example includes production. There are no extraction costs, so that there is essentially a single exhaustible resource, and no depreciation. F satisfies: there exists ψ such that $F(k, v) \leq \psi v$ for all $(k, v) \geq 0$. Then, for any given (k_0, s_0) any feasible consumption trajectory z is integrable since

$$\int_0^t z(\tau)\, d\tau \leq \int_0^t \psi v(\tau)\, d\tau + k_0 \leq \psi s_0 + k_0$$

   Moreover, also the converse is true. Any consumption pattern in L_1 can be produced by a proper choice of k_0.

c) The next example shows that the limited availability of the exhaustible resources does not preclude positive maintained consumption. Suppose that there is only one exhaustible resource which can costlessly be extracted ($m = 1, G_1 \equiv 0$). There is no depreciation ($\mu = 0$). The consumers have identical isoelastic utility functions ($\eta_h(c) = \eta < 0$ all h) and $\rho_h = \rho$ (all h). Suppose, finally, that F is CES with elasticity of substitution larger than unity. In that case there exists a constant ψ such that $F(x, 1)/x \to \psi$ as $x \to \infty$ meaning that the average product of capital is bounded away from zero. Dasgupta and Heal (1974) show that if $\psi > \rho > \psi(1 + \eta)$, there exists an equilibrium with an asymptotic growth rate of consumption equal to $(\rho - \psi)/\eta$, which is positive. Therefore, equilibrium consumption profiles are not in L_1, nor in L_∞.

d) We maintain the assumptions of the previous example except that depreciation is positive and F is CES with elasticity of substitution smaller than unity. We define $\varphi := \max F(1, x)$, the maximum product per unit of capital. Take $\mu > \varphi$, and $\alpha = (\mu - \varphi)/2$. Then it follows from (2.2) with $z(t) = e^{-\alpha t}$ inserted, that $\dot{k} \leq (\varphi - \mu)k - e^{-\alpha t}$. So, k becomes negative within finite time. Hence the consumption profile $e^{-\alpha t}$, although in L_1, cannot be produced. Obviously, with extraction costs included this conclusion holds a fortiori.

In the CES-cases discussed above unbounded marginal efficiency in production gives rise to a commodity space with empy interior and bounded marginal efficiency gives rise to a commodity space with non-empty interior. This is not what one would like to have if the aim is to apply the standard general equilibrium theory.

e) A final example shows that with decreasing returns to scale aggregate net production over time might be unbounded even with a positive rate of depreciation and convex extraction costs. Suppose F is Cobb-Douglas and $G_j(v_j) = av_j$ for all j with a constant (so, virtually there is only one exhaustible resource). Then net production over a period of time of length T is given by

$$\int_0^T [(k - av)^\alpha v^\beta - \mu k]\, dt$$

where α and β are positive constants with $\alpha + \beta < 1$. Choose for $t \in [0, T]$

$$v(t) := s_0/T$$

$$k(t) := \left(\frac{\alpha}{\mu}\right)^{1/(1-\alpha)} v(t)^{\beta/(1-\alpha)} + av(t)\ .$$

Then the value of the integral given above is

$$(1-\alpha)\left(\frac{\alpha}{\mu}\right)^{\alpha/(1-\alpha)} s_0^{\beta/(1-\alpha)} T^{(1-\alpha-\beta)/(1-\alpha)} - a\mu s_0.$$

and this expression goes to infinity as T goes to infinity. So, there exists a feasible consumption pattern that is not in L_1. With strictly convex extraction costs the same result can be obtained, as long as $G'(0)$ is bounded from above.

We shall now make some additional assumptions on F and G_j that will enable us to establish the existence of an equilibrium.

Assumption 4. $F(0, y) = F(x, 0)$ for all $(x, y) \in \mathbb{R}_+^2$. F is linearly homogeneous. For all $x > 0$, $F_2(x, y) \to \infty$ as $y \to 0$.
G_j is strictly convex (all j), $G_j'(0) \geq a$ for some $a > 0$ (all j).
$\mu > 0$.

So, we assume that both inputs are necessary, that F displays constant returns to scale and that the resource is "essential" (see Dasgupta and Heal, 1974). The extraction functions are strictly convex and marginal extraction costs are bounded away from zero. The rate of depreciation is positive.

Under this additional assumption we can show boundedness of total production over time.

Theorem 2.1. There exists a scalar $\bar{g} > 0$ such that

(i) $\int_0^\infty [F(k - \Sigma G_j(v_j), \Sigma v_j) - \mu k]\, dt \le \bar{g}\, \Sigma s_{0j}$
(ii) $k(t) + \int_0^t z(\tau)\, d\tau \le k_0 + \bar{g}\, \Sigma s_{0j}$

for any (k, s, v, z) with k and s absolutely continuous and v Lebesgue integrable such that (2.2)–(2.5) are satisfied.

Proof. Consider $g : \mathbb{R}_+ \to \mathbb{R}$ defined by $g(k) = F(k, 1) - \mu k$. Clearly $g(0) = 0$. In view of the linear homogeneity of F, we have $g(k) = k[F(1, 1/k) - \mu]$ for all $k > 0$. So, since $\mu > 0$ and $F(0, y) = 0$ for all $y \ge 0$ if follows that $g(k) \le 0$ for k sufficiently large. Hence $F(k, 1) - \mu k \le \bar{g}$ for some $\bar{g} > 0$ and all $k \ge 0$. Take some v with $\Sigma v_j > 0$. Then

$$\frac{1}{\Sigma v_j}[F(k - \Sigma G_j(v_j), \Sigma v_j) - \mu k] \le \frac{\Sigma v_j}{\Sigma v_j}\left[F\left(\frac{k}{\Sigma v_j}, 1\right) - \mu \frac{k}{\Sigma v_j}\right] \le \bar{g}$$

Hence $\dot{k} + z = F(k - \Sigma G_j(v_j), \Sigma v_j) - \mu k \le \bar{g}\, \Sigma v_j$. Since $\int_0^\infty \Sigma v_j\, dt \le \Sigma s_{0j}$, this proves part i) of the theorem. Part ii) of the theorem follows from integration. □

Clearly, homogeneity is not sufficient to be able to produce any consumption pattern in L_1, by a proper choice of the (initial) inputs. For that we need that the technology is productive enough. A set of sufficient conditions would be the following. There is j (say $j = 1$) and $a > 0$ such that $G_1(v) = av$ for all $v \ge 0$ and there exist $\lambda > 0$ and $\varphi > 0$ such that $F(1 - a\lambda, \lambda) - \mu = \varphi$. To prove the claim, take some $c \in L_1$ and construct k and v (omitting the index 1) as follows.

$$k(t) = e^{\varphi t} \int_t^\infty e^{-\varphi\tau} c(\tau)\, d\tau, \quad t \ge 0$$

$$v(t) = \lambda k(t)$$

Then $\dot{k} = F(k - G(v), v) - \mu k - c$. Choose s_0 as follows.

$$s_0 = \frac{\lambda}{\varphi a} \int_0^\infty (1 - e^{-\varphi\tau}) c(\tau)\, d\tau$$

Then (2.3) is satisfied since

$$\int_0^\infty k(s)\, ds = \int_0^\infty d\tau \int_0^\tau e^{\varphi(t-s)} c(s)\, ds = \frac{1}{\varphi} \int_0^\infty c(\tau)(1 - e^{-\varphi\tau})\ d\tau$$

3 Existence of a general equilibrium

We introduce the following notation. x_h^T is a consumption bundle of consumer h in the economy with horizon $T \in \mathbb{N}$. x_h^T will always mean $(0, 0, c_h^T)$, where $c_h^T(t) = 0$ for $t > T$. The initial endowment vector of consumer h is just ω_h. A production vector is denoted by y^T and we say that $y^T = (-k_0, -s_0, z^T)$ belongs to the production set Y^T if $y^T \in Y$ (see definition 2.1) and $z^T(t) = 0$ for all $t > T$. A price vector is denoted by $\pi^T = (1, q^T, p^T) = (1, q_1^T, \ldots, q_m^T, p^T)$.

The price of the stock of capital is set equal to unity. The value of a bundle, say z, at prices π^T is denoted by $\pi^T \cdot z$.

The assumptions on preferences and technology guarantee the existence of an equilibrium in the finite horizon economies [as was shown by Van Geldrop and Withagen (1994) using a Negishi approach]. We state this without proof in

Proposition 3.1. For all $T > 0$ there are $x_h^T \in X_h^T$ (all h), $y^T \in Y^T$ and $\pi^T = (1, q^T, p^T)$ where $p^T : [0, \infty) \to \mathbb{R}_+$ is a continuous mapping with $p^T(0) = 1$ and $p^T(t) = 0$ for $t \geq T$, so that the following holds:

$$\sum_h x_h^T \leq y^T + \sum_h \omega_h$$

$$\pi^T \cdot y^T \geq \pi^T \cdot z \quad \text{for all } z \in Y^T$$

$$\pi^T \cdot x_h^T \leq \pi^T \cdot \omega_h \quad (\text{all } h)$$

$$\pi^T \cdot z \leq \pi^T \cdot x_h^T \text{ and } z \in X_h^T \text{ implies } U_h(x_h^T) \geq U_h(z) \quad (\text{all } h) \qquad \square$$

We first derive some properties of the finite horizon equilibria.

For a fixed $T < \infty$ the problem of consumer h can be formulated as:

$$\max_{c_h} \int_0^T e^{-\rho_h t} u_h(c_h(t))\, dt$$

$$\text{subject to } \int_0^T p^T(t) c_h(t)\, dt \leq \pi^T \cdot \omega_h$$

It is straightforward to see from the application of Pontryagin's maximum principle that the following holds:

Lemma 3.1. There exists $b^T := (b_1^T, b_2^T, \ldots, b_\ell^T) \in \mathbb{R}_+^\ell$, such that, for all $t \leq T$, $e^{-\rho_h t} u_h'(c_h^T(t)) = b_h^T p^T(t)$ (all h). $\square$

Subject to (2.2)–(2.5) the producer chooses z to maximize

$$\int_0^T p^T(t) z(t)\, dt - k_0 - q^T \cdot s_0$$

According to Pontryagin's maximum principle there exist absolutely continuous $\lambda^T : [0, T] \to \mathbb{R}$, $k^T : [0, T] \to \mathbb{R}_+$ and piece-wise continuous $v^T : [0, T] \to \mathbb{R}_+^m$ such that for all $t \leq T$

$$p^T(t) = \lambda^T(t) \tag{3.1}$$

$$F_1(k^T(t) - \Sigma G_j(v_j^T(t)), \Sigma v_j^T(t)) - \mu = -\dot{\lambda}^T(t)/\lambda^T(t) \tag{3.2}$$

and such that

$$\lambda^T(t) F(k^T(t) - \Sigma G_j(v_j), \Sigma v_j) - \Sigma q_j^T v_j \tag{3.3}$$

is maximized with respect to v subject to (2.4)–(2.5).

We can now prove that the growth rate of equilibrium prices is uniformly bounded.

Lemma 3.2. There exists $\beta > 0$ such that $\mu - \beta \leq \dot{p}^T(t)/p^T(t) \leq \mu$ for all T and all $t < T$.

Proof. Since $F_1 \geq 0$ along an equilibrium, it is clear from (3.1) and (3.2) that $\dot{p}^T(t)/p^T(t) \leq \mu$.

Consider, for $x \geq 0, h(x) := F(x,1)/(a+x)$, where a is defined in Assumption 4. For all $x \geq 0$ we have $h(x) \geq 0$. Also $h(0) = 0$ since $F(0,1) = 0$ and $h(x) \to 0$ as $x \to \infty$ because $\frac{1}{x}F(x,1) = F(1,1/x) \to 0$ as $x \to \infty$. Hence h reaches a maximum for $x = \bar{x}$ (>0), defined by $F(\bar{x},1) - (a+\bar{x})F_1(\bar{x},1) = 0$. Define $\beta := F_1(\bar{x},1)$ and suppose that $\dot{p}^T(t)/p^T(t) < \mu - \beta$ for some T and some $t < T$. It follows from (3.1) and (3.2) that (omitting T and t) $F_1(k - \Sigma G_j(v_j), \Sigma v_j) > \beta$ and $x := (k - \Sigma G_j(v_j))/\Sigma v_j < \bar{x}$, in view of the concavity of F.

We must have $v_j > 0$ for at least one j. Then (3.3), together with the linear homogeneity of F, implies for this j:

$$\begin{aligned} q_j &= \lambda[F_2(k - \Sigma G_j(v_j), \Sigma v_j) - G_j'(v_j)F_1(k - \Sigma G_j(v_j), \Sigma v_j)] \\ &= \lambda[F(x,1) - (G_j'(v_j) + x)F_1(x,1)] \leq \lambda[F(x,1) - (a+x)F_1(x,1)] \\ &< \lambda\left[\frac{a+x}{a+\bar{x}}F(\bar{x},1) - (a+x)F_1(\bar{x},1)\right] = 0 \end{aligned}$$

But this contradicts that $q_j \geq 0$. □

It is shown next that the equilibrium stocks of capital and net investments are uniformly bounded.

Lemma 3.3. There exist $\overline{T}$, $\gamma > 0$ and $\delta > 0$ such that for all $T > \overline{T}$ and all $t < T$

i) $0 \leq k^T(t) \leq \gamma$
ii) $-\delta \leq \dot{k}^T(t) \leq \delta$

Proof.
ad i) Obviously $k^T(t) \geq 0$. The uniform boundedness from above follows from theorem 2.1 part ii).
ad ii) $\dot{k}^T(t)$ is bounded from above due to the facts that $z^T(t)$ is positive, $k^T(t)$ is bounded and $\Sigma G_j(v_j^T(t)) \leq k^T(t)$. $\dot{k}^T(t)$ can be made arbitrarily small only if $z^T(t)$ is arbitrarily large. Take some fixed h. It follows from lemma 3.1 that $-\rho_h + \eta_h(c_h^T(t))\dot{c}_h^T(t)/c_h^T(t) = \dot{p}^T(t)/p^T(t)$

Hence (omitting the index h)

$$\dot{c}^T/c^T = \frac{-\rho - \dot{p}^T/p^T}{-\eta(c^T)} \qquad (*)$$

It follows from Lemma 3.2 that $\rho + \dot{p}^T/p^T$ is bounded. Moreover, $\eta(c)$ is bounded away from zero. Hence, there are $\gamma > 0$ and $\overline{T} > 0$ such that $|\dot{c}^T(t)/c^T(t)| \leq \gamma$ for all $T > \overline{T}$ and all $t \leq T$. Hence $e^{\gamma t}c^T(t)$ is increasing and $e^{-\gamma t}c^T(t)$ is decreasing. Take arbitrary T and $t_0 \in [0, T]$. Omitting T as an index we have $c(t) \geq e^{\gamma(t-t_0)}c(t_0)$ for $0 \leq t \leq t_0$ and $c(t) \geq e^{\gamma(t_0-t)}c(t_0)$ for $t_0 \leq t \leq T$

Hence,

$$\begin{aligned}\int_0^T c(t)\,dt &\geq c(t_0)\int_0^{t_0} e^{\gamma(t-t_0)}\,dt + c(t_0)\int_{t_0}^T e^{\gamma(t_0-t)}\,dt \\ &= c(t_0)\left[\int_0^{t_0} e^{-\gamma s}\,ds + \int_0^{T-t_0} e^{-\gamma s}\,ds\right] \\ &=: c(t_0)\varphi(t_0).\end{aligned}$$

Clearly $\varphi'(0) > 0$, $\varphi'(T) < 0$ and $\varphi''(t_0) < 0$. Hence

$$\varphi(t_0) \geq \varphi(0) = \varphi(T) = \int_0^T e^{-\gamma s}\,ds \geq \int_0^1 e^{-\gamma s}\,ds \text{ for } T \geq 1.$$

It follows from Theorem 2.1 part ii that $c(t_0) \leq M$ for some $M > 0$. □

We now proceed to the existence of a general equilibrium in the infinite horizon economy by means of a sequence of lemmata. Formally, the statements made in these lemmata apply to subsequences but in order to avoid abundance of notation, we will denote them by T. Moreover, the symbol $\to$ is always used for point-wise convergence.

Lemma 3.4. There exists $b \in \mathbb{R}^{\ell}_{++}$ such that $b^T \to b$ as $T \to \infty$

Proof. Fix some arbitrary $h \in \{1, 2, \ldots, \ell\}$. It will be shown that b_h^T is uniformly bounded from above and from below (away from zero).

Since u_h is concave there exist constants A and B such that $u_h(c) \leq A + Bc$ for all $c \geq 0$. From the boundedness of c_h^T established in the previous lemma, there is $M > 0$ and $\overline{T} > 0$ such that for all $T > \overline{T}$

$$\begin{aligned}M &\geq \int_0^T e^{-\rho_h s}(A + Bc_h^T(s))\,ds \geq \int_0^T e^{-\rho_h s}u_h'(c_h^T(s))c_h^T(s)\,ds \\ &= \int_0^T b_h^T p^T(s)c_h^T(s)\,ds = b_h^T\pi^T \cdot \omega_h \geq b_h^T\overline{k}_h \qquad (*)\end{aligned}$$

Since $\overline{k}_h > 0$, b_h^T is uniformly bounded from above. Since $u'(c_h^T(0)) = b_h^T p^T(0) = b_h^T$ and $c_h^T(0) \leq \bar{c}$ for some $\bar{c}$ we have that b_h^T is bounded away from zero. □

Lemma 3.5. There exists $\pi = (1, q, p)$ with $p(0) = 1$ and p absolutely continuous on $[0, \infty)$ such that

i) $q^T \to q$ as $T \to \infty$
ii) $p^T \to p$ as $T \to \infty$

Proof.

ad i) Summation of the inequalities of the previous lemma over h yields

$$lM \geq \sum_h b_h^T \pi^T \cdot \omega_h$$

Recall that $\pi^T = (1, q_1^T, q_2^T, \ldots, q_m^T, p^T)$. Since b_h^T is bounded away from zero and $s_{0j} > 0$ (all j), q_j^T is uniformly bounded from above. Of course $q_j^T \geq 0$ for all j so that the sequence q^T has a convergent subsequence.

ad ii) Define $\hat{p}^T(t) := e^{-\mu t} p^T(t)$. In view of lemma 3.2 the functions $\hat{p}^T$ and $\dot{\hat{p}}^T$ can on $[0, \infty)$ be considered as uniformly bounded L_∞ functions. By Alaoglu's theorem (see e.g. Luenberger, 1969) there exists $f \in L_\infty$ and a subsequence (again denoted by T) such that $\dot{\hat{p}}^T$ converges to f "weak star". Choose $t_0 \in [0, \infty)$. Then, for $T > t_0$, we have

$$\hat{p}^T(t_0) = 1 + \int_0^{t_0} \dot{\hat{p}}^T(s)\, ds$$

from which it follows that

$$\hat{p}(t_0) := \lim \hat{p}^T(t_0) = 1 + \int_0^{t_0} f(s)\, ds$$

Hence $\hat{p}^T \to \hat{p}$ and

$$\hat{p}(t) = 1 + \int_0^{t_0} f(s)\, ds$$

so that $\hat{p}(0) = 1$, $\dot{\hat{p}} = f$ (a.e.) and $\hat{p}$ and hence p is absolutely continuous. □

Lemma 3.6. There exist $c_h \in L_\infty$ (all h) and $z \in L_\infty$ such that

i) $c_h^T \to c_h$ as $T \to \infty$ (all h)
ii) $z^T \to z$ as $T \to \infty$

Proof. This is clear from lemma 3.1, the convergence of b_h^T and $p^T(t)$, and from the fact that $z^T = \Sigma c_h^T$. □

It remains to be shown that $(\{x_h\}_h, y, \pi)$, with $y := (-k_0, -s_0, z)$, constitutes a general equilibrium for the infinite horizon economy.

Clearly $c_h(t) \geq 0$ for all h and all t; therefore $x_h = (0, 0, c_h) \in X_h$. The question whether $y \in Y$ or not is more difficult to handle. In the literature on infinite dimensional commodity spaces it is generally *assumed* that the production sets are closed in the topology of the commodity space, but this need not be the topology of point-wise convergence. So, it might be expected that there arises a problem here. In the case at hand it can be dealt with by means of the maximum theorem.

Lemma 3.7.

i) $x_h \in X_h$ (all h)
ii) $y \in Y$

Proof.

ad ii) Fix $T > 0$ and $t < T$. Let $k^T(t)(>0)$, $\lambda^T(t)$ and q^T be given. Consider the problem of maximizing (3.3) subject to (2.4) and (2.5). The constraint set, given by (2.4) and (2.5) is non-empty, compact valued (because $k^T(t)$ is bounded) and continuous. The maximand is continuous and has a compact range, the latter because $\lambda^T(t)$ and $k^T(t)$ are bounded. Since G_j is strictly convex for all j, the solution $v^T(t)$ of the problem is unique. It follows from the maximum theorem (see e.g. Berge, 1963) that $v^T(t)$ is a continuous function of the parameters $(k^T(t), \lambda^T(t), q^T)$. It follows from the previous lemmata that (k^T, λ^T, q^T) converges point-wise to (k, λ, q), and hence v^T converges point-wise to some v, by the maximum theorem. Also z^T converges point-wise. Since F is continuous, this proves that $(-k_0, -s_0, z) \in Y$. □

Lemma 3.8.

$$0 = \pi \cdot y \geq \pi \cdot \bar{y} \quad \text{for all } \bar{y} \in Y.$$

Proof. Suppose $\bar{y} := (-k_0, -s_0, \bar{z}) \in Y$ and $\pi \cdot \bar{y} > 0$. Hence

$$\int_0^\infty p(t)\bar{z}(t)\,dt > k_0 + q \cdot s_0$$

Then there exists $\bar{T} > 0$ such that

$$\lim_{T\to\infty} \int_0^{\bar{T}} p^T(t)\bar{z}(t)\,dt = \int_0^{\bar{T}} p(t)\bar{z}(t)\,dt > \lim_{T\to\infty} (k_0 + q^T \cdot s_0)$$

For T large enough and $T > \bar{T}$ we have

$$\int_0^{\bar{T}} p^T(t)\bar{z}(t)\,dt > k_0 + q^T \cdot s_0$$

and hence

$$\int_0^{T} p^T(t)\bar{z}(t)\,dt > k_0 + q^T \cdot s_0$$

contradicting the zero profit condition in the finite horizon economy. The proof of $\pi \cdot y = 0$ is straightforward and is omitted here. □

Lemma 3.9.

i) $\pi \cdot x_h = \pi \cdot \omega_h$ (all h)

ii) $\pi \cdot \bar{x} \leq \pi \cdot \omega_h$ with $z \in X_h \Rightarrow U_h(\bar{x}) \geq U_h(z)$ (all h)

Proof. Take some fixed $t \geq 0$ and h. Then $p^T(t)c_h^T(t) - p(t)c_h(t) = p^T(t) \times (c_h^T(t) - c_h(t)) + (p^T(t) - p(t))c_h(t) \to 0$ as $T \to \infty$ in view of point-wise convergence. Furthermore $p^T(t)c_h^T(t)$ is bounded from above, uniformly with respect to t and T, because $\pi^T \cdot \omega_h$ is uniformly bounded. By Lebesgue's dominated convergence theorem we have $\pi^T \cdot \omega_h = \pi^T \cdot x_h^T \to \pi \cdot x_h$. Now take $\bar{x} \in X_h$ such that $\pi \cdot x \geq \pi \cdot \bar{x}$. In view of the concavity of u_h we have

$$U_h(x_h) - U_h(\bar{x}) \geq \int_0^\infty b_h p(t)(c_h(t) - \bar{c}(t))\, dt \geq 0 \qquad \square$$

In view of the previous lemmata we can state.

Theorem 3.1. $(x_1, x_2, \ldots x_l, y, \pi)$ constitutes a general competitive equilibrium in the infinite horizon economy.

4 Conclusions

In the field of environmental and resource economics there is growing interest in economies which have the capacity to produce a limited amount of goods only. The question arises whether or not in such economies there are general competitive equilibria. It has been shown here that for a prototype model the answer is in the affirmative under a set of rather weak assumptions. The method employed, deriving the infinite horizon equilibrium from the sequence of finite horizon equilibria, may prove helpful in other existence problems as well.

References

Aliprantis, C., Brown, D., Burkinshaw, O.: Existence and optimality of competitive equilibria. Berlin Heidelberg New York: Springer 1989

Berge, C.: Topological spaces. New York: MacMillan 1963

Bewley, T.: Existence of equilibria in economies with infinitely many commodities. Journal of Economic Theory **43**, 514–540 (1972)

Chiarella, C.: Trade between resource-poor and resource-rich economies as a differential game. In: Kemp, M., Long, N. (eds.) Exhaustible resources, optimality and trade, pp. 219–246. Amsterdam: North-Holland 1980

Dasgupta, P., Heal, G.: The optimal depletion of exhaustible resources. Review of Economic Studies, Symposium, pp. 3–28 (1974)

Elbers, C., Withagen, C.: Trading in exhaustible resources in the presence of conversion costs: a general equilibrium approach. Journal of Economic Dynamics and Control **8**, 197–209 (1984)

Van Geldrop, J., S. Jilin, C. Withagen: Existence of general equilibria in economies with natural exhaustible resources and an infinite horizon. Journal of Mathematical Economics **20**, 225–248 (1991)

Van Geldrop, J., Withagen, C.: General equilibrium in an economy with exhaustible resources and an unbounded horizon. Journal of Economic Dynamics and Control **18**, 1011–1035 (1994)

Luenberger, D.: Optimization by vector space methods. New York: Wiley 1969

Mas-Colell, A.: The price equilibrium existence problem in topological vector lattices. Econometrica **54**, 1039–1054 (1986)

Mas-Colell, A., Zame, W.: Equilibrium theory in infinite dimensional spaces. In: Hildenbrand, W., Sonnenschein, H. (eds.) Handbook of Mathematical Economics, vol. IV, pp. 1835–1898. Amsterdam: North-Holland 1991

Mitra, T.: Efficient growth with exhaustible resources in a neoclassical model. Journal of Economic Theory **17**, 114–129 (1991)

Mitra, T.: On optimal depletion of exhaustible resources: existence and characterization results. Econometrica **48**, 1431–1450 (1980)

Stiglitz, J.: Growth with exhaustible natural resources: efficient and optimal growth paths. Review of Economic Studies, Symposium, pp. 123–137 (1974)

Toman, M.: Existence and optimality of dynamic competitive equilibria with a non-renewable resource. Resources and Energy **9**, 1–19 (1986)

Yannelis, N., Zame, W.: Equilibria in Banach lattices without ordered preferences. Journal of Mathematical Economics **15**, 75–110 (1986)

Zame, W.: Competitive equilibrium in production economies with an infinite-dimensional commodity space. Econometrica **55**, 1075–1108 (1987)