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UNDECIDABLE RELATIVIZATIONS OF ALGEBRAS OF RELATIONS

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Abstract. In this paper we show that relativized versions of relation set algebras and cylindric set algebras have undecidable equational theories if we include coordinatewise versions of the counting operations into the similarity type. We apply these results to the guarded fragment of first-order logic.

§1. Introduction.

Relativized algebras of relations are extensively investigated in the literature, cf., e.g., [HMT, HMTAN, Ma82, Mo93, Né91]. In general, relativized versions of algebras of relations have a nicer behavior from the computational point of view than the original versions.

In this paper, we concentrate on (un)decidability. We show that if we include coordinatewise versions of the counting operations into the similarity type, then the expressive power is strong enough to interpret the tiling problem into the equational theories of relativized relation set algebras and cylindric-relativized set algebras of dimension (at least) three. Thus these equational theories must be undecidable.

Finally, in the last section, we apply these results to logic: the corresponding versions of the guarded fragment of first-order logic and of arrow logic are undecidable.

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1.1. Relativization.

Relativization of an algebra amounts to intersecting all its elements with a fixed set (usually an element of the algebra or a subset of the unit) and to defining the operations using this set as the unit of the new algebra.

It turned out that if we relativize (set) algebras of relations with arbitrary, symmetric and/or reflexive elements, then we get a class of algebras with nice algebraic properties. For instance, while relation (set) algebras and cylindric (set) algebras of dimension at least three have undecidable equational theories, the sets of equations valid in the above relativizations are decidable.

Traditionally, during relativization we keep the original similarity type — in the case of relation algebras: Booleans, composition, converse, identity. As a consequence, some operations that are definable in the original version are not available after relativization. An example is the global counting operations *once*

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and *twice* and their coordinatewise versions. The question of which operations of the clone of the original algebras can be included into the similarity type of the relativized versions such that the nice properties of the relativized algebras are preserved naturally arises.

Elsewhere, cf. [MMN], we showed that (some of) the nice properties are preserved even if we consider a similarity type including the global counting operations besides the usual operations. In particular, if we add the global counting operations as basic operations to relativized relation set algebras [MMN, Mi95] and to cylindric-relativized set algebras [Mi98] then we get classes of algebras with decidable equational theories. See also [AHN] for a general characterization of operations that can be included without the loss of decidability.

However, there are more definable operations in a relation set algebra that become undefinable after relativization. An example is the coordinatewise version of the counting operation *twice*, expressing that there are two different pairs with the same vertical (or horizontal) coordinate.

It is well known that cylindric algebras correspond to first-order logic, cf. [HMT]. After relativizing cylindric set algebras we cannot express that a certain relation is a function. Thus it is a natural approach to define such versions of cylindric-relativized set algebras that are able to express functionality. For instance, we may include the coordinatewise version of the operation *at most one*.

In this paper, we show that including (one of) the vertical and horizontal counting operations in the similarity type yields relativized relation set algebras with undecidable equational theory. The undecidability result for relativized relation set algebras will follow from (the proof of) a similar undecidability result for relativizations of algebras of relations with higher arity: cylindric-relativized set algebras of dimension (at least) three with a coordinatewise version of *at most one* have undecidable equational theory.

The idea of the undecidability proof is to interpret the tiling problem into the equational theories of our algebras. While interpreting the tiling problem into relation algebras is very intuitive, the interpretation into cylindric algebras is more involved. That is why we sketch the proof for relation algebras and work out the details for the cylindric case (from this the relation algebra case easily follows).

1.2. Tiling. We will interpret the undecidable tiling problem into the equational theories of algebras of relations. Recently [Ma97] showed how to interpret the tiling problem into the theory of some weakened, axiomatically defined versions of relation algebras. It turned out that his idea can be used in the representable case as well — we will use a "semantical" version of that argument. Another interesting application of the tiling problem for relation algebras is in [HH97]: representability is undecidable for finite relation algebras.

Let us recall what the tiling problem is. By a tile we mean a square with a color on each side. Tiling a grid amounts to covering the surface such that the colors of the adjacent tiles are matching (e.g., if a tile has color c on its right-hand side, then the tile on its right must have color c on its left-hand side). One version of the tiling problem is the following:

$$\text{Given a finite set } T \text{ of tiles, can } T \text{ tile } \omega \times \omega?$$

This problem is undecidable (in fact, co-r.e. complete), cf. [Ro71]. Now we give a more formal definition.

DEFINITION 1.1. Let C be a set (of colors). By a *tile* t we mean a four-tuple of elements of C: $t = (c_0, c_1, c_2, c_3) \in {}^4C$. Given a tile t, we will denote c_0, c_1, c_2, c_3 by $\mathrm{left}(t), \mathrm{right}(t), \mathrm{up}(t), \mathrm{down}(t)$, respectively.

Let T be a set of tiles. We say that T *tiles* $\omega \times \omega$ if there is a function $\tau : \omega \times \omega \longrightarrow T$ such that, for every $(n, m) \in \omega \times \omega$,

$$\begin{aligned} \mathrm{right}(\tau(n,m)) &= \mathrm{left}(\tau(n+1,m)) \\ \mathrm{up}(\tau(n,m)) &= \mathrm{down}(\tau(n,m+1)). \end{aligned}$$

§2. **Relativized relation algebras.** In this section we define an expansion of relativized relation set algebras with a coordinatewise counting operation. We sketch how to prove the undecidability of its equational theory by interpreting the tiling problem. A self-contained proof can be obtained by straightforward modification of the proof of the cylindric case, Theorem 3.3. While this section gives insight for interpreting the tiling problem, it is not necessary for understanding the latter sections.

First we recall the definition of a (relativized) relation set algebra.

DEFINITION 2.1. By a *relation set algebra*, an Rs, we mean an algebra $\mathscr{A} = (A, 0, 1, \cdot, -, ;, {}^\smile, 1')$ such that $A \subseteq \mathscr{P}(W)$ (the powerset of W) for some set $W = U \times U$, $0 = \emptyset$, $1 = W$, $\cdot$ is intersection, $-$ is complement w.r.t. W, $;$ is relation composition, ${}^\smile$ is relation converse, and $1'$ is the identity relation on W. More formally, for all elements $x, y \in A$,

$$\begin{aligned} x\,;y &= \{(u,v) \in W : (u,w) \in x \text{ and } (w,v) \in y \text{ for some } w\} \\ x^\smile &= \{(u,v) \in W : (v,u) \in x\} \\ 1' &= \{(u,v) \in W : u = v\}. \end{aligned}$$

We denote the class of relation set algebras by Rs. Given an $\mathscr{A} \in$ Rs, W and U as above we call W the *unit* of $\mathscr{A}$ and U the *base* of $\mathscr{A}$.

The class **Rl**Rs of *relativized relation set algebras* is defined by allowing any $W \subseteq U \times U$ as unit in the definition of Rs.

The varieties generated by Rs and **Rl**Rs are usually denoted by RRA and **SRl**RRA, respectively. It is easy to see that *relativizing* an $\mathscr{A}' \in$ Rs, i.e., intersecting every element in A' with some fixed element $W \in A'$, yields an $\mathscr{A} \in$ **Rl**Rs. Conversely, every $\mathscr{A} \in$ **Rl**Rs can be obtained from an $\mathscr{A}' \in$ Rs by relativization and taking subalgebras.

In relativized algebras, the behavior of the operators may be different than in the original version. For instance, composition is associative in Rs, while in **Rl**Rs this does not hold in general (because some pairs may be missing from the unit).

The equational theory of Rs is undecidable [TG87], but that of **Rl**Rs is decidable [Ma95]. This last fact stays true when we add all counting operations (see below) to **Rl**Rs [Mi95].

In Rs we can term define the following operations:

$$\begin{aligned} \mathsf{D}_0 x &= \{(u,v) \in W : (u',v) \in x \text{ for some } u' \neq u\} \\ \mathsf{D}_1 x &= \{(u,v) \in W : (u,v') \in x \text{ for some } v' \neq v\} \\ \mathsf{D} x &= \{(u,v) \in W : (u',v') \in x \text{ for some } (u',v') \neq (u,v)\}, \end{aligned}$$

by setting $\mathsf{D}_0 x = (-1');x$, $\mathsf{D}_1 x = x;(-1')$, and $\mathsf{D}x = (1;\mathsf{D}_1 x)+(\mathsf{D}_0 x;1)$. On the other hand, this does not hold for **RlRs**; none of D, D_1 and D_0 is definable in **RlRs**. For instance, let $A = \{\emptyset, \{(0,1)\}\}$, $B = \{\emptyset, \{(0,1),(2,1)\}\}$ and $\mathscr{A}, \mathscr{B}$ be in **RlRs** with universes A and B, respectively. Then $\mathscr{A}$ and $\mathscr{B}$ are isomorphic, but the intended meaning of $\mathsf{D}_0(1) = \mathsf{D}_0(\{(0,1)\})$ in $\mathscr{A}$ is $\emptyset$, but $\mathsf{D}_0(1) = \mathsf{D}_0(\{(0,1),(2,1)\}) = \{(0,1),(2,1)\}$ in $\mathscr{B}$.

These *difference* operators provide a limited ability to count: using them we can define the operators *at most once*, k^1, and the coordinatewise version *at most once in the ith coordinate*, k^1_i. In general these operations are defined as follows on an $\mathscr{A}$ with unit $W \subseteq {}^2U$: for any $x \in A$,

$$
\begin{aligned}
\mathsf{k}^n x &= \begin{cases} 1 & \text{if } |x| \le n \\ 0 & \text{otherwise} \end{cases}\\
\mathsf{k}^n_0 x &= \{(u,v) \in W : \text{there exist at most } n \text{ distinct } w \in U \text{ such that } (w,v) \in x\}\\
\mathsf{k}^n_1 x &= \{(u,v) \in W : \text{there exist at most } n \text{ distinct } w \in U \text{ such that } (u,w) \in x\}.
\end{aligned}
$$

We call the k^n $(n \in \omega)$ operators (*global*) *counting operations*, and the k^n_i $(n \in \omega, i < 2)$ *coordinatewise counting operations*. We note that, in an expanded **RlRs**, *at most one* is definable as $\mathsf{k}^1_i x = -\mathsf{D}_i(\mathsf{D}_i x \cdot x)$ (with or without the index i). As mentioned above, the expansion of **RlRs** with all global counting operations is decidable [Mi95], and has the finite base property [AHN], i.e., every non-valid equation fails in an algebra on a finite base. Here we show that adding only k^1_1 (*at most one vertically*) destroys these properties.

Definition 2.2. $\mathbf{RlRs}^+$ denotes the class of all **RlRs** algebras expanded with an operation k^1_1 as defined above.

Theorem 2.3. 1. $\mathbf{RlRs}^+$ *does not have the finite base property, i.e., there is a non-valid equation that is valid in every algebra with finite base.*
2. *The equational theory of* $\mathbf{RlRs}^+$ *is undecidable.*

Sketch of proof. We prove the weaker 1 because it provides all the ingredients for the proof of 2 in a simple manner. We then only sketch the proof of 2, because below we provide a very similar proof for the more difficult case of Crs_3^+, cf. Theorem 3.3, and a full proof for $\mathbf{RlRs}^+$ is available in [Ma97].

For 1 we propose the following infinity axiom ($\mathsf{c}^\partial_0 x$ abbreviates $-(1;-x)$):

$$
\begin{aligned}
&(t_a) \quad 1' \cdot -(1;f)\\
&(t_b) \quad \mathsf{c}^\partial_0[(f \cdot 1^\smile);1^\smile]\\
&(t_c) \quad \mathsf{c}^\partial_0 \mathsf{k}^1_1 f^\smile.
\end{aligned}
$$

We show that $t_a \cdot t_b \cdot t_c = 0$ is not valid, and that if $t_a \cdot t_b \cdot t_c \neq 0$ in a $\mathbf{RlRs}^+$ with unit W, then W must contain the graph of a non-total and surjective function.

Let $\mathscr{A}$ be the full $\mathbf{RlRs}^+$ with unit $\omega \times \omega$ and let f be interpreted as the successor function. Then $(0,0) \in t_a$ and $(f \cdot 1^\smile);1^\smile$ and $\mathsf{k}^1_1 f^\smile$ contain every pair $(k,0)$ $(k \in \omega)$, as is easy to see. In $\mathscr{A}$, the term $\mathsf{c}^\partial_0 R$ equals $\{(x,y) :$ for every $z \in \omega$, $(z,y) \in R\}$, whence $(0,0) \in t_a \cdot t_b \cdot t_c$.

To see that the term forces the base set to be infinite, let it be satisfied in a $\mathbf{RlRs}^+$ with unit $W \subseteq U \times U$, at a pair (u_0,v_0): $(u_0,v_0) \in t_a \cdot t_b \cdot t_c$. By $t_a \neq 0$, then $u_0 = v_0$. Let

$$K = \{u \in U : (u_0,u),(u,u_0) \in W\}.$$

We show that $f^{\smile}$ (restricted to K) is a non-total, surjective function from K to K. By $(u_0, u_0) \in t_a$, $f^{\smile}$ is not total on K. By $t_b \neq 0$, $f^{\smile}$ is surjective, because for every $u \in K$, $(u, u_0) \in (f \cdot 1^{\smile}) ; 1^{\smile}$. Finally, $f^{\smile}$ is a function on K, since $(u_0, u_0) \in t_c$ implies that, for every $u \in K$, $(u, u_0) \in \mathsf{k}^1_1 f^{\smile}$. Hence K must be infinite. Thus we have shown 1.

The proof gives us two of the three crucial ingredients for our undecidability proof. The first is that using the fact that $(u_0, u_0) \in W$ and, for all $u_i \in K$, $\{(u, u_0), (u_0, u)\} \subseteq W$, we can by c^{∂}_0 ensure "locally" that certain relations hold at each (u, u_0). The second is that using this ability we can say — with the help of k^1_1 — that there exist functions from K to K. Finally, as we will see in the proof for Crs^+_3, the crucial point of our undecidability proof is that we can express that we have two total *commuting* functions. This is easily expressed using composition by (assuming we have $t_b \neq 0$ and $t_c \neq 0$ for two variables u and r) $\mathsf{c}^{\partial}_0[(r ; u \cdot u ; r) ; 1]$.

Let $T = \{\tau_i : i \in I\}$ be a set of tiles. We define the following terms:

$$\begin{array}{ll} (s_0) & \mathsf{c}^{\partial}_0[(r ; u \cdot u ; r) ; 1] \\ (s_1) & \mathsf{c}^{\partial}_0(\mathsf{k}^1_1 r \cdot \mathsf{k}^1_1 u) \\ (t_0) & \mathsf{c}^{\partial}_0 \sum\{\tau_i : i \in I\} \cdot 1' \end{array}$$

and for every $\tau_i \in T$,

$$\begin{array}{ll} (t^i_1) & \mathsf{c}^{\partial}_0(-\tau_i + \prod\{-\tau_j : i \neq j \in I\}) \\ (t^i_2) & \mathsf{c}^{\partial}_0(-\tau_i + [r ; (\sum\{\tau_j : \mathrm{right}(\tau_i) = \mathrm{left}(\tau_j)\})]) \\ (t^i_3) & \mathsf{c}^{\partial}_0(-\tau_i + [u ; (\sum\{\tau_j : \mathrm{up}(\tau_i) = \mathrm{down}(\tau_j)\})]). \end{array}$$

Let s_T be the term $s_0 \cdot s_1 \cdot t_0 \cdot \prod\{t^i_j : i \in I, 1 \leq j \leq 3\}$. Now the proof can easily be finished in the same way as with Crs^+_3 in the proof of Theorem 3.3 by showing that $s_T \neq 0$ if and only if T can tile $\omega \times \omega$. ⊣

COROLLARY 2.4. *The equational and universal theories of* $\mathbf{R1Rs}^+$ *are r.e. complete.*

SKETCH OF PROOF. By Theorem 2.3, the complexity of these theories are at least r.e.

That these theories are r.e. can be proved by using pseudo-axiomatizations of these theories: one can define many-sorted structures and recursively axiomatize them in a suitable first-order language, whence there exist recursive enumerations of the valid equations and first-order formulas. See [HMT] 4.2.27–32 and [Né91] for how this method works in the case of Cs_α (defined below). ⊣

Let us finish this section with an open problem:

Is the variety generated by $\mathbf{R1Rs}^+$ finitely axiomatizable?

§3. Cylindric algebras. In this section we define strengthenings of cylindric-relativized set algebras by expanding the language with counting operations (both global and coordinatewise).

DEFINITION 3.1. 1. Let U be a set, α be an ordinal, and W be a non-empty subset of α-long sequences from U, i.e., let $W \subseteq {}^\alpha U$. We define

$$\mathscr{A} = (\mathscr{P}(W), 0, 1, \cdot, -, \mathsf{c}_i, \mathsf{d}_{ij})_{i,j<\alpha} \in \text{ full } \mathsf{Crs}_\alpha$$

if the following hold: $0 = \emptyset$, $1 = W$, $\cdot$ is intersection, $-$ is complement w.r.t. W, and for every $x \subseteq W$ and $i, j < \alpha$,

$$\begin{aligned} \mathsf{c}_i x &= \{a \in W : \text{for some } b \in x \text{ and for every } j \neq i,\ a(j) = b(j)\} \\ \mathsf{d}_{ij} &= \{a \in W : a(i) = a(j)\}. \end{aligned}$$

We define $\mathsf{Crs}_\alpha = \mathbf{S}$ full Crs_α, i.e., we take subalgebras of the elements of the class full Crs_α. We call Crs_α the class of *cylindric-relativized set algebras of dimension* α.

2. The class Cs_α of *cylindric set algebras of dimension* α is defined by requiring that the unit W be a Cartesian space on a base set U $(W = {}^\alpha U)$ in the definition of Crs_α.

We note that the class Crs_α is a variety, and the variety generated by Cs_α is usually denoted by RCA_α.

Let $\mathscr{A}$ be an element of Crs_α, and assume that $W \subseteq {}^\alpha U$ is the unit of $\mathscr{A}$. The counting operations c^n $(n \in \omega)$ are defined in the following way: for every $x \in A$ and $a \in W$,

$$a \in \mathsf{c}^n x \iff |x| \geq n$$

or, equivalently

$$\mathsf{c}^n x = \begin{cases} 1 & \text{if } |x| \geq n \\ 0 & \text{otherwise} \end{cases}$$

where $|x|$ denotes the cardinality of x. Their coordinatewise versions c_i^n $(i < \alpha, n \in \omega)$ are defined as

$$a \in \mathsf{c}_i^n x \iff \begin{array}{l} \text{there are at least } n \text{ different } b_1, \ldots, b_n \in x \\ \text{such that, for every } j \neq i \text{ and } 1 \leq k \leq n,\ a(j) = b_k(j). \end{array}$$

We note that c_i^1 coincides with c_i. Similarly, we will usually drop the upper index 1 from c^1.

The presence of c_i^2 enables us to express uniqueness in the ith coordinate: $\mathsf{c}_i^1!$. More precisely, let

$$\mathsf{k}_i^1 x = \{a \in W : \text{ there is at most one } b \in x \text{ such that } b(j) = a(j) \text{ for every } j \neq i\}$$

in a set algebra with unit W. Then we can define $\mathsf{k}_i^1 x$ as $-\mathsf{c}_i^2 x$ and $\mathsf{c}_i^1!x$ as $\mathsf{c}_i x \cdot \mathsf{k}_i^1 x$. Thus we can express that certain relations are in fact functions, and this will be crucial in interpreting the tiling problem. Note also that the coordinatewise difference operator is definable by $\mathsf{D}_1 x = (\mathsf{c}_1 x \cdot \mathsf{k}_1^1 x \cdot - x) + -\mathsf{k}_1^1 x$.

The following defined operations in cylindric-relativized set algebras will be useful. Let $i, j < \alpha$. We define the *dual cylindrification* $\mathsf{c}_i^\partial x$ as $-\mathsf{c}_i - x$, and the *substitution* as

$$\mathsf{s}_j^i x = \begin{cases} \mathsf{c}_i(\mathsf{d}_{ij} \cdot x) & \text{if } i \neq j \\ x & \text{otherwise.} \end{cases}$$

The situation is very similar to the relational case. The class Cs_α has undecidable equational theory whenever $\alpha > 2$, cf. [HMT] 4.2.18. Its relativized version Crs_α and its expansion with the global counting operations have decidable equational theories, cf. [Né86, Né95] and [Mi98]. On the other hand, expansions with the coordinatewise counting operations are undecidable whenever $\alpha > 2$, by Theorem 3.3 below. However, in dimension 2, these expansions are decidable, Theorem 3.2.

The main results of this paper are the following two theorems. Their proofs are in the subsequent subsections.

	Crs_α	Crs_α + global counting	Crs_α + coordinatewise counting	Cs_α	Cs_α + coordinatewise counting
$\alpha = 2$	+ [Né86]	+ [Mi98]	+ Theorem 3.2	+ [HMT]	+ [GOR, Ma97]
$\alpha > 2$	+ [Né86]	+ [Mi98]	– Theorem 3.3	– [HMT]	– [HMT]

TABLE 1. (Un)decidability of the equational theories of classes of cylindric algebras.

THEOREM 3.2. *The equational theory of the expansion of* Crs_2 *with the global and coordinatewise counting operations* $\mathsf{c}^n, \mathsf{c}_i^n$ $(n \in \omega, i < 2)$

1. *is decidable, but*
2. *does not have the finite base property, i.e., there is a non-valid equation which is valid in every algebra with finite base.*

THEOREM 3.3. *Let* $\alpha \geq 3$. *The equational theory of the expansion of* Crs_α *with the coordinatewise counting operations* c_i^n $(n \in \omega, i < \alpha)$ *is undecidable. In fact, the expansion* Crs_3^+ *of* Crs_3 *with a single operation* k_1^1 *has an undecidable equational theory.*

We summarized these results in Table 1. We note that, similarly to the relation algebra case, the undecidable equational theories are in fact r.e. complete.

3.1. Dimension 2. Next we prove that expanding the language with counting operations in dimension 2 does not ruin decidability, though it enables us to express infinity.

PROOF OF THEOREM 3.2. 1: We will show how to decide the validity of an equation in expanded relativized algebras given a decision algorithm for validity of the same expansion of Cs_2 (provided by [GOR] or [Ma97]). First let us note that an equation $\rho = \sigma$ is valid iff $\mathsf{c}(\rho \oplus \sigma) = 1$ is not satisfiable (where $\oplus$ is symmetric difference). Then it suffices to define a recursive translation t such that an equation e is satisfiable in an expanded relativized algebra iff $t(e)$ is satisfiable in an expanded Cs_2. Let z be a new variable. We define a translation of terms as follows: for variable x and terms σ, τ,

$$\begin{aligned}
t(x) &= x \cdot z \\
t(1) &= 1 \cdot z \\
t(\mathsf{d}_{ij}) &= \mathsf{d}_{ij} \cdot z \\
t(\sigma \cdot \tau) &= t(\sigma) \cdot t(\tau) \cdot z \\
t(-\sigma) &= -t(\sigma) \cdot z \\
t(\mathsf{c}_i \sigma) &= \mathsf{c}_i t(\sigma) \cdot z \\
t(\mathsf{c}_i^n \sigma) &= \mathsf{c}_i^n t(\sigma) \cdot z.
\end{aligned}$$

Let e be $\sigma = \tau$. We define $t(e)$ as $t(\sigma) = t(\tau)$.

Let us assume that e is satisfied in a relativized $\mathscr{A}$ with base U and unit W. Let $\mathscr{B}$ be the full algebra with the Cartesian square unit $U \times U$. We evaluate the variables of e in $\mathscr{B}$ as in $\mathscr{A}$, and let z have the value W. Then an easy induction shows that the value of a term σ in $\mathscr{A}$ coincides with that of $t(\sigma)$ in $\mathscr{B}$. From this follows that $t(e)$ is satisfied in $\mathscr{B}$.

Now assume that $t(e)$ is satisfied in an expanded Cs_2 $\mathscr{A}$ under a certain evaluation. Let W be the value of z, and let $\mathscr{B}$ be the relativization of $\mathscr{A}$ by W, i.e., $B = \{a \cap W : a \in A\}$. Then the above argument shows that $\mathscr{B}$ satisfies e.

2: Let e be an equation of the form $\rho = \sigma$. Then e is satisfiable iff $\mathsf{c}(\rho \oplus \sigma) = 1$ is not valid. Thus it suffices to show that there is a satisfiable equation e such that satisfiability of e implies the infinity of the base.

Let us define the following terms:

$$\begin{aligned} \rho_0 &= \mathsf{c}(\mathsf{d}_{01} \cdot -\mathsf{c}_1^1 f) \\ \rho_1 &= -\mathsf{c}\mathsf{c}_1^2 f \\ \rho_2 &= -\mathsf{c} - (\mathsf{c}_0(f \cdot \mathsf{c}_1 \mathsf{d}_{01})). \end{aligned}$$

The equation e is defined as $\rho_0 \cdot \rho_1 \cdot \rho_2 = 1$.

Let $\mathscr{A}$ be the full algebra with unit $\omega \times \omega$ and

$$f = \{(n+1, n) : n \in \omega\}.$$

It is easy to check that e holds in $\mathscr{A}$.

Let $\mathscr{A}$ be with unit W such that e holds in $\mathscr{A}$. Let $K = \{x : (x, x) \in W\}$. By ρ_1, for every $x \in K$, there is at most one $y \in K$ such that $(x, y) \in f$; let $F(x) = y$ if such a y exists. Then F is a partial function $F : K \longrightarrow K$. By ρ_0, F is not total. By ρ_2, F is onto. It follows that K is infinite. ⊣

Remark 3.4. We note that the above idea can be used to prove that the *existential* theory of Crs_2 does not have the finite base property. That is, there is a non-valid existential sentence that is valid in every finite Crs_2, cf. [Mi97] for details. Note that, on the other hand, Crs_α (α finite) has the finite base property for *universal* sentences, cf. [AHN].

3.2. Dimensions higher than 2. Now we turn to proving Theorem 3.3.

Let $T = \{\tau_i : i \in I\}$ be a given finite set of tiles. Let us recall that for a given tile $\tau_i \in T$ we denote its colors by left(τ_i), right(τ_i), up(τ_i), and down(τ_i). For every $\tau_i \in T$, let τ_i be a variable, and let r and u be variables as well.

The idea of the undecidability proof is that we can "code up" the tiling of $\omega \times \omega$ into an equation. We will map $\omega \times \omega$ onto ω and evaluate the τ_i's on sequences of the form $(n, 0, 0, \dots)$ ($n \in \omega$). Using two variables r and u we will define right- and up-successors. These will be commuting total functions, and we will make sure that the evaluation of the τ_i's is in correspondence with these successor functions so that adjacent colors match.

Proof of Theorem 3.3. We prove the theorem for the expansion of Crs_3 by k_1^1 (that is definable by c_1^2). Let us denote this expansion by Crs_3^+. Obvious modifications in the proof yield the same result for higher dimensions and larger similarity types.

We define (a kind of) composition of two elements x and y as

$$x\,;y = \mathsf{c}_2(\mathsf{c}_0(\mathsf{d}_{02} \cdot \mathsf{c}_2 y) \cdot \mathsf{c}_1(\mathsf{d}_{12} \cdot \mathsf{c}_2 x)) = \mathsf{c}_2(\mathsf{s}_2^0 \mathsf{c}_2 y \cdot \mathsf{s}_2^1 \mathsf{c}_2 x),$$

cf. [HMT] 5.3.7. The idea of the above definition is to consider x and y as binary relations, and using an extra coordinate, to express their composition, cf. Remark 3.7 after the proof.

Consider the set of terms below. Their intuitive meaning is as follows. For a given sequence $(n, 0, 0)$, s_0 guarantees that there are m, l such that m is the right-successor of n, l is the up-successor of n and the up-successor of m and the right-successor of l coincide. The uniqueness of the up- and right-successors is guaranteed by s_1. The role of t_0 and t_1^i is to evaluate the τ_i's on the diagonal d_{12} in a disjoint way. Finally, t_2^i and t_3^i ensure that the successor functions and the evaluation of the tiles make the colors of adjacent tiles match. We define

$$\begin{array}{ll} (s_0) & \mathsf{c}_0^\partial[([(r \cdot \mathsf{d}_{02}) \,;\, (u \cdot \mathsf{d}_{02})] \cdot [(u \cdot \mathsf{d}_{02}) \,;\, (r \cdot \mathsf{d}_{02})]) \,;\, \mathsf{d}_{12}] \\ (s_1) & \mathsf{c}_0^\partial \mathsf{s}_0^1 \mathsf{s}_0^2(\mathsf{k}_1^1 r \cdot \mathsf{k}_1^1 u) \\ (t_0) & \mathsf{c}_0^\partial(\sum\{\tau_i : i \in I\}) \cdot \mathsf{d}_{01} \cdot \mathsf{d}_{12} \end{array}$$

and for every $\tau_i \in T$,

$$\begin{array}{ll} (t_1^i) & \mathsf{c}_0^\partial(-\tau_i + \prod\{-\tau_j : i \neq j \in I\}) \\ (t_2^i) & \mathsf{c}_0^\partial(-\tau_i + [(r \cdot \mathsf{d}_{02}) \,;\, (\sum\{\tau_j : \mathrm{right}(\tau_i) = \mathrm{left}(\tau_j)\} \cdot \mathsf{d}_{12})]) \\ (t_3^i) & \mathsf{c}_0^\partial(-\tau_i + [(u \cdot \mathsf{d}_{02}) \,;\, (\sum\{\tau_j : \mathrm{up}(\tau_i) = \mathrm{down}(\tau_j)\} \cdot \mathsf{d}_{12})]). \end{array}$$

Let s_T be the term $s_0 \cdot s_1 \cdot t_0 \cdot \prod\{t_j^i : i \in I, 1 \leq j \leq 3\}$. We will show that the equation $s_T = 0$ is valid in Crs_3^+ if and only if T cannot tile $\omega \times \omega$, yielding the undecidability of the validity problem of equations. Note that the equation $s_T = 0$ is not valid in Crs_3^+ iff there is an $\mathscr{A} \in \mathsf{Crs}_3^+$ and a sequence (a, b, c) in the unit of $\mathscr{A}$ which is in the value $(s_T)^{\mathscr{A}}$ of the term s_T in $\mathscr{A}$. (If no confusion is likely we will omit the superscript $\mathscr{A}$.)

First assume that T tiles $\omega \times \omega$. We have to show that s_T is satisfiable in an $\mathscr{A} \in \mathsf{Crs}_3^+$, i.e., that there is a sequence (x, y, z) in the unit W of $\mathscr{A}$ which is in the value of the term s_T in $\mathscr{A}$. Let $\mathscr{A}$ be the full Cs_3 with unit $W = {}^3\omega$ expanded with k_1^1. Let $f : \omega \times \omega \longrightarrow \omega$ be a bijection such that $f(0, 0) = 0$. Let every $\tau_i \in T$ be evaluated according to the given tiling of $\omega \times \omega$: $(x, 0, 0) \in \tau_i$ iff there are $n, m \in \omega$ such that $f(n, m) = x$ and τ_i tiles (n, m). We evaluate u and r as follows. For every $(x, y, z) \in W$, we let $(x, y, z) \in r$ iff $x = z$ and there are $n, m \in \omega$ such that $f(n, m) = x$ and $f(n + 1, m) = y$. Since every (n, m) has a unique right-successor $(n + 1, m)$, for a given $(x, 0, 0)$ there is a unique y such that $(x, y, x) \in r$. We define u similarly. We claim that $(0, 0, 0) \in s_T$.

The term t_0 is satisfied at $(0, 0, 0)$, since $(0, 0, 0) \in \mathsf{d}_{01} \cdot \mathsf{d}_{12}$ and, by the surjectivity of f, for every $x \in \omega$, $(x, 0, 0) \in \tau_i$ for some $i \in I$.

We show that $(0, 0, 0) \in s_0$. Indeed, let $x \in \omega$ be arbitrary, and assume that $x = f(n, m)$ for some $n, m \in \omega$. Further, let $f(n+1, m) = y$, $f(n, m+1) = z$ and $f(n + 1, m + 1) = v$. Then $(x, y, x) \in r \cdot \mathsf{d}_{02}$, $(y, v, y) \in u \cdot \mathsf{d}_{02}$, $(x, z, x) \in u \cdot \mathsf{d}_{02}$ and $(z, v, z) \in r \cdot \mathsf{d}_{02}$ by the definition of r and u. By unfolding the definition of composition, we get that $(x, v, 0) \in [(r \cdot \mathsf{d}_{02}) \,;\, (u \cdot \mathsf{d}_{02})] \cdot [(u \cdot \mathsf{d}_{02}) \,;\, (r \cdot \mathsf{d}_{02})]$. Since $(v, 0, 0) \in \mathsf{d}_{12}$, we have $(x, 0, 0) \in ([(r \cdot \mathsf{d}_{02}) \,;\, (u \cdot \mathsf{d}_{02})] \cdot [(u \cdot \mathsf{d}_{02}) \,;\, (r \cdot \mathsf{d}_{02})]) \,;\, \mathsf{d}_{12}$.

Next we check s_1. Again let x, y, z be as in the previous paragraph. By the injectivity of f, y is the unique element of ω such that $(x, y, x) \in r$. Thus $(x, x, x) \in \mathsf{k}_1^1 r$, and similarly $(x, x, x) \in \mathsf{k}_1^1 u$. Hence $(x, 0, 0) \in \mathsf{s}_0^1 \mathsf{s}_0^2(\mathsf{k}_1^1 r \cdot \mathsf{k}_1^1 u)$.

Since in the given tiling of $\omega \times \omega$ every $(n, m) \in {}^2\omega$ is covered by a unique tile, t_1^i holds at $(0, 0, 0)$ for every $i \in I$. So far we have seen that, for every $x \in \omega$, $(x, 0, 0) \in \tau_i$ for a unique $i \in I$.

Finally, we check t_2^i — the proof of t_3^i is completely analogous. Let $(x, 0, 0) \in \tau_i$, $x = f(n, m)$, $y = f(n+1, m)$ and $(y, 0, 0) \in \tau_j$. Then right$(\tau_i) =$ left(τ_j), by the evaluation of τ_i and τ_j. Since $(x, y, x) \in r$, we get that $(x, 0, 0) \in (r \cdot \mathsf{d}_{02}) ; (\tau_j \cdot \mathsf{d}_{12})$.

To prove the other direction, let us assume that s_T is satisfied in an $\mathscr{A} \in \mathsf{Crs}_3^+$. We will show that T can tile $\omega \times \omega$.

Let W be the unit and U be the base of $\mathscr{A}$, and let $(k, l, m) \in W$ be in the value of s_T. Let us fix such a k and denote it by 0. By t_0, $(0, l, m) \in \mathsf{d}_{01} \cdot \mathsf{d}_{12}$, i.e., $k = l = m = 0$. We define

$$K = \{x \in U : (x, 0, 0) \in W\}.$$

Let $x \in K$ be arbitrary. Then $(x, 0, 0) \in \tau_i$ for some $i \in I$, by t_0. By t_2^i, $(x, 0, 0) \in (r \cdot \mathsf{d}_{02}) ; (\sum\{\tau_j : \text{right}(\tau_i) = \text{left}(\tau_j)\} \cdot \mathsf{d}_{12})$. Unfolding the definition of composition, we get that there is a $y \in U$ such that $(y, 0, 0) \in \sum\{\tau_j : \text{right}(\tau_i) = \text{left}(\tau_j)\}$ and $(x, y, x) \in r$. On the other hand, by s_1, there is at most one y such that $(x, y, x) \in r$. Thus we can define a function $\mathsf{Right} : K \longrightarrow K$ by letting $\mathsf{Right}(x)$ be the unique y for which $(x, y, x) \in r$ and $(y, 0, 0) \in W$. Similarly we define Up: $\mathsf{Up}(x) = z$ iff z is the unique element of K such that $(x, z, x) \in u$. We are ready to formulate the following lemma.

LEMMA 3.5. Right *and* Up *are commuting functions on* K, *i.e.*, $\mathsf{Right}, \mathsf{Up} : K \longrightarrow K$ *and, for every* $x \in K$, $\mathsf{Right}(\mathsf{Up}(x)) = \mathsf{Up}(\mathsf{Right}(x))$.

PROOF. Let $x \in K$ be arbitrary. By s_0,

$$(x, 0, 0) \in ([(r \cdot \mathsf{d}_{02}) ; (u \cdot \mathsf{d}_{02})] \cdot [(u \cdot \mathsf{d}_{02}) ; (r \cdot \mathsf{d}_{02})]) ; \mathsf{d}_{12}.$$

Then we have $y, z, v \in U$ such that $(x, y, x) \in r$, $(y, v, y) \in u$, $(x, z, x) \in u$ and $(z, v, z) \in r$. The argument we used above the lemma ensures that $y, z, v \in K$, i.e., $y = \mathsf{Right}(x)$, $v = \mathsf{Up}(y)$, $z = \mathsf{Up}(x)$ and $v = \mathsf{Right}(z)$. Hence $\mathsf{Up}(\mathsf{Right}(x)) = \mathsf{Right}(\mathsf{Up}(x))$. ⊣

We define a tiling τ of $\omega \times \omega$ as follows: for every $(n, m) \in \omega \times \omega$ and $i \in I$, let

$$\tau(n, m) = \tau_i \iff (\mathsf{Right}^n \mathsf{Up}^m(0), 0, 0) \in \tau_i,$$

where $\mathsf{Right}^0(0) = \mathsf{Up}^0(0) = 0$, $\mathsf{Right}^{k+1}\mathsf{Up}^l(0) = \mathsf{Right}(\mathsf{Right}^k\mathsf{Up}^l(0))$ and $\mathsf{Right}^k\mathsf{Up}^{l+1}(0) = \mathsf{Right}^k\mathsf{Up}^l(\mathsf{Up}(0))$ for every $k, l \in \omega$.

LEMMA 3.6. *The function* τ *defined above is a tiling of* $\omega \times \omega$.

PROOF. Let $(n, m) \in \omega \times \omega$ be arbitrary and $x = \mathsf{Right}^n\mathsf{Up}^m(0)$. Since Right and Up are functions with range K, there exists, by t_0, a $\tau_i \in T$ such that $(x, 0, 0) \in \tau_i$, and by t_1^i such a τ_i must be unique. Hence, τ is indeed a function with domain $\omega \times \omega$ and with range T.

It remains to check that adjacent tiles have matching colors. Let $(n, m) \in \omega \times \omega$ be arbitrary. Let $\tau(n, m) = \tau_i$ and $x = \mathsf{Right}^n\mathsf{Up}^m(0)$. Then $(x, 0, 0) \in \tau_i$.

Let $\tau(n+1, m) = \tau_j$, i.e., $(\mathsf{Right}(x), 0, 0) \in \tau_j$. Recall that $\mathsf{Right}(x)$ is the unique element of U such that $(x, \mathsf{Right}(x), x) \in r$. Then by t_2^i, right$(\tau_i) =$ left(τ_j).

To prove that up$(\tau(n, m)) =$ down$(\tau(n, m+1))$ one needs the following:

$$\mathsf{Up}(\mathsf{Right}^n\mathsf{Up}^m(0)) = \mathsf{Right}^n\mathsf{Up}^{m+1}(0).$$

This can be proved by an easy induction using Lemma 3.5. Then, using t_3^i instead of t_2^i, the above argument gives the desired result. ⊣

This finishes the proof of Theorem 3.3. ⊣

REMARK 3.7. From the above proof we can see that any subclass of Crs_α^+ which contains an expanded full Cs_α with a countable base set has an undecidable equational theory. Similar remark applies to the $\mathbf{RlRs}^+$ case.

We note that the above definition of composition works properly (i.e., according to the intuition) only in algebras with Cartesian space units. More precisely, if we consider the relation-algebraic reduct of a cylindric set algebra of dimension three, it turns out to be a relation set algebra, cf. [HMT] 5.3.16. On the other hand, there is a Crs_3 such that its relation-algebraic reduct is not in the class $\mathbf{RlRs}$ (for instance, consider a Crs_3 with a unit $W = \{(a,c,c),(a,b,b),(b,c,c)\}$). That is why we could not prove the above theorem by reducing it to the relation algebra case.

§4. **Logical applications.** In [MMN] and [Mi95] we raised the question of how to find computationally well-behaved versions of well-investigated logics. The following strategy proved to be fruitful: (1) weakening the logic by widening the class of models such that this version of the logic has nice properties and (2) strengthening the weakened version by (re-)introducing connectives without losing the nice properties.

For instance, we may consider *relativized versions of first-order logic*, where we restrict the set of available evaluations of the variables to an arbitrary non-empty subset of all possible valuations. This logic corresponds to cylindric-relativized set algebras, Crs, and is decidable [Né95] even if we expand the signature by the graded modalities [Mi98]. Applying the results of the previous sections shows a limit of the strategy described above: adding the counting quantifiers (the coordinatewise versions of the graded modalities) yields undecidable relativized logics.

There is also another route towards decidability in first-order logic and that is to consider only certain syntactic fragments, but keep the standard semantics. As van Benthem observed [vB96] the distinction between these two routes is relative. It is easy to translate the formulas from the weakened first-order logics to formulas inside the decidable, so called *Guarded Fragment*. What Theorem 3.3 then implies is that any expansion of the Guarded Fragment with some operation sufficient to express that a relation behaves as a partial function must be undecidable. We will now briefly review the connection between (relativized) cylindric set algebras and first-order logic, and show how to obtain the mentioned result on the Guarded Fragment.

Let L_n^r denote the *restricted version of the n-variable fragment of first-order logic with equality*: the language does not contain function symbols or constants, all variables occurring in a formula are from the set $\{v_0, \ldots, v_{n-1}\}$, and the atomic formulas are of the form $R(v_0, v_1, \ldots, v_{n-1})$ or $v_i = v_j$. So we only have n-ary predicate symbols, and the variables always occur in the same order. The cylindric-algebraic terms and the L_n^r formulas are just syntactic variants, by the following (bijective) translation:

$$\begin{array}{rcl} x^t & = & X(v_0, v_1, \dots, v_{n-1}) \\ (-\tau)^t & = & \neg\tau^t \\ (\tau \cdot \sigma)^t & = & \tau^t \wedge \sigma^t \\ \mathsf{d}_{ij} & = & v_i = v_j \\ (\mathsf{c}_i\tau)^t & = & \exists v_i \tau^t, \end{array}$$

where X is a predicate symbol (different for different variables). In fact $(\cdot)^t$ is truth-preserving, namely

$$\begin{array}{rcl} \mathsf{Cs}_n \models \tau = \sigma & \iff & \models \sigma^t \leftrightarrow \tau^t \\ \models \varphi & \iff & \mathsf{Cs}_n \models (\varphi^t)^{-1} = 1 \end{array}$$

where $(.^t)^{-1}$ denotes the inverse of $(.)^t$. Let $(\cdot)^{t'}$ be the same as $(\cdot)^t$, except for

$$(\mathsf{c}_i\tau)^{t'} = \exists v_i(V(v_0, \dots, v_{n-1}) \wedge \tau^{t'}),$$

where V is a new fixed predicate symbol. Then (cf. [vB96] Corollary 9.14)

$$\mathsf{Crs}_n \models \sigma = \tau \iff \models V(v_0, \dots, v_{n-1}) \rightarrow (\sigma^{t'} \leftrightarrow \tau^{t'}).$$

The interesting thing about the range of $(\cdot)^{t'}$ is that every occurrence of a quantifier occurs relativized by $V(v_0, \dots, v_{n-1})$ (this predicate corresponds to the available evaluations in a relativized model). These formulas all belong to the *Guarded Fragment*, defined as follows. We expand (the n-variable fragment of) first-order logic (with equality but without function symbols or constants) with polyadic quantifiers $\exists\vec{v}$ ($\vec{v}$ a vector of variables). A formula of this language is called *guarded* if it is generated from atoms using the Booleans and "guarded quantification"

$$\exists\vec{v}(G\vec{v}\vec{x} \wedge \varphi(\vec{v}\vec{x})),$$

where G is a predicate symbol, and the variables occurring in $\vec{v}$ and $\vec{x}$ may occur in any order and with any multiplicity in both G and φ, though they are the only variables which occur there free. All these formulas together form the Guarded Fragment.

The Guarded Fragment is decidable [vB96], thus it is a decidable extension of Crs_n, by the effective translation $(\cdot)^{t'}$. Note that the translation even goes into the Guarded Fragment of restricted first-order logic L_n^r.

No truth preserving translation of the term $\mathsf{k}_1^1\tau$ can go to the Guarded Fragment, otherwise we could decide Crs_3^+, which we cannot by Theorem 3.3. Thus if we expand the Guarded Fragment with an operation $[\mathsf{func}]_1$, with meaning

$$\mathcal{M} \models [\mathsf{func}]_1\varphi\ [a] \text{ iff there exists at most one } d \in M \text{ such that } \mathcal{M} \models \varphi\ [a(a(1) \mapsto d)],$$

then that expansion is undecidable, because we could interpret Crs_3^+ in it, using the above $(\cdot)^{t'}$ which now translates $\mathsf{k}_1^1\tau$ to $[\mathsf{func}]_1\tau^{t'}$. This even holds if we restrict the application of $[\mathsf{func}]_1$ to predicate symbols only, because in the given encoding of the tiling problem we only applied k_1^1 to variables. Moreover, this also holds for the restriction of the Guarded Fragment to three variables, and not containing the polyadic quantifiers $\exists\vec{v}$, since these are not needed in the translation of Crs_3^+.

On the other hand, Crs_n expanded with all the global counting (or graded) modalities is decidable [Mi95], and even has the finite base property [AHN]. The translation of these operators does not arrive in the guarded fragment. It seems likely that the Guarded Fragment can be expanded in this direction as well, without loss of decidability.

We note that in two dimensions the situation is better: the expansion of L_2^r with all the counting quantifiers is decidable ([GOR] and [Ma97]), but does not have the finite base property, by Theorem 3.2.

Finally, we note that, in a similar fashion, the undecidability result for relation algebras yields undecidability of (relativized) arrow logic (cf. [MMP]) if we expand the signature with coordinatewise counting quantifiers.

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