Geometric interpretation of the Pancharatnam connection and non-cyclic polarization changes

Thomas van Dijk,[1] Hugo F. Schouten,[1] and Taco D. Visser[1,2,*]

[1]*Department of Physics and Astronomy, and Laser Centre, VU University, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands*
[2]*Department of Electrical Engineering, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands*
**Corresponding author: T.D.Visser@tudelft.nl*

Received April 12, 2010; revised June 15, 2010; accepted July 1, 2010; posted July 13, 2010 (Doc. ID 126904); published August 12, 2010

If the state of polarization of a monochromatic light beam is changed in a cyclical manner, the beam acquires—in addition to the usual dynamic phase—a *geometric phase*. This geometric or *Pancharatnam–Berry phase* equals half the solid angle of the contour traced out on the Poincaré sphere. We show that such a geometric interpretation also exists for the *Pancharatnam connection*, the criterion according to which two beams with different polarization states are said to be in phase. This interpretation offers what is to our knowledge a new and intuitive method to calculate the geometric phase that accompanies non-cyclic polarization changes.

OCIS codes: 350.1370, 260.5430, 260.6042.

In 1984 Berry pointed out that a quantum system whose parameters are cyclically altered does not return to its original state but acquires, in addition to the usual dynamic phase, a so-called geometric phase [1]. It was soon realized that such a phase is not just restricted to quantum systems, but also occurs in contexts such as Foucault's pendulum [2]. Also the polarization phenomena described by Pancharatnam [3] represent one of its manifestations. The polarization properties of a monochromatic light beam can be represented by a point on the Poincaré sphere [4]. When, with the help of optical elements such as polarizers and retarders, the state of polarization is made to trace out a closed contour on the sphere, the beam acquires a geometric phase. This *Pancharatnam–Berry phase*, as it is nowadays called, is equal to half the solid angle of the contour subtended at the origin of the sphere [5–10].

In this work we show that such a geometric relation also exists for the so-called *Pancharatnam connection*, the criterion according to which two beams with different polarization states are in phase, i.e., their superposition produces a maximal intensity. This relation can be extended to arbitrary (e.g., non-closed) paths on the Poincaré sphere and allows us to study how the phase builds up for such non-cyclic polarization changes. Our work offers a geometry-based alternative to the algebraic work presented in [11,12].

The state of polarization of a monochromatic beam can be represented as a two-dimensional Jones vector [13] with respect to an orthonormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$ as

$$\mathbf{E} = \cos\alpha\,\hat{\mathbf{e}}_1 + \sin\alpha\,\exp(\mathrm{i}\theta)\hat{\mathbf{e}}_2, \tag{1}$$

with $0 \le \alpha \le \pi/2$, $-\pi \le \theta \le \pi$, and $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$ $(i,j=1,2)$. The angle α is a measure of the relative amplitudes of the two components of the electric vector $\mathbf{E}$, and the angle θ denotes their phase difference. Two different states of polarization, A and B, can hence be written as

$$\mathbf{E}_A = (\cos\alpha_A, \sin\alpha_A e^{\mathrm{i}\theta_A})^T, \tag{2}$$

$$\mathbf{E}_B = e^{\mathrm{i}\gamma_{AB}}(\cos\alpha_B, \sin\alpha_B e^{\mathrm{i}\theta_B})^T. \tag{3}$$

Since only relative phase differences are of concern, the overall phase of $\mathbf{E}_A$ in Eq. (2) is taken to be zero. According to Pancharatnam's connection [5] these two states are in phase when their superposition yields a maximal intensity, i.e., when

$$|\mathbf{E}_A + \mathbf{E}_B|^2 = |\mathbf{E}_A|^2 + |\mathbf{E}_B|^2 + 2\,\mathrm{Re}(\mathbf{E}_A \cdot \mathbf{E}_B^*) \tag{4}$$

reaches its greatest value, implying that

$$\mathrm{Im}(\mathbf{E}_A \cdot \mathbf{E}_B^*) = 0, \tag{5}$$

$$\mathrm{Re}(\mathbf{E}_A \cdot \mathbf{E}_B^*) > 0. \tag{6}$$

These two conditions uniquely determine the phase γ_{AB}, except when the states A and B are orthogonal.

Let us now consider a sequence of three polarization states with each successive state being in phase with its predecessor. As the initial state we take the basis state X with Jones vector $\mathbf{E}_X = (1,0)^T$. It follows immediately that *any polarization state* A with Jones vector $\mathbf{E}_A$ as defined by *Eq. (2) is in phase with* X. Consider now a third state B. This state is in phase with A provided that the angle γ_{AB} in Eq. (3) satisfies relations (5) and (6). Clearly, B is not in phase with X, but rather with $e^{\mathrm{i}\gamma_{AB}}X$. Apparently the total geometric phase that is accrued by following the closed circuit XAB equals γ_{AB}. This observation allows us to make use of Pancharatnam's classic result which relates the accumulated geometric phase to the solid angle of the geodesic triangle XAB [3]. According to this result then, the angle (phase) γ_{AB} between the states A and B

for which they are in phase is given by half the solid angle Ω_{XAB} of the triangle *XAB* subtended at the center of the Poincaré sphere, i.e.,

$$\gamma_{AB} = \Omega_{XAB}/2. \tag{7}$$

The solid angle Ω_{XAB} is taken to be positive (negative) when the circuit *XAB* is traversed in a counterclockwise (clockwise) manner. Thus we have $-2\pi \leq \Omega_{XAB} \leq 2\pi$, and hence $-\pi \leq \gamma_{AB} \leq \pi$. Hence we arrive at the following geometric interpretation of Pancharatnam's connection: The phase γ_{AB} for which the superposition of two beams with polarization states *A* and *B* yields a maximum intensity equals half the solid angle subtended by their respective Stokes vectors and the Stokes vector corresponding to the basis state *X*. We emphasize that γ_{AB} is defined with respect to a certain basis. We return to this point later.

Several consequences follow from the geometric interpretation. First, consider a state *B* that lies on the great circle through the points *A* and *X*. As illustrated in Fig. 1, two cases can be distinguished. If *B* is not on the geodesic that connects –*A* and –*X*, then the curves *XA*, *AB*, and *BX* cancel each other [see Fig. 1(a)], i.e., $\gamma_{AB} = \Omega_{XAB}/2 = 0$. If *B* does lie on the geodesic connecting –*A* and –*X* [see Fig. 1(b)], then these three curves together constitute the entire great circle and hence $\gamma_{AB} = \Omega_{XAB}/2 = \pi$. Consequently, we arrive at

Corollary 1. *All polarization states that lie on the great circle that runs through A and X and which are not part of the geodesic curve that connects –A and –X are in phase with state A. All other states on the great circle are out of phase with state A.*

(We exclude the pathological case $A = \pm X$.)

The great circle through *A* and *X* divides the Poincaré sphere into two hemispheres. For all states *B* on one hemisphere, the path *XAB* runs clockwise. For *B* on the other hemisphere, the path *XAB* always runs counterclockwise. Thus we find

Corollary 2. *The great circle that runs through A and X divides the Poincaré sphere into two halves, one on which all states have a positive phase with respect to A, and one on which all states have a negative phase with respect to A.*

Thus far we have not specified the basis vectors in which the Jones vectors are expressed. The two most commonly used are the Cartesian representation and the helicity representation. The Stokes vectors corresponding to the basis state *X* are (1,0,0) and (0,0,1) in these two bases, respectively. Our results so far are valid for any choice of representation. For computational ease, however, we will from now on make use of the Cartesian basis.

Given two different polarization states *A* and *B*, we may inquire about the set $\{B'\}$ of all states which have the same phase difference γ_{AB} with respect to *A* as *B* has. We begin by noticing that the solid angle Ω_{ABC} subtended at the origin of the Poincaré sphere by three unit vectors **A**, **B**, and **C** satisfies the equation [14]

$$\tan\left(\frac{\Omega_{ABC}}{2}\right) = \frac{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})}{1+\mathbf{B}\cdot\mathbf{C}+\mathbf{A}\cdot\mathbf{C}+\mathbf{A}\cdot\mathbf{B}}. \tag{8}$$

On taking **A**, **B**, and **C** as the Stokes vectors corresponding to states *A*, *B*, and *X*, i.e., $\mathbf{C} = (1,0,0)$, Eqs. (7) and (8) yield

$$\tan\,\gamma_{AB} = \frac{A_yB_z - A_zB_y}{1 + B_x + A_x + A_xB_x + A_yB_y + A_zB_z}. \tag{9}$$

For γ_{AB} and **A** fixed, we thus find that the three components of **B** must satisfy the relation

$$c_xB_x + c_yB_y + c_zB_z + D = 0, \tag{10}$$

with the coefficients c_x, c_y, c_z, and D given by

$$c_x = \tan\,\gamma_{AB}(1 + A_x), \tag{11}$$

$$c_y = \tan\,\gamma_{AB}A_y + A_z, \tag{12}$$

$$c_z = \tan\,\gamma_{AB}A_z - A_y, \tag{13}$$

$$D = c_x. \tag{14}$$

The solutions of Eq. (10) form a plane. In addition, the vector **B** must be of unit length, ensuring that it lies on the Poincaré sphere. The intersection of the plane and the sphere is a circle that runs through *B*. Finding two other points on this circle defines it uniquely. It can be verified by substitution that the Stokes vectors –**A** and –**X** both satisfy Eq. (10). Hence, for all states on the circle that runs through *B*, –*A*, and –*X*, the phase γ_{AB} has the same value, mod π. Since the plane defined by Eq. (10) does, in general, not include the origin of the Poincaré sphere, this circle is not a great circle. This is illustrated in Fig. 2, where the circle through *B* is drawn as dashed. The dashed circle intersects the great circle through *A* and *X* at the points –*A* and –*X*. According to Corollary 2, γ_{AB} changes sign at these points. Since Eq. (9) defines the

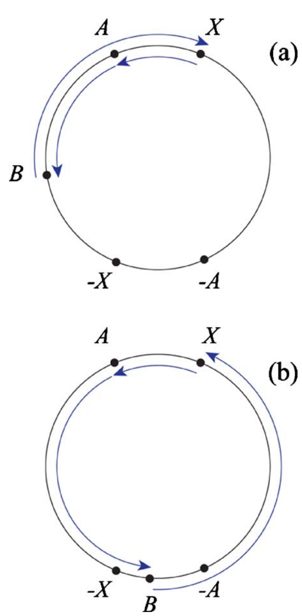

Fig. 1. (Color online) The great circle through *A*, *B*, and basis state *X*. (a) If state *B* does not lie on the segment between –*A* and –*X*, then the sum of the three geodesics *XA*, *AB*, and *BX* is zero. (b) If *B* lies on the segment between –*A* and –*X*, then the sum of the three geodesics equals the great circle.

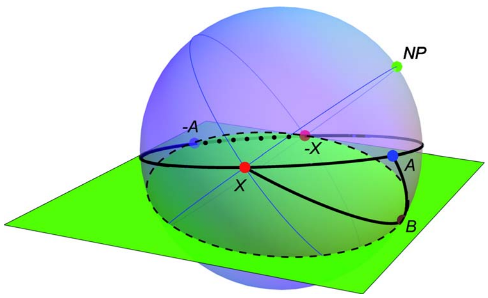

Fig. 2. (Color online) Illustration of the intersection of the plane given by Eq. (10) and the Poincaré sphere. This intersection is a circle (indicated by the dashed curve) that runs through the points $-A$, $-X$, and B. All points on the circle segment that runs from $-A$ to B to $-X$ constitute the set $\{B'\}$ of states that have the same phase difference γ_{AB} with respect to A as the state B. The great circle through A and X is shown as a solid-dotted curve.

phase modulo π, it follows that γ_{AB} undergoes a π phase jump at these points. We thus arrive at

Corollary 3. *Consider the circle through $-A$, $-X$, and B. It consists of two segments, both with end points $-A$ and $-X$. The segment which includes B equals the set $\{B'\}$ of states such that $\gamma_{AB'} = \gamma_{AB}$. The other segment represents states for which $\gamma_{AB'} = \gamma_{AB} \pm \pi$.*

It can be shown that the plane-sphere intersection is always a circle, and not just a single point, if the pathological case $A = \pm X$ is excluded. If, for a fixed state A, the state B is being varied, the plane given by Eq. (10) rotates along the line connecting $-A$ and $-X$.

We now demonstrate how our geometric interpretation implies that for a fixed state A the phase γ_{AB} may vary in different ways when the state B is moved across the Poincaré sphere. We specify the position of B by spherical coordinates (ϕ, θ), where $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$ represent the azimuthal angle and the angle of inclination, respectively. If A is taken to be at the south pole and $B = B(\phi)$ lies on the equator, then

$$\gamma_{AB} = \frac{\Omega_{XAB}}{2} = \frac{1}{2}\int_{\pi/2}^{\pi}\int_{0}^{\phi} \sin\,\theta \mathrm{d}\phi' \mathrm{d}\theta = \frac{1}{2}\phi. \qquad (15)$$

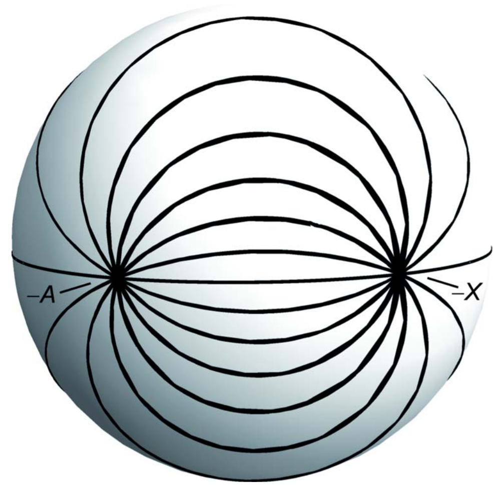

Fig. 4. (Color online) Contours of equal phase of γ_{AB} for the case that the state A is taken to be (0.6,0,0.8). Two singular points with opposite topological charges can be seen at $-A$ and $-X$.

Clearly, the phase varies linearly with the angle ϕ in this case.

Let us now consider the contours of equal phase γ_{AB} as shown in Fig. 3. It is seen that the intersections of the contours with the equator are not equidistant. Hence in this case the phase depends in a nonlinear way on the angle ϕ.

The singular behavior, finally, of the phase is a direct consequence of the fact that two anti-podal states A and $-A$ do not interfere with each other [see the remark below Eq. (6)]. From Eq. (8) it follows that the phase is antisymmetric under the interchange of the points $\mathbf{C} = \mathbf{X}$ and $\mathbf{A}$. Hence we expect two singular points, namely, $-A$ and $-X$, with opposite topological charges (± 1). This is illustrated in Figs. 4 and 5. We note that the existence of singular points is in agreement with the "hairy ball theorem" due to Brouwer [15], according to which there is no nonvanishing continuous tangent vector field on a sphere in $\mathbb{R}^3$. This implies that $\nabla \gamma_{AB}$ has at least one zero, in this case at the two singularities.

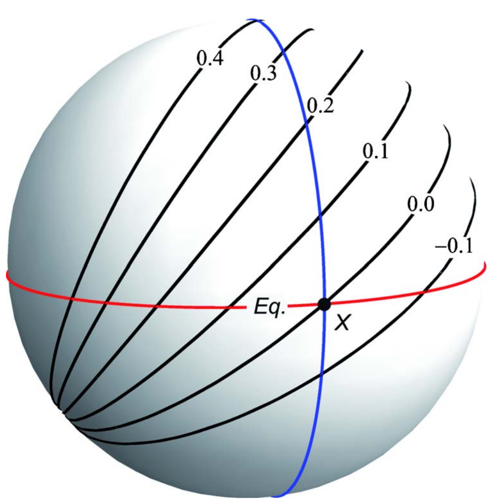

Fig. 3. (Color online) Selected contours of the phase γ_{AB} for the case $\mathbf{A} = (0, 0.8, 0.6)$. The basis state X, the equator (*Eq.*), and the meridian through X are also shown.

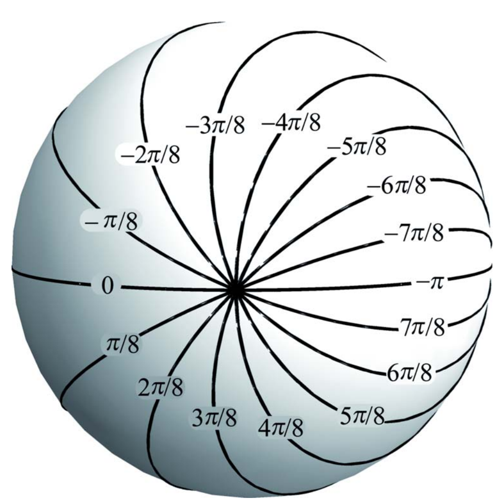

Fig. 5. (Color online) Contours of equal phase of γ_{AB} for the case that the state A is taken to be (0,0,1). The singularity at $-A$ is seen to have a topological charge of +1.

Let us now apply our results for the Pancharatnam connection to study the geometric phase for an arbitrary, i.e., non-closed, path ABC on the Poincaré sphere. The successive states are assumed to be in phase. Therefore the geometric phase accumulated on this path equals

$$\gamma_{ABC} \equiv \gamma_{AB} + \gamma_{BC} = (\Omega_{XAB} + \Omega_{XBC})/2 = \Omega_{XABC}/2, \quad (16)$$

where Ω_{XABC} is the *generalized solid angle* of the path $X \rightarrow A \rightarrow B \rightarrow C \rightarrow X$. Ω_{XABC} can consist of two triangles (see Fig. 6), whose contribution is positive or negative depending on their handedness.

Now we keep states A and C fixed and study how the geometric phase γ_{ABC} changes when state B is varied. We will show that this change, in contrast to γ_{AB}, is independent of the choice of basis vectors. Consider the phase γ'_{ABC} in a non-Cartesian basis (for example, the helicity basis) whose first basis state we call N. We then have, in analogy to Eq. (16),

$$\gamma'_{ABC} \equiv \gamma'_{AB} + \gamma'_{BC} = (\Omega_{NAB} + \Omega_{NBC})/2 = \Omega_{NABC}/2. \quad (17)$$

Also,

$$\Omega_{NABC} - \Omega_{XABC} = \Omega_{NABC} + \Omega_{CBAX} = \Omega_{NAXC}. \quad (18)$$

The justification of the last step of Eq. (18) is illustrated in Fig. 7. It follows on using Eqs. (16)–(18) that

$$\gamma'_{ABC} - \gamma_{ABC} = \Omega_{NAXC}/2. \quad (19)$$

The term $\Omega_{NAXC}/2$ is a constant, independent of B, i.e., the geometric phase in both representations differs by a constant only. Hence the variation of the geometric phase with B is independent of the choice of the basis, as it should be for an observable quantity. This is in contrast to γ_{AB}, which explicitly depends on the choice of basis, as is evident from Eqs. (2) and (3).

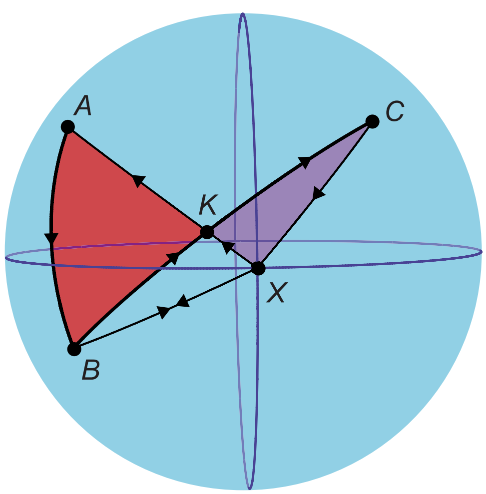

Fig. 6. (Color online) Illustration of the generalized solid angle Ω_{XABC}. In going from state A to state B, the beam acquires a geometric phase equal to half the solid angle Ω_{XAB}, which is positive. In going from B to C the acquired phase equals half the solid angle Ω_{XBC}, which is negative. Since the triangle BKX does not contribute, this is equivalent to the generalized solid angle Ω_{XABC}, which equals half the solid angle of the triangle ABK (positive), plus half the solid angle of the triangle XKC (negative).

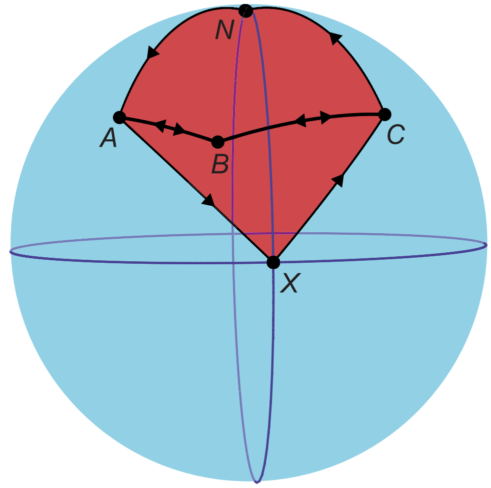

Fig. 7. (Color online) Illustration of the equality $\Omega_{NABC} + \Omega_{CBAX} = \Omega_{NAXC}$. Such a construction can be made for any choice of states.

The behavior of γ_{ABC} on varying B can be linear [16], nonlinear [17], or singular [18–20], as we have also shown for γ_{AB}. However γ_{AB} has singularities at $B=-A$ and $B=-X$. The first is due to the orthogonality of A and $-A$, while the second is a consequence of the choice of representation. The phase γ_{ABC} is singular only at $B=-A$ and $B=-C$, and not at $B=-X$.

In conclusion, we have shown how the Pancharatnam connection may be interpreted geometrically. Our work offers a geometry-based approach to calculate the Pancharatnam–Berry phase associated with non-cyclic polarization changes. As such it is an alternative to the algebraic treatments presented in [11,12]. Our approach can be extended to the description of geometric phases in quantum mechanical systems.

ACKNOWLEDGMENTS

The authors wish to thank Laura de Graaff for technical assistance. This research is supported by NWO (Netherlands Organization for Scientific Research) and FOM (Foundation for Fundamental Research on Matter).

REFERENCES

1. M. V. Berry, "Quantal phase factors accompanying adiabatic changes," Proc. R. Soc. London, Ser. A **392**, 45–57 (1984).
2. M. V. Berry, "Anticipations of the geometric phase," Phys. Today **43**(12), 34–40 (1990).
3. S. Pancharatnam, "Generalized theory of interference, and its applications," Proc. Indian Acad. Sci., Sect. A **44**, 247–262 (1956).
4. M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge Univ. Press, 1999).
5. M. V. Berry, "The adiabatic phase and Pancharatnam's phase for polarized light," J. Mod. Opt. **34**, 1401–1407 (1987).
6. R. Bhandari, "Polarization of light and topological phases," Phys. Rep. **281**, 1–64 (1997).
7. P. Hariharan, "The geometric phase," in *Progress in Optics*, E. Wolf, ed., Vol. 48 (Elsevier, 2005), pp. 149–193.
8. R. Bhandari and J. Samuel, "Observation of topological phase by use of a laser interferometer," Phys. Rev. Lett. **60**, 1211–1213 (1988).

9. Z. Bomzon, G. Biener, V. Kleiner, and E. Hasman, "Space-variant Pancharatnam–Berry phase optical elements with computer-generated subwavelength gratings," Opt. Lett. **27**, 1141–1143 (2002).
10. G. Biener, Y. Gorodetski, A. Niv, V. Kleiner, and E. Hasman, "Manipulation of polarization-dependent multivortices with quasi-periodic subwavelength structures," Opt. Lett. **31**, 1594–1596 (2006).
11. J. Samuel and R. Bhandari, "General setting for Berry's phase," Phys. Rev. Lett. **60**, 2339–2342 (1988).
12. T. F. Jordan, "Berry phases for partial cycles," Phys. Rev. A **38**, 1590–1592 (1988).
13. R. C. Jones, "A new calculus for the treatment of optical systems," J. Opt. Soc. Am. **31**, 488–493 (1941).
14. F. Eriksson, "On the measure of solid angles," Math. Mag. **63**, 184–187 (1990).
15. L. E. J. Brouwer, "Über Abbildung von Mannigfaltigkeiten," Math. Ann. **LXXI**, 97–115 (1912).
16. T. H. Chyba, L. J. Wang, L. Mandel, and R. Simon, "Measurement of the Pancharatnam phase for a light beam," Opt. Lett. **13**, 562–564 (1988).
17. H. Schmitzer, S. Klein, and W. Dultz, "Nonlinearity of Pancharatnam's topological phase," Phys. Rev. Lett. **71**, 1530–1533 (1993).
18. R. Bhandari, "Observation of Dirac singularities with light polarization. I," Phys. Lett. A **171**, 262–266 (1992).
19. R. Bhandari, "Observation of Dirac singularities with light polarization. II," Phys. Lett. A **171**, 267–270 (1992).
20. T. van Dijk, H. F. Schouten, W. Ubachs, and T. D. Visser, "The Pancharatnam–Berry phase for non-cyclic polarization changes," Opt. Express **18**, 10796–10804 (2010).