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Abstract

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game. A solution for TU-games assigns a set of payoff distributions to every TU-game.

In the literature various models of games with restricted cooperation can be found. So, instead of allowing all subsets of the player set N to form, it is assumed that the set of feasible coalitions is a subset of the power set of N. In this paper we consider such sets of feasible coalitions that are closed under union, i.e. for any two feasible coalitions also their union is feasible. Properties of solutions (the core, the nucleolus, the prekernel and the Shapley value) are given for games on union closed systems.

Keywords: TU-game, restricted cooperation, union closed system, core, prekernel, nucleolus.

AMS subject classification: 91A12, 5C20
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1 Introduction

A *cooperative game with transferable utility*, or simply a TU-game, is a finite set of players and for any subset (coalition) of players a worth representing the total payoff that the coalition can obtain by cooperating. A (single-valued) solution is a function that assigns to every game a payoff vector which components are the individual payoffs of the players.

In its classical interpretation, a TU-game describes a situation in which the players in every coalition S of N can cooperate to form a feasible coalition and earn its worth. In the literature various restrictions on coalition formation are developed.[1] For example, in the *(communication) graph games* of Myerson (1977) a coalition is feasible if it is connected in a given (communication) graph. Games in which the collection of feasible coalitions forms an *antimatroid*[2] are considered in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003, 2004). A well-known example of an antimatroid is the collection of feasible coalitions induced by a acyclic *permission structure*, i.e. players need permission from (some of) their superiors in a hierarchical structure when they want cooperate with others. Games with a permission structure are considered in e.g. Gilles, Owen and van den Brink (1992), van den Brink and Gilles (1996), Gilles and Owen (1994) and van den Brink (1997). A model that generalizes both the communication graph games as well as the games on antimatroids are the games on *augmenting systems*, see Bilbao (2003), Bilbao and Ordóñez (2009) and Algaba, Bilbao and Slikker (2010).

In this paper we consider games with restricted cooperation given by a collection of feasible coalitions that is closed under union, meaning that for any pair of feasible coalitions also their union is feasible. Since such collections are more general than antimatroids, the class of games on union closed systems contains the class of games on antimatroids. In van den Brink, Katsev and van der Laan (2010) two single-valued solutions for games on union closed systems that generalize the Shapley value are defined and characterized. The first solution is based on games with a permission structure, the other directly applies the Shapley value to some restricted game. Both solutions generalize the Shapley value in the sense that they are equal to the Shapley value when the union closed system is the power set of player set N.

The restricted game considered in van den Brink *et al.* (2010) is defined by assigning to each coalition the worth of its maximal feasible subset in the union closed system. In this paper we apply several well-known solution concepts as the core, nucleolus, prekernel and Shapley value to this restricted game. We show some interesting properties of these

[1] For a survey on we refer to Bilbao (2000).

[2] A collection of feasible coalitions $\mathcal{A} \subseteq 2^N$ is an antimatroid if it (i) contains the $\emptyset$, (ii) is *union closed* (when $S, T \in \mathcal{A}$, then also $S \cup T \in \mathcal{A}$), and (iii) satisfies *accessibility* (for every $S \in \mathcal{A}$ there is $i \in S$ such that $S \setminus \{i\} \in \mathcal{A}$), see Dilworth (1940) and Edelman and Jamison (1985).

solutions on the class of games on union closed systems, in particular for monotone games. We also give a sufficient condition to guarantee that the nucleolus is the unique point in the intersection of the prekernel and the core.

This paper is organized as follows. Section 2 is a preliminary section on cooperative TU-games. In Section 3 we introduce games on union closed systems. Section 4 discusses properties of some solutions for monotone games on union closed systems. Finally, Section 5 gives special attention to the prekernel.

2 TU-games and solutions

A situation in which a finite set of players can obtain certain payoffs by cooperating can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N \subset \mathbb{N}$ is a finite set of n players and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the worth of coalition S, i.e., the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. For ease of notation we write $v(i) = v(\{i\})$ for $i \in N$. A player $i \in N$ is called a *veto* player if $v(S) = 0$ if $i \notin S$ and a game v is *veto-rich* if it contains at least one veto player. Since we take the player set N to be fixed, we denote the game (N, v) just by its characteristic function v. For each nonempty $T \subseteq N$, the *unanimity* game u_T is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. It is well-known that the unanimity games form a basis for $\mathcal{G}^N$. For every $v \in \mathcal{G}^N$ it holds that $v = \sum_{T\subseteq N,\ T\neq\emptyset} \Delta_T(v)u_T$, where $\Delta_T(v) = \sum_{S\subseteq T}(-1)^{|T|-|S|}v(S)$ are the *Harsanyi dividends*, see Harsanyi (1959).

We denote the collection of all characteristic functions on N by $\mathcal{G}^N$ and $n = |N|$ denotes the cardinality of N. A game $v \in \mathcal{G}^N$ is *monotone* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$. We denote by $\mathcal{G}^N_m$ the class of all monotone TU-games on N.

A payoff vector is a vector $x \in \mathbb{R}^n$ assigning a payoff x_i to every $i \in N$. In the sequel, for $S \subseteq N$ we denote $x(S) = \sum_{i\in S} x_i$. The set of *efficient* payoff vectors of a game $v \in \mathcal{G}^N$ is given by

$$X(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N)\}$$

and the *imputation set* is the set of efficient and *individually rational* payoff vectors given by

$$I(v) = \{x \in X(v) \mid x_i \geq v(i) \text{ for every } i \in N\}.$$

A (set-valued) solution is a mapping $F\colon \mathcal{G}^N \to \mathbb{R}^n$ that assigns a (possibly empty) set $F(v) \subset \mathbb{R}^n$ of payoff vectors to every $v \in \mathcal{G}^N$. A solution F is said to be *single-valued* if it assigns to every $v \in \mathcal{G}^N$ a single payoff vector $F(v) \in \mathbb{R}^n$. Notice that $F = X$ and $F = I$

are set-valued solutions assigning to every v the set of efficient payoff vectors, respectively the imputation set. The most well-known set-valued solution is the *core* C, assigning to every $v \in \mathcal{G}^N$ the set

$$C(v) = \{x \in X(v) \mid x(S) \geq v(S) \text{ for every } S \subset N\}.$$

Since $I(v) \subseteq C(v)$, the core is the set of stable imputations in the sense that no coalition S can improve by separating from the grand coalition N.

A collection $\mathcal{B}$ of subsets B of N is said to be a *balanced collection* when the system of equations

$$\sum_{B \in \mathcal{B}} \lambda_B e^B = e^N$$

has a positive solution, where for any $B \subseteq N$ the vector $e^B \in \mathbb{R}^n$ is defined by $e^B_j = 1$ when $j \in B$ and $e^B_j = 0$ otherwise. A game $v \in \mathcal{G}^N$ is *balanced* if

$$\sum_{j=1}^{m} \lambda_j^B \; v(S_j) \leq v(N)$$

for every balanced collection $\mathcal{B} = \{S_1, \ldots, S_m\} \in \mathcal{B}$. A well-known result states that the core of a game is non-empty if and only if the game is *balanced*, see Bondareva (1962) or Shapley (1967). Notice that every veto-rich monotone game has a non-empty core (any payoff vector that assigns worth $v(N)$ to the veto players is in the core) and thus is balanced.

Two other well-known solutions are the (pre)nucleolus and the (pre)kernel. To define the (pre)nucleolus of a game $v \in \mathcal{G}^N$, let $x \in \mathbb{R}^n$ be a payoff vector. Then the *excess* $e(S, x)$ of coalition $S \subseteq N$ is defined by

$$e(S, x) = v(S) - x(S).$$

Further, let $E(x)$ be the $(2^n - 2)$-component vector that is composed of the excesses of all coalitions $S \subset N,\ S \neq \emptyset$, in a non-increasing order, so $E_1(x) \geq E_2(x) \geq \ldots \geq E_{2^n-2}(x)$. Then the *prenucleolus* $PN(v)$ of a game $v \in \mathcal{G}^N$ is the unique efficient payoff vector which lexicographically minimizes the vector-valued function $E(\cdot)$ over the set of efficient payoff vectors. Formally,

$$PN(v) = x \text{ such that } x \in X(v) \text{ and } E(x) \preceq_L E(y) \text{ for all } y \in X(v),$$

where $\preceq_L$ denotes the lexicographic order of vectors. The *nucleolus* $Nuc(v)$ of a game $v \in \mathcal{G}^N$ is the unique imputation which lexicographically minimizes the vector-valued function $E(\cdot)$ over the imputation set, so

$$Nuc(v) = x \text{ such that } x \in I(v) \text{ and } E(x) \preceq_L E(y) \text{ for all } y \in I(v).$$

Both the prenucleolus and the nucleolus are single-valued solutions.

To define the prekernel and the kernel of a game $v \in \mathcal{G}^N$ we first introduce the notion of *complaint*. For a payoff vector $x \in \mathbb{R}^n$, the complaint of player $i \in N$ against another player $j \in N$ is given by

$$s_{ij}(x) = \max_{\{S \subseteq N | i \in S,\ j \notin S\}} (v(S) - x(S)).$$

The *prekernel* PK assigns to every $v \in \mathcal{G}^N$ the set of efficient payoff vectors

$$PK(v) = \{x \in X(v) | s_{ij}(x) = s_{ji}(x) \text{ for all } i, j \in N\}$$

and the *kernel* K assigns to every $v \in \mathcal{G}^N$ the set of imputations

$$K(v) = \{x \in I(v) |\ [s_{ij}(x) = s_{ji}(x)] \text{ or } [s_{ij}(x) > s_{ji}(x) \text{ and } x_j = v(j)] \text{ for all } i, j \in N\}.$$

Finally we define the *least core* $LC(v)$ of a game $v \in \mathcal{G}^N$. For an efficient payoff vector $x \in X(v)$, the *excess* $e_v(x)$ of x is defined by

$$e_v(x) = \max_{\{S \in 2^N | S \neq \emptyset, N\}} e(S, x) = \max_{\{S \in 2^N | S \neq \emptyset, N\}} (v(S) - x(S)),$$

i.e. for any coalition $S \neq \emptyset, N$, its payoff $x(S)$ is at least equal to its own worth $v(S)$ minus the excess $e_v(x)$ with equality for at least one of these coalitions. Further the *gain* $e(v)$ of v is defined as the largest negative excess, thus

$$e(v) = \max_{x \in X(N,v)} -e_v(x).$$

Notice that $e_v(x) \leq 0$ when $x \in C(v)$ and $e(v) \geq 0$ if and only if $C(v) \neq \emptyset$. Then the least core, introduced by Maschler, Peleg and Shapley (1979), see e.g. also Einy, Holzman and Monderer (1999), is defined as the solution LC that assigns to game v the set of efficient payoff vectors

$$LC(v) = \{x \in X(v) | x(S) \geq v(S) + e(v) \text{ for every } S \neq \emptyset, N\}.$$

Observe that $LC(v) \subseteq C(v)$ if $C(v) \neq \emptyset$, with $LC(v) = C(v)$ when $e(v) = 0$. We also have that $Nuc(v) \in LC(v)$ and that $LC(v) \subseteq I(v)$ when $v \in \mathcal{G}^N_m$.

3 Games on union closed systems

We now consider tuples (v, Ω), where v is a TU-game on player set N and $\Omega \subseteq 2^N$ is a collection of subsets of N. We call such a tuple a *game with restricted cooperation*. In such a game the collection of subsets Ω restricts the cooperation possibilities of the players in N. We say that a coalition $S \in 2^N$ is *feasible* if and only if $S \in \Omega$. In this paper we only consider sets of feasible coalitions that are closed under union.

Definition 3.1 *A collection $\Omega \subseteq 2^N$ is a union closed system of coalitions if*
1. $\emptyset, N \in \Omega$,
2. If $S,\ T \in \Omega$, then $S \cup T \in \Omega$.

In the sequel we denote the collection of all union closed systems in 2^N by $\mathcal{C}^N$.

Example 3.2
1. Both $\Omega = \{\emptyset, N\}$ and $\Omega = 2^N$ are union closed systems, the first one is the smallest union closed system and the second one is the largest union closed system of subsets of N, i.e. $\{\emptyset, N\} \subseteq \Omega \subseteq 2^N$ for every union closed system Ω of subsets of N.

2. For some $k \in \{1, \ldots |N|\}$, the collection of coalitions $\Omega = \{S \subseteq N \mid |S| \geq k\} \cup \{\emptyset\}$ is closed under union. More generally, let the collection $\mathcal{P} = \{P^1, \ldots, P^m\}$ of nonempty subsets of N be a partition of N, and for every $P^k,\ k \in \{1, \ldots, m\}$, let $q_k \in \{1, \ldots, |P^k|\}$ be a quotum meaning that a nonempty coaliton $S \subseteq N$ can form S contains at least q_k players from P^k for every $k = 1, \ldots, m$. The collection of feasible coalitions

$$\Omega = \{S \subseteq N \mid |S \cap P_k| \geq q_k \text{ for all } k \in \{1, \ldots, m\}\} \cup \{\emptyset\}$$

is closed under union.

3. Let D be an acyclic directed graph on player set N (representing for instance some hierarchical structure), and let Ω be the collection of subsets of N such that $S \in \Omega$ whenever for every $i \in S$ also all predecessors of i in the digraph D belong to S. Then Ω is union closed. Also the collection Ω of subsets of N such that $S \in \Omega$ whenever for every $i \in N$ having a predecesssor in D at least one of the predecessors is in S is union closed. For given D these collections are called the collection of conjunctive, respectively disjunctive, feasible coalitions and are an antimatroid, see Algaba *et al.* (2003). Every antimatroid is a union closed system by definition[3]. □

For notational convenience we require in Definition 3.1 that the grand coalition N is feasible. The results in this paper can be modified to hold without this requirement if in the axioms we distinguish between players that belong to at least one feasible coalition and those that do not belong to any feasible coalition. Note that union closedness implies that the grand coalition is feasible if every player belongs to at least one feasible coalition. So, instead of assuming that $N \in \Omega$ we could do with the weaker *normality* assumption

[3] In fact, an augmenting system is an antimatroid if and only if it is closed under union, see Bilbao (2003) and Algaba, Bilbao and Slikker (2010). Note that examples 1 and 2 above are union closed systems that are not antimatroids.

stating that every player belongs to at least one feasible coalition. We give some definitions and properties for union closed systems.

Definition 3.3 *For two players $i, j \in N$, $i \neq j$, player i is a superior of player j in $\Omega \in \mathcal{C}^N$, if $i \in S$ for every $S \in \Omega$ such that $j \in S$. In that case player j is a subordinate of i.*

Corollary 3.4 *If i is a superior of j in Ω and k is a superior of i in Ω then k is a superior of j in Ω.*

Further, for $i \in N$ and $\Omega \in \mathcal{C}^N$ define

$$S_i^\Omega = \{j \in N \mid j = i \text{ or } i \text{ is a superior of } j\},$$

i.e. $S_i^\Omega \subseteq N$ denotes the set containing player i and all subordinates of i in Ω. Then the next proposition says that when Ω is a union closed system, for every $i \in N$ the complement of i and all its subordinates is in Ω.

Proposition 3.5 *When $\Omega \in \mathcal{C}^N$, then $N \setminus S_i^\Omega \in \Omega$ for every $i \in N$.*

Proof. Let U be the union of all feasible sets not containing i. Since $\Omega \in \mathcal{C}^N$, it follows that $U \in \Omega$. Further, by definition of U we have that $i \notin U$. Consider a player $j \notin U$ with $j \neq i$. It holds that any feasible set without i does not contain j. So i is a superior of j and thus $j \in S_i^\Omega$. Hence $N \setminus U \subseteq S_i^\Omega$. On the other hand, consider some player $j \in S_i^\Omega$. If $j = i$ then $j \notin U$ by definition of U. If $j \neq i$, then any feasible set containing j also contains i. Hence $j \notin U$, which shows that $S_i^\Omega \subseteq N \setminus U$. Hence $N \setminus S_i^\Omega = U \in \Omega$. $\square$

A set $S \subseteq N$ of players can attain its value $v(S)$ if $S \in \Omega$. When $S \notin \Omega$ then coalition S can not be formed and so the set S of players can not realize its worth $v(S)$. For a tuple (v, Ω), let $\sigma_\Omega \colon 2^N \to \Omega$ be given by $\sigma_\Omega(S) = \cup_{\{E \in \Omega \mid E \subseteq S\}} E$, i.e. $\sigma_\Omega(S)$ is the largest feasible subset of S in the system Ω. By union closedness this largest feasible subset is unique. We then define the *restricted* game $r_{v,\Omega} \in \mathcal{G}^N$ of (v, Ω) by

$$r_{v,\Omega}(S) = v(\sigma_\Omega(S)),$$

i.e. the restricted game is a standard TU-game that assigns to each coalition $S \subseteq N$ the worth of its largest feasible subset. Notice that when $\Omega = \{\emptyset, N\}$, then $\sigma_\Omega(N) = N$ and $\sigma_\Omega(S) = \emptyset$ for all $S \neq N$ and thus $r_{v,\Omega}(N) = v(N)$ and $r_{v,\Omega}(S) = v(\emptyset) = 0$ for every $S \neq N$. Thus the restricted game $r_{v,\Omega}$ is a multiple of the unanimity game of N, being a game in which every player is a veto-player. When $\Omega = 2^N$ then $\sigma_\Omega(S) = S$ and $r_{v,\Omega}(S) = v(S)$ for every $S \subseteq N$. In this case the restricted game $r_{v,\Omega}$ coincides with v.

Next we generalize some inheritance properties of the restricted game which generalizes known results for games with a permission tructure and games on antimatroids.

Proposition 3.6 *Let $\Omega \in \mathcal{C}^N$ and let $v \in \mathcal{G}^N$ be a monotone game. Then*

1. *the restricted game $r_{v,\Omega}$ is monotone;*

2. *if v is superadditive then $r_{v,\Omega}$ is superadditive;*

3. *if v is balanced then $r_{v,\Omega}$ is balanced;*

4. *if v is convex and Ω is closed under intersection (i.e. $S,\ T \in \Omega$ implies that $S \cap T \in \Omega$), then $r_{v,\Omega}$ is convex.*

Proof. Let $\Omega \in \mathcal{C}^N$ and let $v \in \mathcal{G}^N$ be a monotone game.

1. By definition of σ_Ω it is obvious that $S \subseteq T$ implies that $\sigma_\Omega(S) \subseteq \sigma_\Omega(T)$, and thus by monotonicity of v, $S \subseteq T$ implies that $r_{v,\Omega}(S) = v(\sigma_\Omega(S)) \leq v(\sigma_\Omega(T)) = r_{v,\Omega}(T)$, showing monotonicity of $r_{v,\Omega}$.

2. By union closedness, $\sigma_\Omega(S) \cup \sigma_\Omega(T) \in \Omega$ for all $S, T \in \Omega$. Since $\sigma_\Omega(S) \cup \sigma_\Omega(T) \subseteq S \cup T$, we then have $\sigma_\Omega(S) \cup \sigma_\Omega(T) \subseteq \sigma_\Omega(S \cup T)$. If $S \cap T = \emptyset$ then $\sigma_\Omega(S) \cap \sigma_\Omega(T) = \emptyset$, and thus $r_{v,\Omega}(S) + r_{v,\Omega}(T) = v(\sigma_\Omega(S)) + v(\sigma_\Omega(T)) \leq v(\sigma_\Omega(S) \cup \sigma_\Omega(T)) \leq v(\sigma_\Omega(S \cup T)) = r_{v,\Omega}(S \cup T)$, where the first inequality follows from superadditivity of v. This shows superadditivity of $r_{v,\Omega}$.

3. This follows from Proposition 4.1 and the obvious fact that $C(v) \subseteq C^*(v, \Omega)$.

4. In 2 we already showed that by union closedness $\sigma_\Omega(S) \cup \sigma_\Omega(T) \subseteq \sigma_\Omega(S \cup T)$. Similar, by intersection closedness $\sigma_\Omega(S) \cap \sigma_\Omega(T) \in \Omega$ for all $S, T \in \Omega$. Since $\sigma_\Omega(S) \cap \sigma_\Omega(T) \subseteq S \cap T$, we have $\sigma_\Omega(S) \cap \sigma_\Omega(T) \subseteq \sigma_\Omega(S \cap T)$. But then $r_{v,\Omega}(S) + r_{v,\Omega}(T) = v(\sigma_\Omega(S)) + v(\sigma_\Omega(T)) \leq v(\sigma_\Omega(S) \cup \sigma_\Omega(T)) + v(\sigma_\Omega(S) \cap \sigma_\Omega(T)) \leq v(\sigma_\Omega(S \cup T)) + v(\sigma_\Omega(S \cap T)) = r_{v,\Omega}(S \cup T) + r_{v,\Omega}(S \cap T)$, where the first inequality follows from convexity of v. This shows convexity of $r_{v,\Omega}$.

□

A solution for games on union closed systems is a mapping F that assigns a set of payoff vectors $F(v, \Omega) \subset \mathbb{R}^n$ to every $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$. In this paper we only consider solutions for games on union closed systems that assign to each tuple $(v, \Omega) \in \mathcal{G}^N \times \mathcal{C}^N$ the set of payoff vectors $F(r_{v,\Omega})$ of a solution $F \colon \mathcal{G}^N \to \mathbb{R}^n$, i.e. a solution for games on a union closed system assigns the set of payoff vectors that is assigned by a solution F on $\mathcal{G}^N$ to the restricted game $r_{v,\Omega}$. For ease of notation we denote $F(v, \Omega) = F(r_{v,\Omega})$.

4 Properties of solutions for the class of monotone games on union closed systems

In this section we apply the solutions for TU games given in Section 2 and consider their properties for games on union closed systems, in particular we consider the relation between the payoffs of some player j and its superior i for monotone games on union closed systems. Notice that when v is monotone, it holds that for every $\Omega \in \mathcal{C}^N$ also the restricted game $r_{v,\Omega}$ is monotone (by Proposition 3.6). Further it should be noticed that $r_{v,\Omega}(\{j\}) = v(\emptyset) = 0$ when j has a superior, because $\{j\}$ is not feasible when j has a superior.

First we consider the core and the least core of the restricted game. When we take as solution F the core of a game, then we obtain

$$C(v, \Omega) = C(r_{v,\Omega}) = \{x \in X(r_{v,\Omega}) \mid x(S) \geq v(\sigma_\Omega(S)),\ S \subset N\}.$$

For a tuple (v, Ω), let $C^*(v, \Omega)$ be given by

$$C^*(v, \Omega) = \{x \in X(v) | x(S) \geq v(S) \text{ for any } S \in \Omega \text{ and } x_j \geq 0 \text{ for any } j \in N\}.$$

i.e. $C^*(v, \Omega)$ is the set of nonnegative efficient payoff vectors satisfying the core inequalities corresponding to the feasible coalitions in Ω. It turns out that this set is equal to the core on the class of monotone games on union closed systems.

Proposition 4.1 *For every $v \in \mathcal{G}_m^N$ and $\Omega \in \mathcal{C}^N$ we have $C(v, \Omega) = C^*(v, \Omega)$.*

Proof. Since $N \in \Omega$ we have that $\sigma_\Omega(N) = N$ and thus $v(\sigma_\Omega(N)) = v(N)$ and $X(r_{v,\Omega}) = X(v)$. Let $x \in C(v, \Omega)$. When the singleton player set $\{j\} \in \Omega$, then $\sigma_\Omega(\{j\}) = \{j\}$ and thus $v(\sigma_\Omega(\{j\})) = v(\{j\}) \geq 0$, since $v \in \mathcal{G}_m^N$. Otherwise $\sigma_\Omega(\{j\}) = \emptyset$ and thus $v(\sigma_\Omega(\{j\})) = v(\emptyset) = 0$. Hence for every $x \in C(v, \Omega)$ we have that $x_j \geq v(\sigma_\Omega(\{j\})) \geq 0$ for every $j \in N$. Further, since $\sigma_\Omega(S) = S$ if $S \in \Omega$, the inequalities $x(S) \geq v(\sigma_\Omega(S))$ for every $S \subset N$, imply that $x(S) \geq v(S)$ for every $S \in \Omega$. Thus, $x \in C^*(v, \Omega)$.

Next, let $x \in C^*(v, \Omega)$. Obviously, $x(S) \geq v(\sigma_\Omega(S))$ for every $S \in \Omega$. Since $\sigma_\Omega(S) \subseteq S$ and $x_j \geq 0$ for all $j \in N$, we have for any $S \subset N$ and $v \in \mathcal{G}_m^N$ that $x(S) \geq x(\sigma_\Omega(S)) \geq v(\sigma_\Omega(S))$ for every $S \subset N$. Thus, $x \in C(v, \Omega)$.

□

Next recall that the least core of a monotone game is contained in the imputation set of the game. Since the restricted game of a monotone game is also monotone, it follows for $v \in \mathcal{G}_m^N$ that $x \in I(r_{v,\Omega})$ for every $x \in LC(r_{v,\Omega})$. Since, for every $j \in N$, either $\{j\}$ is feasible in Ω and thus $x_j \geq v(\{j\}) \geq 0$, or j is not feasible and $x_j \geq r_{v,\Omega}(\{j\}) = v(\emptyset) = 0$, we have the following proposition.

Proposition 4.2 *Let $v \in \mathcal{G}_m^N$ be monotone, and $\Omega \in \mathcal{C}^N$. Then $x_j \geq 0$ for every $x \in LC(r_{v,\Omega})$ and $j \in N$.*

For a monotone game v it is straightforward that for two union closed systems Ω_1 and Ω_2 such that $\Omega_1 \subseteq \Omega_2$, we have $r_{v,\Omega_1}(S) \leq r_{v,\Omega_2}(S)$ for every $S \in 2^N$. Therefore the next proposition follows immediately without proof.

Proposition 4.3 *Let v be monotone and Ω_1, Ω_2 be two union closed systems such that $\Omega_1 \subseteq \Omega_2$. Then $C(v, \Omega_2) \subseteq C(v, \Omega_1)$.*

Since $r_{v,\Omega} = v$, and thus $C(v, \Omega) = C(v)$ when $\Omega = 2^N$, Proposition 4.3 yields that $C(v, \Omega) \neq \emptyset$ for any $\Omega \in \mathcal{C}^N$ and $v \in \mathcal{G}_m^N$ with non-empty core.

In the following, let i and j be two fixed players such that i is a superior of j in Ω (and thus $r_{v,\Omega}(j) = 0$). For a vector x with $x_j > 0$ and some number $0 \leq a \leq x_j$, we denote for fixed i and j the vector x^a by[4]

$$\begin{cases} x_i^a = x_i + a, \\ x_j^a = x_j - a, \\ x_k^a = x_k & \text{when } k \neq i, j. \end{cases} \tag{4.1}$$

Clearly, since $x_j^a = x_j - a \geq 0 = r_{v,\Omega}(\{j\})$ we have that $x^a \in I(N, r_{v,\Omega})$ when $x \in I(N, r_{v,\Omega})$. Moreover, for $S \subset N$

$$\begin{cases} x^a(S) = x(S) + a > x(S) & i \in S,\ j \notin S, \\ x^a(S) = x(S) - a < x(S) & j \in S,\ i \notin S, \\ x^a(S) = x(S) & \text{otherwise.} \end{cases}$$

So, for every $S \in \Omega$ it is true that $x^a(S) \geq x(S)$ because i is a superior of j and thus there does not exist $S \in \Omega$ with $j \in S$ and $i \notin S$. We now have the following proposition.

Proposition 4.4 *Let (v, Ω) be a monotone game on a union closed system and, for a vector x and two players i and j such that i is a superior of j, let x^a be as defined in equation (4.1). Then*
(i) if $x \in C(v, \Omega)$, then $x^a \in C(v, \Omega)$ for all $a \in (0, x_j]$.
(ii) if $x \in LC(v, \Omega)$ and $x_i < x_j$, then $x^a \in LC(v, \Omega)$ for all $a \in (0, \frac{1}{2}(x_j - x_i)]$.

Proof. To prove (i), recall from Proposition 4.1 that

$$C(v, \Omega) = \{x \in X(v)\ x(S) \geq v(S) \text{ for any } S \in \Omega \text{ and } x_j \geq 0 \text{ for any } j \in N\}.$$

[4] There is some abuse of notation, actually x^a also depends on i and j.

Clearly, for every $S \in \Omega$ we have that $x^a(S) \geq x(S) \geq v(S)$. Further, we have for every $k \neq j$ that $x_k^a \geq x_k \geq 0$ and that $x_j^a = x_j - a \geq 0$. Since i is a superior of j and thus $\{j\} \notin \Omega$, it follows that $x^a \in C(v, \Omega)$.

To prove (ii), notice that $x_i^a \leq x_j^a$ for all $a \in (0, \frac{1}{2}(x_j - x_i)]$. Suppose that x^a is not in $LC(v, \Omega)$. Then there exists a coalition $S \subset N$ such that

$$x^a(S) - r_{v,\Omega}(S) < e(r_{v,\Omega}). \tag{4.2}$$

Since $x \in LC(v, \Omega)$ we have that

$$x(S) - r_{v,\Omega}(S) \geq e(r_{v,\Omega}).$$

Hence $x(S) > x^a(S)$, implying that S contains j but not i. Let $T = S \backslash \{j\}$ and $S' = T \cup \{i\}$. Then

$$x^a(S) - r_{v,\Omega}(S) = x^a(S) - v(\sigma_\Omega(T))$$

because $i \notin S$ and thus $j \notin \sigma_\Omega(S)$. Hence

$$x^a(S) - r_{v,\Omega}(S) = x^a(S) - v(\sigma_\Omega(T)) \geq x^a(T) + x_j^a - v(\sigma_\Omega(T \cup \{i\})) \geq$$

$$x^a(T) + x_i^a - v(\sigma_\Omega(T \cup \{i\})) = x^a(S') - r_{v,\Omega}(S'),$$

where the second inequality follows because $x_i^a \leq x_j^a$. So, with equation (4.2) it follows that

$$x^a(S') - r_{v,\Omega}(S') \leq x^a(S) - r_{v,\Omega}(S) < e(r_{v,\Omega}),$$

which contradicts that $x(S') - r_{v,\Omega}(S') \geq e(r_{v,\Omega})$. The latter inequality must hold for $x \in LC(v, \Omega)$, since $x^a(S') > x(S')$ because $i \in S'$ and $j \notin S'$. □

From Part (i) of Proposition 4.4 we obtain the following corollary, saying that when the core of the resticted game is not empty, there exist core stable payoff vectors that give zero payoff to every player j that has a superior in Ω.

Corollary 4.5 *If $C(r_{v,\Omega}) \neq \emptyset$, then there exists $x \in C(r_{v,\Omega})$ such that $x_j = 0$ for every j that has a superior.*

The final proposition in this section states that for monotone games on a union closed system, a player gets at most the same payoff as its superior when applying the nucleolus to the restricted game. It should be noticed that when v is monotone, $Nuc_j(v, \Omega) \geq 0$ for all j, because $Nuc(v, \Omega)$ is in the least core of $r_{v,\Omega}$ and thus also in $I(N, r_{v,\Omega})$.

Proposition 4.6 *Let (v, Ω) be a monotone game on a union closed system. Then for every two players i and j such that i is a superior of j it holds that $Nuc_i(v, \Omega) \geq Nuc_j(v, \Omega)$.*

Proof. Let $w \in \mathcal{G}^N$ be a game such that for every $S \subseteq N \setminus \{i, j\}$ it holds that $w(S \cup \{i\}) \geq w(S \cup \{j\})$. Then we know from Peleg and Südholter (2003, Theorem 5.3.5) that $x_i \geq x_j$ for every x in the prekernel of w. Since the nucleolus of a game is in the prekernel of a game, it is sufficient to show that for every $S \subseteq N \setminus \{i, j\}$ it holds that $r_{v,\Omega}(S \cup \{i\}) \geq r_{v,\Omega}(S \cup \{j\})$ when i a superior of j. Indeed, in that case we have that

$$r_{v,\Omega}(S \cup \{i\}) = v(\sigma_\Omega(S \cup \{i\})) \geq v(\sigma_\Omega(S)) = v(\sigma_\Omega(S \cup \{j\})) = r_{v,\Omega}(S \cup \{j\}),$$

where the second equality follows from the fact that $i \notin S$ and there does not exist a feasible set containing j but not i. □

In van den Brink *et al.* (2010) it is shown that also the Shapley value of the restricted game satisfies this property.[5]

5 The prekernel of monotone games on union closed systems

In this section we focus on the prekernel for games on union closed systems. Arin and Feltkamp (1997) proved that the kernel of a game $v \in \mathcal{G}^N$ consists of only one point (and coincides with the nucleolus), when the game is veto-rich and $I(v)$ is non-empty. When in the tuple (v, Ω) there exists a player $i \in N$ such that $i \in S$ for every $S \in \Omega$, then i is a veto-player in the restricted game $r_{v,\Omega}$. When $v \in \mathcal{G}^N_m$ we have that $I(r_{v,\Omega}) \neq \emptyset$ and thus it follows from Arin and Feltkamp (1997) that the kernel of $r_{v,\Omega}$ has the nucleolus of $r_{v,\Omega}$ as its unique element. It is also well-known that for every game (N, v) with $|N| \leq 3$, the intersection of the prekernel and the core consists of at most one point. In this section we generalize these results and give a sufficient condition to guarantee that the prekernel and the core of a monotone game on a union closed system have at most one point in common. Of course, when such a point exists, then it is the nucleolus of the restricted game. We first introduce some new notions.

Definition 5.1 *For two players $i, j \in N$, $i \neq j$, player i is a strong superior of player j in $\Omega \in \mathcal{C}^N$ if i is a superior of j and j is not a superior of i.*

Definition 5.2 *A player $i \in N$ is a free player in $\Omega \in \mathcal{C}^N$ if i has no superiors; player $i \in N$ is a weakly free player in $\Omega \in \mathcal{C}^N$ if i has no strong superiors.*

[5] In this paper the Shapley value for games on union closed systems is introduced and characterized as the so-called union rule.

Notice that a free player is also a weakly free player and that a weakly free player i is a superior of j when j is a superior of i. For $\Omega \in \mathcal{C}^N$, we denote the set of weakly free players by

$$W_\Omega = \{i \in N \mid i \text{ is a weakly free player in } \Omega\}.$$

The next proposition gives three properties of the set W_Ω.

Proposition 5.3
1. For every player $j \notin W_\Omega$, there is a player $i \in W_\Omega$, such that i is a strong superior of j.
2. When j is a superior of a player $i \in W_\Omega$, then i is a superior of j.
3. When j is a superior of a player $i \in W_\Omega$, then $j \in W_\Omega$.

Proof.
1. Consider some player $i_0 \in N$. If i_0 is not in W_Ω, then i_0 has a strong superior, say i_1. Then, either $i_1 \in W_\Omega$ and thus i_0 has a strong superior in W_Ω, or not. In the latter case i_1 has a strong superior, say i_2. When i_2 is not in W_Ω, it also has a strong superior. Continuing this we get a sequence of players $i_0, i_1, i_2, \ldots, i_m$ such that for $h = 1, \ldots, m-1$, player i_{h+1} is a strong superior of i_h and thus $i_h \notin W_\Omega$ and either $i_m \in W_\Omega$ or $m \geq 2$ and $i_m = i_k$ for some $k = 0, \ldots, m-2$. In the latter case, by Corollary 3.4 every pair i_j, i_ℓ with $j, \ell \in \{k, k+1, \ldots, m-1\}$ are superiors of each other, contradicting that i_{h+1} is strong superior of i_h, $h = k, \ldots, m-1$. Hence every next player in the sequence is different from all preceding players. Since the number of players is finite, this case can not happen and thus within a finite number of steps some player $i_m \in W_\Omega$ is generated. By Corollary 3.4 i_m is a superior of i_0. When i_0 is a superior of i_m, then again by Corollary 3.4 we have that i_0 is a superior of i_1, contradicting that i_1 is a strong superior of i_0. Hence $i_m \in W_\Omega$ and is a strong superior of i_0.
2. By definition, i is a superior of j, since otherwise j is a strong superior of i, which contradicts that $i \in W_\Omega$.
3. Suppose $j \notin W_\Omega$. Then by the first property, j has a strong superior k in W_Ω. By Corollary 3.4 player k is also a superior of i, and thus by property 2 we have that player i is also a superior of k. However this implies that also j is a superior of k, contradicting that k is a strong superior of j. □

The first property of Proposition 5.3 yields the following corollary.

Corollary 5.4 *For every $\Omega \in \mathcal{C}^N$, $W_\Omega \neq \emptyset$.*

Next, for $i \in W_\Omega$, define $T_\Omega(i) = \{j \in N \mid j = i \text{ or } j \text{ is a superior of } i\}$ and let $\mathcal{T}_\Omega$ be the collection of sets defined by

$$\mathcal{T}_\Omega = \{T_\Omega(i) \mid i \in W_\Omega\}.$$

Notice that for every $j \in T_\Omega(i) \setminus \{i\}$, also i is a superior of j, because $i \in W_\Omega$, and thus $T_\Omega(i) \subseteq S_i^\Omega = \{j \in N | j = i \text{ or } i \text{ is a superior of } j\}$. The next proposition describes the set W_Ω.

Proposition 5.5 *The collection $\mathcal{T}_\Omega$ is a partition of the set W_Ω.*

Proof. First, by Property 3 of Proposition 5.3 we have that $j \in W_\Omega$ when $j \in T_\Omega(i)$ for some $i \in W_\Omega$ and thus $T_\Omega(i) \subseteq W_\Omega$. Next, let $R \subseteq W_\Omega \times W_\Omega$ be the binary relation on W_Ω defined by $(j, i) \in R$ if and only if $j \in T_\Omega(i)$. It is sufficient to show that this relation is an equivalence relation on W_Ω, i.e. the relation is reflexive, symmetric and transitive. First, by definition $(i, i) \in R$ for all $i \in W_\Omega$, so R is reflexive. Second, for $j \neq i$, when $(j, i) \in R$, then j is a superior of i. By Property 2 of Proposition 5.3 then also i is a superior of j and thus $(i, j) \in R$, showing that R is symmetric. Third, when $(k, j) \in R$ and $(j, i) \in R$, then k is a superior of j and j of i and thus, by Corollary 3.4, also k is a superior of i. Hence, $(k, i) \in R$ and thus R is transitive. Since R is an equivalence relation, it follows that the sets $T_\Omega(i)$, $i \in W_\Omega$, are equivalence classes of W_Ω and thus the collection $\mathcal{T}_\Omega$ partitions W_Ω. □

Proposition 5.5 implies that $j \in T_\Omega(i)$ if and only if $i \in T_\Omega(j)$. When, for two different agents $i, j \in W_\Omega$, i is not a superior of j, then $T_\Omega(i)$ and $T_\Omega(j)$ are two different equivalence classes.

Proposition 5.6 *Let Ω be a union closed system. When $j \in W_\Omega$ is a superior of $i \in N$ then every $k \in T_\Omega(j)$ is a superior of i.*

Proof. For $i \in W_\Omega$ the proposition follows from Proposition 5.5, because $T_\Omega(i) = T_\Omega(j)$ when j is a superior of i. Let $i \notin W_\Omega$ and $j \in W_\Omega$ be a superior of i. Then every $k \neq j$ in $T_\Omega(j)$ is a superior of j and thus a superior of i by Corollary 3.4. □

Proposition 5.7 *Let (v, Ω) be a game on a union closed system. When $\mathcal{T}_\Omega$ consists of only one set, then every player in W_Ω is a veto-player in the restricted game $r_{v,\Omega}$.*

Proof. First, when $\mathcal{T}_\Omega$ consists of only one set, say T, then, by Proposition 5.5, $T = W_\Omega$. So, $T_\Omega(i) = W_\Omega$ for every $i \in W_\Omega$ and thus by definition of $T_\Omega(i)$ and Proposition 5.6 every player $k \in W_\Omega$ is a superior to every other i in W_Ω. Moreover, by Property 1 of Proposition 5.3 every player not in W_Ω has a player i in W_Ω as its superior, and thus, again by Proposition 5.6, every player in W_Ω is a superior of every player not in W_Ω. So, every player in W_Ω is a superior of every other player in N, so that every $S \in \Omega$ contains all players in W_Ω. □

Notice that $T_\Omega(i) = \{i\}$ when i is a free player. So, every free player i gives a single element equivalence class $T_\Omega(i) = \{i\}$ in the partition $\mathcal{T}_\Omega$ of W_Ω. When there is a free player i and $\mathcal{T}_\Omega$ consists of only one set, then $W_\Omega = \{i\}$. In the sequel we call the number of sets in $\mathcal{T}_\Omega$ the *weakly free player cardinality* of Ω. Since by Corollary 5.4 the set of weakly free players is not-empty, this cardinality is at least one. It follows from Proposition 5.7 that $r_{v,\Omega}$ is a veto-rich game when this cardinality is equal to one. Then the next corollary follows from Arin and Feltkamp (1997).

Corollary 5.8 *Let (v, Ω) be a game on a union closed system. Then the kernel $K(r_{v,\Omega})$ contains the nucleolus $Nuc(r_{v,\Omega})$ as its unique element when the weakly free player cardinality is one.*

To generalize this, we use the famous theorem of Kohlberg (1971) giving a sufficient and necessary condition for a payoff vector to be in the prenucleolus of a game. For game $v \in \mathcal{G}^N$, a payoff vector $x \in \mathbb{R}^n$ and real number α, let $\mathcal{B}(\alpha, x)$ be the collection of coalitions given by $\mathcal{B}(\alpha, x) = \{S \in N \mid e(S, x) \geq \alpha\}$.

Theorem 5.9 *(***Kohlberg, 1971***) For game $v \in \mathcal{G}^N$, a payoff vector x is in $PN(v)$ if and only if for any real number α the collection of coalitions $\mathcal{B}(\alpha, x)$ is either balanced or empty.*

In Katsev and Yanovskaya (2010) an analogue of this theorem for the prekernel is proved in terms of 2-balancedness. We first give the notion of k-balancedness for $2 \leq k \leq n$.

Definition 5.10 *A collection $\mathcal{S}$ of coalitions $S \in 2^N$ is k-balanced if for every coalition $K \subseteq N$ with $|K| = k$ the collection $\mathcal{S}_K = \{S' \subset K \mid S' = S \cap K, S \in \mathcal{S}\}$ is balanced on K.*

Theorem 5.11 *(***Katsev and Yanovskaya, 2010***) For $v \in \mathcal{G}^N$, a payoff vector x is in $PK(v)$ if and only if for any real number α the collection of coalitions $\mathcal{B}(\alpha, x)$ is either 2-balanced or empty.*

Recall from the standard definition of balancedness that when a collection $\mathcal{S}_K$ is balanced on K, then there exist strictly positive weights λ_T, $T \in \mathcal{S}_K$, such that for every $i \in K$ the total weight of the sets $T \in \mathcal{S}_K$ that contain i is equal to one. From this the following corollary follows immediately.

Corollary 5.12 *Let $K = \{i, j\} \subseteq N$ be a two-player coalition and $\mathcal{S}$ be a collection of coalitions $S \in 2^N$ such that $\mathcal{S}_K$ is balanced on K. When $\mathcal{S}$ contains a set T such that $i \in T$ and $j \notin T$, then $\mathcal{S}$ contains a coalition T' such that $j \in T'$ and $i \notin T'$.*

Also notice that a k-balanced collection $\mathcal{S}$ is balanced when $k = n$. Moreover it should be noticed that when $|N| = 3$, any 2-balanced collection is also balanced. The next lemma generalizes this fact and will be used to prove the main result of this section.

Lemma 5.13 *For a union closed system Ω with weakly free player cardinality of at most three, let $\mathcal{B} \subset 2^N$ be a 2-balanced collection that only contains feasible sets in Ω and singletons. Then $\mathcal{B}$ is balanced.*

Proof. Let $c \in \{1, 2, 3\}$ be the weakly free player cardinality of Ω. Without loss of generality, let the players be numbered in such way that $W_\Omega \supset \{1, \ldots, c\}$ and that $T_\Omega(k)$, $k = 1, \ldots, c$, are the equivalence classes of $\mathcal{T}_\Omega$. By property 2 of Proposition 5.3, every player $j \neq k$ in $T_\Omega(k)$ has player k as its superior. Also, by property 1 of Proposition 5.3 and by Proposition 5.6, every player $j \in N \setminus W_\Omega$ has at least one of the players k, $k \in W_\Omega$ as one of its superiors. For $k \in W_\Omega$, suppose that there exists j in the set

$$S_k^\Omega \setminus \{k\} = \{i \in N | k \text{ is a superior of } i\}$$

such that there is some T in $\mathcal{B}$ containing k, but not j. Take $K = \{k, j\}$. By the 2-balancedness of $\mathcal{B}$ the collection $\{S \cap K \mid S \in \mathcal{B}\}$ is balanced on K. So, by Corollary 5.12 there exists a set $T' \in \mathcal{B}$ such that $j \in T'$ and $k \notin T'$. Since $\mathcal{B}$ only contains feasible sets and singletons, and k is a superior of j, it follows that $T' = \{j\}$. Let

$$S_k = \cap_{\{S \in \mathcal{B} | k \in S\}} S, \quad k \in W_\Omega.$$

From above it follows that $\{j\} \in \mathcal{B}$ for every $j \notin \cup_{k \in W_\Omega} S_k$. Now, let

$$\mathcal{B}' = \{U \in \mathcal{B} \mid U \cap W_\Omega \neq \emptyset\}$$

and consider the collection of subsets of W_Ω given by

$$\mathcal{B}'' = \{W_\Omega \cap U \mid U \in \mathcal{B}'\}.$$

This is a balanced collection on W_Ω. This is trivial when $c = 1$ and follows by the 2-balancedness of $\mathcal{B}$ when $c = 2$. When $c = 3$ this follows from the fact that every 2-balanced collection on a three player set is balanced. So, for the sets $U \in \mathcal{B}'$ there are weights, say $\lambda_U^{\mathcal{B}}$, such that

$$\sum_{\{U \in \mathcal{B}' | k \in U\}} \lambda_U^{\mathcal{B}} = 1, \quad k \in W_\Omega.$$

Since every feasible set has a nonempty intersection with W_Ω, this yields weight $\lambda_U^{\mathcal{B}} > 0$ for every feasible set $U \in \mathcal{B}$. Moreover,

$$\sum_{\{U \in \mathcal{B}' | j \in U\}} \lambda_U^{\mathcal{B}} = 1, \quad \text{for every } j \in \cup_{k=1,\ldots,c} S_k,$$

since if $j \in S_k$ for some $k = 1, \ldots, c$, then the collection of sets from $\mathcal{B}'$ containing j coincides with the collection of sets from $\mathcal{B}$' containing k. Finally, consider some $j \in N \setminus (\cup_{k=1,\ldots,c} S_k)$. Recall that such a player j has at least one of the players from the set W_Ω as one of its superiors, say player k. So, when j is contained in some set $U \in \mathcal{B}'$, then also $k \in U$. Moreover, there exists at least one $U \in \mathcal{B}'$ containing k and not j, otherwise $j \in S_k$. Therefore,

$$\sum_{\{U \in \mathcal{B}' | j \in U\}} \lambda_U^{\mathcal{B}} < 1 \text{ for every } j \in N \setminus (\cup_{k=1,\ldots,c} S_k),$$

i.e., the total weight of the feasible sets containing such a player j is less than one. However, for every such j we also have that the singleton $\{j\} \in \mathcal{B}$. This yields weight $\lambda_{\{j\}}^{\mathcal{B}} = 1 - \sum_{\{U \in \mathcal{B}' | j \in U\}} \lambda_U^{\mathcal{B}}$ for every singleton set $\{j\} \in \mathcal{B}$, $j \in N \setminus (\cup_{k=1,\ldots,c} S_k)$. Since for every $j \in \cup_{k=1,\ldots,c} S_k$, every set in $\mathcal{B}$ containing j also contains one of the players from $\{1, \ldots, c\}$, there are no other singletons in $\mathcal{B}$. So, we have determined weights for all sets in $\mathcal{B}$ satisfying that

$$\sum_{\{S \in \mathcal{B} | j \in S\}} \lambda_S^{\mathcal{B}} = 1, \text{ for every } j \in N,$$

and thus $\mathcal{B}$ is balanced. □

We are now ready to formulate the main result of this section.

Theorem 5.14 *Let (v, Ω) be a monotone game on a union closed system. Then the intersection of $PK(v, \Omega)$ and $C(v, \Omega)$ consists of at most one point if the weakly free player cardinality of Ω is at most equal to three.*

Proof. Clearly, the statement of the theorem is true when $C(v, \Omega) = \emptyset$. So, we only consider the case that $C(v, \Omega) \neq \emptyset$. Then $PN(v, \Omega) = Nuc(v, \Omega)$ and lies in the core. Suppose there is a payoff vector $y \in PK(v, \Omega) \cap C(v, \Omega)$ with $y \neq x = Nuc(v, \Omega)$. Since $y \neq PN(v, \Omega)$, according to Kohlberg's theorem there is some α for which $\mathcal{B}(\alpha, y)$ is not balanced. Since $x = PN(v, \Omega)$, also according to Kohlberg's theorem we have that $\mathcal{B}(\alpha, x)$ is balanced and thus $\mathcal{B}(\alpha, x) \neq \mathcal{B}(\alpha, y)$. Since for α big enough we have that $\mathcal{B}(\alpha, x) = \mathcal{B}(\alpha, y) = \emptyset$, there exists some value α with the properties that

(i) $\mathcal{B}(\alpha, x) \neq \mathcal{B}(\alpha, y)$ and

(ii) for every $\beta > \alpha$ it is true that either $\mathcal{B}(\beta, x) = \mathcal{B}(\beta, y)$ or both $\mathcal{B}(\alpha, x) = \mathcal{B}(\beta, x)$ and $\mathcal{B}(\alpha, y) = \mathcal{B}(\beta, y)$.

For a coalition S and payoff vector x, let $e(S, x) = r_{v,\Omega}(S) - x(S) = v(\sigma_\Omega(S)) - x(S)$ be the excess of coalition S at x in the restricted game $r_{v,\Omega}$ and let α^* be a value satisfying the two properties (i) and (ii). Now, suppose that there exists $S \in \mathcal{B}(\alpha^*, x)$ such that

$e(S,x) < e(S,y)$. Then, for $\beta = e(S,y) > e(S,x) \geq \alpha^*$, we have that $S \in \mathcal{B}(\beta,y)$ and $S \notin \mathcal{B}(\beta,x)$. So, $\mathcal{B}(\beta,x) \neq \mathcal{B}(\beta,y)$ and $\mathcal{B}(\alpha^*,x) \neq \mathcal{B}(\beta,x)$, which contradicts that property (ii) holds for α^*. Hence

$$e(S,x) \geq e(S,y) \text{ for every } S \in \mathcal{B}(\alpha^*,x). \tag{5.3}$$

Further, for $S \in \mathcal{B}(\alpha^*,x)$, let λ_S be the weight of S in the balanced system of collection $\mathcal{B}(\alpha^*,x)$. Since both x and y are efficient, it follows that

$$\sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S e(S,x) = \sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S(r_{v,\Omega}(S) - x(S)).$$

Since x is efficient and by balancedness we have that $\sum_{\{S\in\mathcal{B}(\alpha^*,x)|i\in S\}} \lambda_S = 1$ for every $i \in N$, it follows that

$$\sum_{S\in\mathcal{B}(\alpha^*,x)} \lambda_S x(S) = \sum_{S\in\mathcal{B}(\alpha^*,x)} \lambda_S \sum_{i\in S} x_i = \sum_{i\in N} x_i \sum_{\{S\in\mathcal{B}(\alpha^*,x)|i\in S\}} \lambda_S = \sum_{i\in N} x_i = r_{v,\Omega}(N),$$

and thus

$$\sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S e(S,x) = \sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S(r_{v,\Omega}(S) - x(S)).$$

$$= \sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S r_{v,\Omega}(S) - r_{v,\Omega}(N).$$

Analogously

$$\sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S e(S,y) = \sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S r_{v,\Omega}(S) - r_{v,\Omega}(N).$$

So,

$$\sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S e(S,x) = \sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S e(S,y).$$

With inequalities (5.3) this implies $e(S,y) = e(S,x)$ for every $S \in \mathcal{B}(\alpha^*,x)$ and thus $\mathcal{B}(\alpha^*,x) \subseteq \mathcal{B}(\alpha^*,y)$.

Now, suppose that also the collection $B(\alpha^*,y)$ is balanced. Then by the same reasoning as above we obtain that $e(S,x) = e(S,y)$ for every $S \in \mathcal{B}(\alpha^*,y)$ and thus also $\mathcal{B}(\alpha^*,y) \subseteq \mathcal{B}(\alpha^*,x)$, which contradicts that $\mathcal{B}(\alpha^*,x) \neq \mathcal{B}(\alpha^*,y)$. Hence $B(\alpha^*,y)$ is not balanced.

On the other hand, by Theorem 5.11 we have that $\mathcal{B}(\alpha^*,y)$ is 2-balanced, since $y \in PK(v,\Omega)$. So, $\mathcal{B}(\alpha^*,y)$ is 2-balanced, but not balanced. Then, according to Lemma 5.13, $\mathcal{B}(\alpha^*,y)$ contains a non-feasible coalition S with $|S| > 1$. By definition of $\sigma_\Omega(S)$ and

Ω being union closed, we have that $r_{v,\Omega}(T) = 0$ for every $T \subseteq S \setminus \sigma_\Omega(S)$. Then for every $i \in S \setminus \sigma_\Omega(S)$ it follows that

$$e(S, y) = r_{v,\Omega}(S) - y(S) = r_{v,\Omega}(\sigma_\Omega(S)) - \sum_{j \in \sigma_\Omega(S)} y_j - \sum_{h \in S \setminus \sigma_\Omega(S)} y_h \leq$$

$$e(\sigma_\Omega(S), y) - y_i = e(\sigma_\Omega(S), y) + e(\{i\}, y),$$

because $y \in C(r_{v,\Omega})$ and thus $y_h \geq r_{v,\Omega}(\{h\}) = 0$ for all $h \in S \setminus \sigma_\Omega(S)$. Since both $e(\sigma_\Omega(S), y) \leq 0$ and $e(\{i\}, y) \leq 0$ (again because $y \in C(r_{v,\Omega})$), it follows that

$$e(S, y) \leq e(\sigma_\Omega(S), y) \text{ and } e(S, y) \leq e(\{i\}), y).$$

Hence both $\sigma(S) \in \mathcal{B}(\alpha^*, y)$ and $\{i\} \in \mathcal{B}(\alpha^*, y)$ for every $i \in S \setminus \sigma_\Omega(S)$. However, then also the collection $\mathcal{B}(\alpha^*, y) \setminus \{S\}$ is 2-balanced and not balanced. Let $NF = \{T \in \mathcal{B}(\alpha^*, y) \mid T$ is non-feasible and $|T| > 1\}$. Repeating the reasoning above for every $T \in NF$ it follows that $\mathcal{B}' = \mathcal{B}(\alpha^*, y) \setminus NF$ is 2-balanced and not balanced. However, since $\mathcal{B}'$ only consists of feasible sets and singletons, this contradicts Lemma 5.13. So, there is no $y \in PK(v, \Omega) \cap C(v, \Omega)$ with $y \neq x = Nuc(v, \Omega)$. □

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