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The Non-symmetric Discrete Algebraic Riccati Equation and Canonical Factorization of Rational Matrix Functions on the Unit Circle

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Abstract. Canonical factorization of a rational matrix function on the unit circle is described explicitly in terms of a stabilizing solution of a discrete algebraic Riccati equation using a special state space representation of the symbol. The corresponding Riccati difference equation is also discussed.

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1. Introduction

Throughout R is a rational $m \times m$ matrix function with no poles on the unit circle $\mathbb{T}$, and we write $R(z) = \sum_{j=-\infty}^{\infty} z^j R_j$ for the Laurent expansion of R on $\mathbb{T}$. The corresponding (block) Toeplitz operator on $\ell_+^2(\mathbb{C}^m)$ is denoted by T, that is,

$$T = \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots \\ R_1 & R_0 & R_{-1} & \cdots \\ R_2 & R_1 & R_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ on } \ell_+^2(\mathbb{C}^m). \tag{1.1}$$

Recall (see, e.g., [9], Section XXIV.3) that R is said to admit a *right canonical factorization* (with respect to $\mathbb{T}$) if R can be factored as

$$R(z) = \Psi(z)\Theta(z), \quad z \in \mathbb{T}, \tag{1.2}$$

where Θ and Ψ are regular $m \times m$ matrix rational functions such that $\Theta(z)$ and $\Theta(z)^{-1}$ have no poles in $\{z \in \mathbb{C} : |z| \leq 1\}$, while $\Psi(z)$ and $\Psi(z)^{-1}$ have no poles in $\{z \in \mathbb{C} : |z| \geq 1\}$ infinity included. It is well-known that R admits such a factorization if and only if the Toeplitz operator T is invertible. Moreover, in that case, $T^{-1} = T_{\Theta^{-1}} T_{\Psi^{-1}}$, where $T_{\Theta^{-1}}$ and $T_{\Psi^{-1}}$ are the Toeplitz operators defined by $\Theta(z)^{-1}$ and $\Psi(z)^{-1}$, respectively.

In this paper we analyze canonical factorization of R using a special state space representation, namely

$$R(z) = R_0 + zC(I - zA)^{-1}B + \gamma\,(zI - \alpha)^{-1}\,\beta. \tag{1.3}$$

Here A and α are square matrices of sizes $n \times n$ and $\nu \times \nu$, say, which are assumed to be stable, that is, the eigenvalues of A and α are contained in the open unit disc. The I's in (1.3) stand for identity matrices of appropriate sizes, and B, C, β, γ in (1.3) are matrices, again of appropriate sizes. We shall refer to (1.3) as a *stable representation* of R.

With the representation (1.3) we associate the algebraic Riccati equation

$$Q = \alpha Q A + (\beta - \alpha Q B)(R_0 - \gamma Q B)^{-1}(C - \gamma Q A). \tag{1.4}$$

We say that Q is a *stabilizing solution* to this Riccati equation if the matrix $R_0 - \gamma Q B$ is invertible, Q is a solution to (1.4), and the matrices

$$A_\circ = A - B(R_0 - \gamma Q B)^{-1}(C - \gamma Q A) \tag{1.5}$$

$$\alpha_\circ = \alpha - (\beta - \alpha Q B)(R_0 - \gamma Q B)^{-1}\gamma \tag{1.6}$$

are both stable. We are now ready to state the main result of this note.

Theorem 1.1. *Let R be an $m \times m$ rational matrix function with no poles on the circle, and let* (1.3) *be a stable representation for R. Then R admits a right canonical factorization with respect to the unit circle if and only if the algebraic Riccati equation* (1.4) *has a stabilizing solution Q, and in that case a canonical factorization $R(z) = \Psi(z)\Theta(z)$ is obtained by taking*

$$\Theta(z) = D + zC_\circ(I - zA)^{-1}B, \quad \Psi(z) = \delta + \gamma(zI - \alpha)^{-1}\beta_\circ. \tag{1.7}$$

Here

$$C_\circ = \delta^{-1}(C - \gamma Q A), \quad \beta_\circ = (\beta - \alpha Q B)D^{-1}, \tag{1.8}$$

and δ and D are any invertible matrices satisfying $\delta D = R_0 - \gamma Q B$. Moreover, the inverses of the factors are given by

$$\Theta(z)^{-1} = D^{-1} - zD^{-1}C_\circ(I - zA_\circ)^{-1}BD^{-1} \tag{1.9}$$

$$\Psi(z)^{-1} = \delta^{-1} - \delta^{-1}\gamma(zI - \alpha_\circ)^{-1}\beta_\circ\delta^{-1}, \tag{1.10}$$

where $A_\circ$ and $\alpha_\circ$ are defined by (1.5) *and* (1.6)*, respectively.*

Finally, if (1.4) *has a stabilizing solution, then this solution is unique and given by*

$$Q = \begin{bmatrix} \beta & \alpha\beta & \alpha^2\beta & \cdots \end{bmatrix} T^{-1} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}, \tag{1.11}$$

where T is the block Toeplitz operator (1.1) *defined by R.*

In Section 2 below we prove Theorem 1.1 and specify this theorem for the case when the Toeplitz operator T is tri-diagonal and for the case when R has no poles on the closed unit disc.

Theorem 1.1 is of particular interest for the case when the values of R on the unit circle are hermitian matrices. In that case, one takes $\alpha = A^*$, $\beta = C^*$, and $\gamma = B^*$. The corresponding Riccati equation is then given by

$$Q = A^*QA + (C^* - A^*QB)(R_0 - B^*QB)^{-1}(C - B^*QA), \tag{1.12}$$

and the rational matrix function $R(z)$ plays the role of the so-called Popov function or spectral density. This symmetric algebraic Riccati equation originates from stochastic realization theory (see the books [3], [5], [6], [12]), and for this symmetric case Theorem 1.1 is basically known. See, for instance, [15] where $R(z)$ is positive definite on the unit circle and the resulting canonical factorization is a spectral factorization (cf., Appendix A1 in [7]). When $R(z)$ is just hermitian on the unit circle symmetric canonical factorization is usually referred to as a J-spectral factorization, and for this type of factorization the relation with the stabilizing solution of (1.12) is covered by Lemma 12.4.1 (iv) in [11]. The existence of a (unique) stabilizing solution of the symmetric algebraic Riccati equation and its connection with the invertibility of the Toeplitz operator can be found in Section 4.7 of [13]. For further references on the symmetric algebraic Riccati equation and a more detailed description of the history of the subject, see the books [3], [11], [13], and [14].

For the non-symmetric case Theorem 1.1 seems to be new. Our proof of Theorem 1.1, which is self-contained, is based on a Schur complement argument as in [6]. This proof and formula (1.11) for the stabilizing solution also may be of interest for the symmetric case.

We see Theorem 1.1 as an addition to [10], where canonical factorization of R and invertibility of T are described explicitly in terms of a different state space representation, namely $R(z) = I + C(zG - A)^{-1}B$, where $zG - A$ is a square matrix pencil which is invertible for z on the unit circle. In [10] canonical factorization is obtained by matching of spectral subspaces of the pencils $zG - A$ and $zG - (A - BC)$ (see also Chapter 2 of [1] for the special case when $G = I$). In Section 3 below we reconsider the example discussed in Section 10 of [10] (cf., Section XXIV.10 in [9]), and we use this example to illustrate Theorem 1.1.

Solving (1.4) by iteration leads to the Riccati difference equation

$$Q_{N+1} = \alpha Q_N A + (\beta - \alpha Q_N B)(R_0 - \gamma Q_N B)^{-1}(C - \gamma Q_N A).$$

As one may expect from formula (1.11) for the stabilizing solution, solving this equation is closely related to inverting Toeplitz operators by the finite section method. This connection is the main topic of Section 4, the final section of the paper.

Other solutions of (1.4), not just the stabilizing ones, are also of interest. For instance, if Q is an arbitrary solution of (1.4), then the rational matrix functions Θ and Ψ defined by (1.7) and (1.8) are analytic on the closed unit disc and $R(z) = \Psi(z)\Theta(z)$. Moreover, this factorization is a so-called pseudo-canonical factorization (see [16] or Section 9.2 in [2]) of R if, in addition, the matrices $A_\circ$ and $\alpha_\circ$ defined by (1.5) and (1.6) have eigenvalues only in the closed unit disc. Note that in this case (1.11) does not hold, and to prove that conversely any pseudo-canonical factorization of R is obtained in this way one has to require additional minimality conditions on the representation (1.3). We will come back to this in a future publication.

Finally, we mention that canonical factorization of rational matrix functions with respect to the real line or the imaginary axis and its connection with continuous time algebraic Riccati equations is well understood; see, e.g., Chapter 5 in [2], and Chapter 19 in [14], and the references in these books.

2. Proof of Theorem 1.1

Let T be the (block) Toeplitz operator on $\ell^2_+(\mathbb{C}^m)$ defined in (1.1). The Toeplitz structure of T allows us to partition T as a 2×2 operator matrix

$$T = \begin{bmatrix} R_0 & \Gamma \\ \Xi & T \end{bmatrix} \text{ on } \begin{bmatrix} \mathbb{C}^m \\ \ell^2_+(\mathbb{C}^m) \end{bmatrix}. \tag{2.1}$$

Here Γ is the row operator and Ξ is the column operator defined by

$$\Gamma = \begin{bmatrix} R_{-1} & R_{-2} & \cdots \end{bmatrix} : \ell^2_+(\mathbb{C}^m) \to \mathbb{C}^m, \quad \Xi = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \end{bmatrix} : \mathbb{C}^m \to \ell^2_+(\mathbb{C}^m).$$

If T is an invertible operator on $\ell^2_+(\mathbb{C}^m)$, then (see, e.g., pages 28, 29 in [2]) the Schur complement $\Delta = R_0 - \Gamma T^{-1}\Xi$ is a well-defined invertible operator on $\mathbb{C}^m$. Moreover, the inverse of T admits the block matrix representation

$$T^{-1} = \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}\Gamma T^{-1} \\ -T^{-1}\Xi\Delta^{-1} & T^{-1} + T^{-1}\Xi\Delta^{-1}\Gamma T^{-1} \end{bmatrix} \text{ on } \begin{bmatrix} \mathbb{C}^m \\ \ell^2_+(\mathbb{C}^m) \end{bmatrix}. \tag{2.2}$$

This yields the following useful result.

Lemma 2.1. *Assume that T is an invertible Toeplitz operator on $\ell^2_+(\mathbb{C}^m)$, and let $\Delta = R_0 - \Gamma T^{-1}\Xi$ be the corresponding Schur complement. Then the following identities hold:*

$$T^{-1} = ST^{-1}S^* + (E - ST^{-1}\Xi)(R_0 - \Gamma T^{-1}\Xi)^{-1}(E^* - \Gamma T^{-1}S^*), \tag{2.3}$$

$$S^*T^{-1} = T^{-1}S^* - T^{-1}\Xi\Delta^{-1}(E^* - \Gamma T^{-1}S^*), \tag{2.4}$$

$$T^{-1}S = ST^{-1} - (E - ST^{-1}\Xi)\Delta^{-1}\Gamma T^{-1}. \tag{2.5}$$

Here E denotes the canonical embedding of $\mathbb{C}^m$ onto the first coordinate space of $\ell^2_+(\mathbb{C}^\nu)$, and S is the block forward shift on $\ell^2_+(\mathbb{C}^m)$.

Note that the identity in (2.3) can be viewed as an algebraic Riccati equation with T^{-1} as the solution. We shall see below (in Part 1 of the proof of Theorem 1.1) that equation (1.4) follows from (2.3) in a straightforward way.

Proof. A simple computation shows that

$$ST^{-1}S^* = \begin{bmatrix} 0 & 0 \\ 0 & T^{-1} \end{bmatrix}.$$

This identity, together with the identity (2.2), yields

$$\begin{aligned} T^{-1} - ST^{-1}S^* &= \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}\Gamma T^{-1} \\ -T^{-1}\Xi\Delta^{-1} & T^{-1}\Xi\Delta^{-1}\Gamma T^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I \\ -T^{-1}\Xi \end{bmatrix} \Delta^{-1} \begin{bmatrix} I & -\Gamma T^{-1} \end{bmatrix}. \end{aligned} \tag{2.6}$$

Next observe that

$$\begin{bmatrix} I \\ -T^{-1}\Xi \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ T^{-1}\Xi \end{bmatrix} = E - ST^{-1}\Xi, \tag{2.7}$$

$$\begin{bmatrix} I & -\Gamma T^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} - \begin{bmatrix} 0 & \Gamma T^{-1} \end{bmatrix} = E^* - \Gamma T^{-1}S^*. \tag{2.8}$$

Using the latter identities in (2.6), we obtain

$$T^{-1} = ST^{-1}S^* + (E - ST^{-1}\Xi)\Delta^{-1}(E^* - \Gamma T^{-1}S^*). \tag{2.9}$$

This is precisely equation (2.5). Multiplying (2.3) by S^* on the left yields (2.4). Likewise multiplying (2.3) by S on the right gives (2.5). $\square$

Proof of Theorem 1.1. The proof is broken up into four parts. In the first two parts we assume that R admits a right canonical factorization, or equivalently, that T is invertible, and we show that the matrix Q defined by (1.11) is a stabilizing solution to the algebraic Riccati equation (1.4). In the third part we start from a stabilizing solution to (1.4) and derive the desired canonical factorization. The final part deals with uniqueness statement.

Part 1. Assume that T is invertible, and let Q be the matrix defined by (1.11). In this part we show that Q is a solution to (1.4). Notice that Q can be written $Q = \omega T^{-1} W$, where W is the observability and ω is the controllability operator defined by

$$W = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} : \mathbb{C}^n \to \ell^2_+(\mathbb{C}^m),$$

$$\omega = \begin{bmatrix} \beta & \alpha\beta & \alpha^2\beta & \cdots \end{bmatrix} \ell^2_+(\mathbb{C}^m) \to \mathbb{C}^\nu.$$

By comparing (1.1) with (2.1), and using the representation (1.3), we obtain

$$\Xi = WB, \quad C = E^*W \quad \text{and} \quad S^*W = WA, \tag{2.10}$$

$$\Gamma = \gamma\varpi, \quad \beta = \omega E \quad \text{and} \quad \omega S = \alpha\varpi. \tag{2.11}$$

Using the first identities in (2.10) and (2.11) together with $Q = \omega T^{-1}W$, we see that

$$\Delta = R_0 - \Gamma T^{-1}\Xi = R_0 - \gamma\omega T^{-1}WB = R_0 - \gamma QB. \tag{2.12}$$

Furthermore, the identities in (2.10) and (2.11) yield

$$(E^* - \Gamma T^{-1}S^*)W = C - \gamma\omega T^{-1}WA = C - \gamma QA, \tag{2.13}$$

$$\omega(E - ST^{-1}\Xi) = \beta - \alpha\omega T^{-1}WB = \beta - \alpha QB. \tag{2.14}$$

Now multiplying equation (2.3) on the left by ω and on the right by W, we obtain that $Q = \omega T^{-1}W$ is a solution to the algebraic Riccati equation (1.4); here we used (2.12), (2.13) and (2.14).

Part 2. As in the previous part $Q = \omega T^{-1}W$. In this part we show that for this choice of Q the matrices $A_\circ$ and $\alpha_\circ$ defined by (1.5) and (1.6) are stable matrices. In fact, we shall only prove that $A_\circ$ is stable. The proof of the stability of $\alpha_\circ$ can be obtained in the same way using a duality argument.

First we note that $S^*T^{-1}W = T^{-1}WA_\circ$. This identity follows from (2.4) together with (2.10) and (2.13). Indeed, we have

$$\begin{aligned} S^*T^{-1}W &= T^{-1}\left(S^* - \Xi\Delta^{-1}(E^* - \Gamma T^{-1}S^*)\right)W \\ &= T^{-1}\left(WA - WB\Delta^{-1}(C - \gamma QA\right) = T^{-1}WA_\circ. \end{aligned}$$

Next, we decompose $\mathbb{C}^n$ as $\mathbb{C}^n = \mathcal{X}_1 \oplus \mathcal{X}_2$, where $\mathcal{X}_2 = \operatorname{Ker} W$ and $\mathcal{X}_1 = \mathbb{C}^n \ominus \operatorname{Ker} W$. Notice that $\mathcal{X}_2$ is an invariant subspace for A, and $C|\mathcal{X}_2 = 0$. We claim that $QA|\mathcal{X}_2 = 0$. This follows from the fact that

$$QA\mathcal{X}_2 \subseteq Q\mathcal{X}_2 = \omega T^{-1}W\mathcal{X}_2 = \{0\}.$$

By using $C|\mathcal{X}_2 = 0$ and $QA|\mathcal{X}_2 = 0$ in (1.5), we see that $A_\circ|\mathcal{X}_2 = A|\mathcal{X}_2$ and $\mathcal{X}_2$ is also an invariant subspace for $A_\circ$. In other words, $A_\circ$ admits a matrix representation of the form

$$A_\circ = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \tag{2.15}$$

where $A_{22} = A|\mathcal{X}_2$ on $\mathcal{X}_2$. Since $\mathcal{X}_2$ is an invariant subspace for A and A is stable, A_{22} is also stable.

Let E_1 be the natural embedding of $\mathcal{X}_1$ into $\mathbb{C}^n = \mathcal{X}_1 \oplus \mathcal{X}_2$. Let W_1 be the one to one operator defined by $W_1 = WE_1$ mapping $\mathcal{X}_1$ into $\ell^2_+(\mathbb{C}^m)$. Using $S^*T^{-1}W = T^{-1}WA_\circ$ with $A_{11} = E_1^*A_\circ E_1$, we obtain

$$\begin{aligned} S^*T^{-1}W_1 &= S^*T^{-1}WE_1 = T^{-1}WA_\circ E_1 \\ &= T^{-1}W_1E_1A_\circ E_1 = T^{-1}W_1A_{11}. \end{aligned}$$

In other words, $S^*T^{-1}W_1 = T^{-1}W_1A_{11}$. Because W_1 is one to one, $T^{-1}W_1$ is also one to one. Notice that $S^{*n}T^{-1}W_1 = T^{-1}W_1A_{11}^n$ for all integers $n \geq 0$. Since S^{*n} converges to zero in the strong operator topology and A_{11} acts on a finite dimensional space, A_{11}^n converges to zero. Therefore A_{11} is stable. Recall that $A_{22} = A|\mathcal{X}_2$ is stable. By consulting the matrix representation for $A_\circ$ in (2.15), we see that $A_\circ$ is stable.

PART 3. In this part Q is a stabilizing solution of (1.4), and we derive the desired canonical factorization. Let $\Theta(z)$ and $\Psi(z)$ be the rational $m \times m$ matrix functions defined by (1.7) and (1.8). First we prove that $R(z) = \Psi(z)\Theta(z)$. Note that

$$C - \gamma QA = \delta C_\circ, \quad \beta - \alpha QB = \beta_\circ D, \quad R_0 - \gamma QB = \delta D. \tag{2.16}$$

Using these identities we see that the Riccati equation (1.4) can be rewritten as a Stein equation (a discrete Lyapunov equation), namely

$$Q = \alpha QA + \beta_\circ C_\circ. \tag{2.17}$$

From the identity (2.17) we see that

$$z\beta_\circ C_\circ = z(Q - \alpha QA) = (zI - \alpha)Q - (zI - \alpha)Q(I - zA) + zQ(I - zA).$$

It follows that

$$\begin{aligned}
&\gamma(zI - \alpha)^{-1}(z\beta_\circ C_\circ)(I - zA)^{-1}B \\
&= \gamma Q(I - zA)^{-1}B - \gamma QB + z\gamma(zI - \alpha)^{-1}QB \\
&= z\gamma QA(I - zA)^{-1}B + \gamma QB + \gamma(zI - \alpha)^{-1}\alpha QB,
\end{aligned}$$

and hence

$$\begin{aligned}
\Psi(z)\Theta(z) &= \big(\delta + \gamma(zI - \alpha)^{-1}\beta_\circ\big)\big(D + zC_\circ(I - zA)^{-1}B\big) \\
&= \delta D + \gamma(zI - \alpha)^{-1}\beta_\circ D + z\delta C_\circ(I - zA)^{-1}B \\
&\quad + \gamma(zI - \alpha)^{-1}(z\beta_\circ C_\circ)(I - zA)^{-1}B \\
&= (\delta D + \gamma QB) + \gamma(zI - \alpha)^{-1}(\beta_\circ D + \alpha QB) + \\
&\quad + z(\gamma QA + \delta C_\circ)(I - zA)^{-1}B.
\end{aligned}$$

From the third identity in (2.16) we see that $\delta D + \gamma QB = R_0$, the second identity in (2.16) yields $\beta_\circ D + \alpha QB = \beta$, and the first identity in (2.16) shows that $\gamma QA + \delta C_\circ = C$. But then we see that $\Psi(z)\Theta(z)$ is equal to the right hand side of (1.3), that is, $R(z) = \Psi(z)\Theta(z)$. It remains to show that this factorization is a right canonical one, i.e., we have to show (see, e.g., [9], Section XXIV.3) that $\Theta(z)$ and $\Theta(z)^{-1}$ have no poles in $\{z \in \mathbb{C} : |z| \le 1\}$, while $\Psi(z)$ and $\Psi(z)^{-1}$ have no poles in $\{z \in \mathbb{C} : |z| \ge 1\}$ infinity included. But these properties follow directly from the stability of the matrices A, α, $A_\circ$, and $\alpha_\circ$. Thus $R(z) = \Psi(z)\Theta(z)$ is a right canonical factorization of R relative to the unit circle.

PART 4. In this part we prove the uniqueness of the stabilizing solution. In fact, we show that the stabilizing solution to (1.4) is given by (1.11). So let Q be a stabilizing solution to (1.4). By the result of the previous part, R admits a right canonical factorization $R(z) = \Psi(z)\Theta(z)$, where Θ and Ψ are given by (1.7). It follows that the block Toeplitz operator T defined by R is invertible. Moreover, its inverse can be expressed explicitly in terms of the Taylor coefficients $\Theta_0^\times, \Theta_1^\times, \Theta_2^\times, \ldots$ of $\Theta(z)^{-1}$ at zero and the Taylor coefficients $\Psi_0^\times, \Psi_1^\times, \Psi_2^\times, \ldots$ of $\Psi(z)^{-1}$ at infinity. In fact (see Theorem XXIV.4.1

in [9]), we have $T^{-1} = T_{\Theta^{-1}} T_{\Psi^{-1}}$, where

$$T_{\Theta^{-1}} = \begin{bmatrix} \Theta_0^\times & 0 & 0 & \cdots \\ \Theta_1^\times & \Theta_0^\times & 0 & \cdots \\ \Theta_2^\times & \Theta_1^\times & \Theta_0^\times & \\ \vdots & \vdots & & \ddots \end{bmatrix}, \quad T_{\Psi^{-1}} = \begin{bmatrix} \Psi_0^\times & \Psi_1^\times & \Psi_2^\times & \cdots \\ 0 & \Psi_0^\times & \Psi_1^\times & \cdots \\ 0 & 0 & \Psi_0^\times & \\ \vdots & \vdots & & \ddots \end{bmatrix}. \tag{2.18}$$

As $\Theta(z)^{-1}$ and $\Psi(z)^{-1}$ are given by (1.9) and (1.10), we see that

$$\Theta_0^\times = D^{-1}, \quad \Theta_j^\times = -D^{-1} C_\circ A_\circ^{j-1} B D^{-1} \quad (j = 1, 2, \ldots),$$
$$\Psi_0^\times = \delta^{-1}, \quad \Psi_j^\times = -\delta^{-1} \gamma \alpha_\circ^{j-1} \beta_\circ \delta^{-1} \quad (j = 1, 2, \ldots).$$

Using these identities and (2.18) we have

$$\omega T_{\Theta^{-1}} = \begin{bmatrix} \widetilde{\beta} & \alpha\widetilde{\beta} & \alpha^2\widetilde{\beta} & \cdots \end{bmatrix}, \quad \text{where}$$

$$\widetilde{\beta} = \beta D^{-1} - \alpha \left(\sum_{j=0}^{\infty} \alpha^j \beta D^{-1} C_\circ A_\circ^j \right) B D^{-1}, \tag{2.19}$$

and

$$T_{\Psi^{-1}} W = \begin{bmatrix} \widetilde{C} \\ \widetilde{C} A \\ \widetilde{C} A^2 \\ \vdots \end{bmatrix}, \quad \text{where}$$

$$\widetilde{C} = \delta^{-1} C - \delta^{-1} \gamma \left(\sum_{j=0}^{\infty} \alpha_\circ^j \beta_\circ \delta^{-1} C A^j \right) A. \tag{2.20}$$

We shall prove that $\widetilde{\beta} = \beta_\circ$ and $\widetilde{C} = C_\circ$. To do this set $\Delta = R_0 - \gamma Q B$. Because Q is a solution to the algebraic Riccati equation (1.4) and $\delta D = \Delta$, we have

$$\begin{aligned} Q &= \alpha Q A + (\beta - \alpha Q B)(R_0 - \gamma Q B)^{-1}(C - \gamma Q A) \\ &= \alpha Q(A - B\Delta^{-1}(C - \gamma Q A)) + \beta \Delta^{-1}(C - \gamma Q A) \\ &= \alpha Q A_\circ + \beta D^{-1} C_\circ. \end{aligned}$$

In other words, $Q = \alpha Q A_\circ + \beta D^{-1} C_\circ$. Since α and $A_\circ$ are both stable, $Q = \sum_{j=0}^{\infty} \alpha^j \beta D^{-1} C_\circ A_\circ^n$. So according to (2.19) and the second identity in (1.8), we see that $\widetilde{\beta} = \beta D^{-1} - \alpha Q B D^{-1} = (\beta - \alpha Q B) D^{-1} = \beta_\circ$. Analogously,

$$\begin{aligned} Q &= \alpha Q A + (\beta - \alpha Q B)\Delta^{-1}(C - \gamma Q A) \\ &= \left(\alpha - (\beta - \alpha Q B)\Delta^{-1}\gamma\right) Q A + (\beta - \alpha Q B)\Delta^{-1} C \\ &= \alpha_\circ Q A + \beta_\circ \delta^{-1} C. \end{aligned}$$

In other words, $Q = \alpha_\circ Q A + \beta_\circ \delta^{-1} C$. Since $\alpha_\circ$ and A are both stable, $Q = \sum_{j=0}^{\infty} \alpha_\circ^j \beta_\circ \delta^{-1} C A^j$. So according to (2.20) and the first identity in (1.8), we see that $C_\circ = \widetilde{C}$.

To complete the proof, recall that Q also satisfies the Lyapunov equation

$$Q = \alpha Q A + \beta_\circ C_\circ.$$

Hence $Q = \sum_{j=0}^{\infty} \alpha^j \beta_\circ C_\circ A^j$. By consulting (2.19) and (2.20) with $\widetilde{\beta} = \beta_\circ$ and $\widetilde{C} = C_\circ$, we obtain

$$\omega T^{-1} W = \omega T_{\Theta^{-1}} T_{\Psi^{-1}} W = \sum_{j=0}^{\infty} \alpha^j \beta_\circ C_\circ A^j = Q.$$

Therefore $Q = \omega T^{-1} W$ and the stabilizing solution Q is unique. □

Next we specify Theorem 1.1 for the simple case when R has no poles on the closed unit disc $\overline{\mathbb{D}}$. In this case the block Toeplitz operator T on $\ell_+^2(\mathbb{C}^m)$ defined by R is block lower triangular, and R admits a representation of the form

$$R(z) = R_0 + zC(I - zA)^{-1}B \quad \text{with } A \text{ a stable } n \times n \text{ matrix.} \tag{2.21}$$

Notice that T defines an invertible operator on $\ell_+^2(\mathbb{C}^m)$ if and only if $\det R(z)$ had no zeros in $\overline{\mathbb{D}}$, or equivalently, R_0 is invertible and $A - BR_0^{-1}C$ is stable.

By choosing $\alpha = 0$ on the zero space $\{0\}$, and $\beta = 0$ from $\mathbb{C}^m$ into $\{0\}$, and $\gamma = 0$ from $\{0\}$ into $\mathbb{C}^m$, we see that (2.21) is of the form (1.3). In this case the corresponding Riccati equation (1.4) is just the equation $Q = 0$, where Q maps $\mathbb{C}^n$ into $\{0\}$. Hence $Q = 0$ is the only solution to (1.4). Moreover, $\alpha_\circ = 0$ and $A_\circ = A - BR_0^{-1}C$. So Q is a stabilizing solution if and only if $A - BR_0^{-1}C$ is stable, or equivalently, T is invertible. By choosing $D = R_0$ and $\delta = I$, we see that $R(z) = \Psi(z)\Theta(z)$ is a right canonical factorization with $\Theta = R$ and $\Psi = I$.

From the simple case considered in the previous paragraphs it already follows that the algebraic Riccati equation (1.4) may not have any stabilizing solution. For example, take R as in (2.21) above, with $A = 0$, $B = -2$, $C = 1$ and $R_0 = 1$. Then $Q = 0$ is the only solution to the corresponding Riccati equation (1.4) and $A_\circ = 2$ is unstable.

The next proposition is a corollary of Theorem 1.1 for the case when the Toeplitz operator is tri-diagonal.

Proposition 2.2. *Assume $R(z) = R_0 + zR_1 + z^{-1}R_{-1}$. Then R admits a right canonical factorization if and only if the equation*

$$Q = R_{-1}(R_0 - Q)^{-1}R_1 \tag{2.22}$$

has a solution Q with $R_0 - Q$ invertible and with

$$A_\circ = -(R_0 - Q)^{-1}R_1 \quad \textit{and} \quad \alpha_\circ = -R_{-1}(R_0 - Q)^{-1} \tag{2.23}$$

being stable matrices. In this case, take any δ and D invertible so that $\delta D = R_0 - Q$. Then the corresponding canonical factorization is given by $R(z) = \Psi(z)\Theta(z)$, where the factors Ψ and Θ and their inverses are determined by

$$\begin{aligned}\Theta(z) &= D + z\delta^{-1}R_1, \quad \Psi(z) = \delta + z^{-1}R_{-1}D^{-1},\\ \Theta(z)^{-1} &= D^{-1} + zA_\circ(I + zA_\circ)^{-1}D^{-1},\\ \Psi(z)^{-1} &= \delta^{-1} + \delta^{-1}(z - \alpha_\circ)^{-1}\alpha_\circ.\end{aligned} \tag{2.24}$$

Proof. To see this we simply set $A = \alpha = 0$ on $\mathbb{C}^m$, and $B = \gamma = I$ on $\mathbb{C}^m$. Moreover, take $C = R_1$ and $\beta = R_{-1}$. Then this proposition follows from Theorem 1.1. □

Put $Y = R_0 - Q$. Then (2.22) can be rewritten as

$$R_0 = Y + R_{-1}Y^{-1}R_1. \tag{2.25}$$

So we can also reformulate the above proposition in terms of Y rather than in terms of Q. The stabilizing solution in this reformulation is the one for which Y is invertible and $A_\circ = -Y^{-1}R_1$ and $\alpha_\circ = -R_{-1}Y^{-1}$ are stable matrices. With R_0 positive definite, and $R_{-1} = R_1^*$, equation (2.25) is studied in [4].

The case when R is a trigonometric polynomial, $R(z) = \sum_{j=-t}^{p} R_j$, is also of special interest. In this case we obtain a representation (1.3) by taking

$$A = \begin{bmatrix} 0 & I & & \\ & \ddots & \ddots & \\ & & 0 & I \\ & & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_p \end{bmatrix}, \quad C = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix},$$

$$\alpha = \begin{bmatrix} 0 & & & \\ I & 0 & & \\ & \ddots & \ddots & \\ & & I & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} R_{-1} & R{-2} & \cdots & R_{-t} \end{bmatrix}.$$

Using (1.11) one computes that for this special representation of R the solution Q of the correponding algebraic Riccati equation is just equal to the $p \times t$ block matrix in the top left corner of the operator T^{-1}.

3. An example

To illustrate how one can use Theorem 1.1, we reconsider the example analyzed in Section 10 of [10] (cf., Section XXIV.10 in [9]). Consider the rational matrix function

$$R(z) = \begin{bmatrix} 1 - z^{-1} & \frac{1}{2}z^{-1} \\ -3z & 1 + z \end{bmatrix}.$$

As in [10] we seek a canonical factorization of R with respect to the unit circle. For this R we have $R_0 = I$ and a stable representation is obtained by taking

$$A = 0,\ B = \begin{bmatrix} -3 & 1 \end{bmatrix},\ C = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \alpha = 0,\ \beta = \begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix},\ \gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In this setting, $Q = q$ is a scalar,

$$R_0 - \gamma q B = \begin{bmatrix} 1+3q & -q \\ 0 & 1 \end{bmatrix}, \quad (R_0 - \gamma q B)^{-1} = \begin{bmatrix} (1+3q)^{-1} & q(1+3q)^{-1} \\ 0 & 1 \end{bmatrix}.$$

The corresponding Riccati equation is now determined by the following scalar equation:

$$\begin{aligned} q = \beta(R_0 - \gamma q B)^{-1} C &= \begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} (1+3q)^{-1} & q(1+3q)^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} - \frac{q}{1+3q} = \frac{1+q}{2+6q}. \end{aligned}$$

Rewriting this leads to the quadratic equation $6q^2 + q - 1 = 0$. The zeros of this equation are $-1/2$ and $1/3$. Since A and α are zero, equations (1.5) and (1.6) yield

$$A_\circ = -B(R_0 - \gamma q B)^{-1} C \quad \text{and} \quad \alpha_\circ = -\beta(R_0 - \gamma q B)^{-1}\gamma.$$

A simple calculation shows that $A_\circ$ and $\alpha_\circ$ are both equal to $-1/(1+3q)$. So the stabilizing solution is obtained with $q = 1/3$, and $A_\circ = \alpha_\circ = -1/2$. We now take $\delta = I$ and $D = R_0 - \gamma q B$, that is

$$D = \begin{bmatrix} 2 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ 0 & 1 \end{bmatrix}.$$

Then we compute the factors $\Theta(z)$ and $\Psi(z)$ in (1.7):

$$\Theta(z) = \begin{bmatrix} 2 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{3} \\ -3z & 1+z \end{bmatrix},$$

and

$$\Psi(z) = I + \frac{1}{z} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix} D^{-1} = I + \frac{1}{z} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2z} & \frac{1}{3z} \\ 0 & 1 \end{bmatrix}.$$

This is exactly the canonical factorization of R derived in Section 10 of [10].

4. Riccati iteration and finite sections

Throughout this section R is a rational $m \times m$ matrix function given by the stable representation (1.3), and (1.4) is the corresponding Riccati equation.

Solving (1.4) by iteration leads to a Riccati difference equation (cf., Appendix A2 in [7]):

$$Q_{N+1} = \alpha Q_N A + (\beta - \alpha Q_N B)(R_0 - \gamma Q_N B)^{-1}(C - \gamma Q_N A). \tag{4.1}$$

Let us assume that starting from an initial condition at $N = k$, at each step of the iteration the matrix $R_0 - \gamma Q_N B$ is invertible. In this way we obtain from (4.1) a sequence $Q_N, Q_{N+1}, Q_{N+2}, \ldots$ of $\nu \times n$ matrices. Moreover, assume that this sequence converges with limit Q and the matrix $R_0 - \gamma Q B$ is invertible. Then Q is a solution to the Riccati equation (1.4) and this solution will be the stabilizing solution of (1.4) provided the matrices $A_\circ$ and $\alpha_\circ$ defined by (1.5) and (1.6) are both stable. In that case the Toeplitz operator defined by R is invertible and a right canonical factorization for R is given by $R(z) = \Psi(z)\Theta(z)$ where Ψ and Θ are defined by (1.7).

Formula (1.11) for the stabilizing solution suggests to define the iterates Q_N in terms of the finite sections of T. By definition the N-*th section* of T is the $N \times N$ block matrix T_N given by

$$T_N = \begin{bmatrix} R_0 & R_{-1} & \cdots & R_{1-N} \\ R_1 & R_0 & \cdots & R_{2-N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N-1} & R_{N-2} & \cdots & R_0 \end{bmatrix}.$$

In what follows we assume that the Toeplitz operator T defined by R is invertible and that the same holds true for its block transpose $T^{\#} = [T_{k-j}]_{j,k=0}^{\infty}$. Note that the latter is equivalent to requiring that $R^{\#}(z) = R(z^{-1})$ admits a right canonical factorization relative to the circle. Thus in the scalar case the invertibility of $T^{\#}$ follows from the invertibility of T and conversely. This is also true in the symmetric case (when $R(z)$ is Hermitian for each z on the unit circle).

Since R is a continuous matrix symbol, we know (see Section VIII.5 in [8]) that invertibility of T and $T^{\#}$ implies (in fact, is equivalent to the statement) that the finite section method for T converges. In particular, in this case, there exists a positive integer k such that T_N is invertible for $N \geq k$ and

$$\lim_{n\to\infty} \omega_n T_N^{-1} W_N = \omega T^{-1} W.$$

Here

$$\omega_j = \begin{bmatrix} \beta & \alpha\beta & \alpha^2\beta & \cdots & \alpha^{j-1}\beta \end{bmatrix}, \quad W_j = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{j-1} \end{bmatrix}, \quad j \geq 1.$$

In checking the invertibility of the finite section the following lemma is useful.

Lemma 4.1. *Assume the N-the section T_N of T is invertible, and put $Q_N = \omega_N T_N^{-1} W_N$. Then T_{N+1} is invertible if and only if $R_0 - \gamma Q_N B$ is invertible, and in that case the matrix $Q_{N+1} = \omega_{N+1} T_{N+1}^{-1} W_{N+1}$ is given by*

$$Q_{N+1} = \alpha Q_N A + (\beta - \alpha Q_N B)(R_0 - \gamma Q_N B)^{-1}(C - \gamma Q_N A).$$

Note that the matrix $R_0 - \gamma QB$ is a square matrix of order m while T_{N+1} is of order $m(N+1)$. Hence, in general, checking the invertibility of $R_0 - \gamma QB$ will be a much easier task than checking the invertibility T_{N+1}.

Proof. Note that T_{N+1} admits the 2×2 block matrix representation:

$$T_{N+1} = \begin{bmatrix} R_0 & \gamma\omega_N \\ W_N B & T_N \end{bmatrix}. \tag{4.2}$$

Moreover, the Schur complement Δ_N of T_N corresponding 2×2 block matrix in (4.2) is given by

$$\Delta_N = R_0 - \gamma\omega_N T_n^{-1} W_N B = R_0 - \gamma Q_N B. \tag{4.3}$$

Hence T_{N+1} will be invertible if and only if $R_0 - \gamma Q_N B$ is invertible. The fact that $Q_{N+1} = \omega_{N+1} T_{N+1}^{-1} W_{N+1}$ is then given by the right hand side of (4.3) follows by proving the analogue of Lemma 2.1 with T being replaced by T_{N+1} and using the same type of arguments as in Part 1 of the proof of Theorem 1.1. $\square$

We summarize the preceding discussion with the following proposition which is a partial converse to the result stated in the second paragraph of this section.

Proposition 4.2. *Let R be given by the stable representation* (1.3), *and consider the Riccati difference equation in* (4.1). *Assume the Toeplitz operator T defined by R and its block transpose $T^\#$ are invertible. Then there exists a positive integer k such that the following holds*

(i) *T_k is invertible;*
(ii) *$R_0 - \gamma Q_N B$ is invertible for all $N \geq k$ where Q_N is the solution to* (4.1) *subject to the initial condition $Q_k = \omega_k T_k^{-1} W_k$;*
(iii) *Q_N converges to Q and $R_0 - \gamma Q B$ is invertible;*
(iv) *the matrices $\alpha_\circ$ and $A_\circ$ are stable.*

In this case, Q is the unique stabilizing solution to the Riccati equation (1.4).

It can happen that Q_n converges to Q and $R_0 - \gamma Q B$ is invertible and $\alpha_\circ$ or $A_\circ$ may not be stable. In fact, this follows from the example considered in the final paragraphs of the previous section. Indeed, take $R(z) = 1 - 2z$, and represent $R(z)$ as in (1.3) with A, B, C matrices of size 1×1, $A = 0$, $B = -2$, $C = 1$, and with $\alpha = 0$ on $\{0\}$, $\beta = 0$ from $\mathbb{C}$ into $\{0\}$, and $\gamma = 0$ from $\{0\}$ into $\mathbb{C}$. Then (4.1) and (1.4) reduce to $Q_N = 0$ and $Q = 0$. Thus $\lim_{N\to\infty} Q_N = Q$, but Q is not a stabilizing solution. In this case, R does not admit a right canonical factorization.

In conclusion we note that for the example considered in Section 3 the Riccati difference equation is given by

$$q_{N+1} = \begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} (1+3q_N)^{-1} & q_N(1+3q_N)^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1+q_N}{2+6q_N}.$$

Starting with the initial condition $q_0 = 0$, we see that the sequence q_N converges to $1/3$. In fact, $q_N = 1/3$ for all $N \geq 11$, and the first eleven values for q_N are given by

$$\frac{1}{2}, \frac{3}{10}, \frac{13}{38}, \frac{51}{154}, \frac{205}{614}, \frac{819}{2458}, \frac{3277}{9830}, \frac{13107}{39322}, \frac{52429}{157286}, \frac{209715}{629146}, \frac{1}{3}.$$

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