Conley index theory for braids and forcing in fourth order conservative systems

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Conley index theory for braids and forcing in fourth order conservative systems

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“If the doors of perception were cleansed
every thing would appear to man as it is, infinite.”

WILLIAM BLAKE

Contents

Preface

The subject of this thesis is developing and applying topological methods to study forcing properties and existence of periodic solutions for fourth order conservative systems.

The introductory Chapter 1 provides an overview of the known results and techniques which are employed in this thesis. Examples are given to explain the basic ideas. The main results are summarized in Chapter 1. We apply our techniques to the Swift-Hohenberg equation and we give a classification of its periodic solutions according to their braid type. The bifurcation diagram for this equation is discussed in terms of this classification.

In Chapter 2 we extend Conley index theory to non-proper braid classes. This means that we can use the Conley index to analyze the invariant set of a non-proper braid class although it is not an isolating neighborhood for the underlying flow. This chapter as well as Chapter 3 is joint work with J.B. van den Berg and R.C.A.M. van der Vorst.

Chapter 3 applies the Conley index theory to obtain forcing results based on the braid type of solutions and establishes a partial order of periodic solutions on the Swift-Hohenberg equation.

CHAPTER 1

Introduction

1.1. Prologue

Differential equations have played an important role in pure and applied mathematics since they were introduced in the mid 17th century. First, applications were made largely to geometry and mechanics. In the 18th century, the study of partial differential equations (PDE's) was initiated in the work of Euler, d'Alambert, Lagrange and Laplace as the principal tool of analytic study of models in the physical science.

In these days physics is far from being the only science where differential equations are used. Every science which tries to study changing quantities is in quest of differential equations because given the initial state they completely determine the evolution of the system. To name a few examples: the reaction and diffusion of chemicals, the dynamics of populations in biology, the development and treatment of diseases in medicine, or the flow of fluids or gases, with applications ranging from fundamental astronomy to meteorology to industrial engineering.

In the late 19th century H. Poincaré stated a prophetic insight that differential equations of mathematical physics will have a significant role within mathematics itself. Indeed through the whole 20th century till now differential equations constitute a bridge between central issues of applied mathematics and physical sciences on the one hand and the development of mathematical ideas in active areas of pure mathematics on the other hand. We will restrict ourself to mention just a few of them: differential geometry, functional analysis, topology, Fourier analysis, algebraic geometry and the theory of chaos. For a more detail discussion of the history of PDE's and approaches to solve (understand) them see [4].

Dynamical systems theory combines analysis, geometry and topology for analyzing (partial) differential equations. Two main ingredients of a dynamical system are the space X which consists of possible states of the system and a rule (differential equation) that governs the evolution of the state. The space X plays a very important role. Understanding the topology of this space may give deep insight into the dynamics of the system. Imagine the rain falling on the mountains. A rain drop which falls on the mountain ridge will flow downwards and eventually join with other rain drops to create a stream running down in the valley, joining with other streams into a river and continuing through the valleys to the sea. Another possible scenario is that the rain drop ends up in a puddle or a lake. A good knowledge about the terrain enables us to predict where the lakes and torrents will be created. However, this approach does not tell us anything about the position of the single drop of water in a river. Hence the information obtained is coarse in this sense but still meaningful. Differential and algebraic topology are used to formalize these ideas.

In this thesis we study dynamical systems via topological invariants. Conley index theory is used to prove the existence of geometrically different solutions for a variety of fourth order differential equations. The main focus is on the extended Fisher-Kolmogorov and Swift-Hohenberg equation introduced in the following section. In the same section we survey the main results of the thesis. Section 1.3 provides a more general setting in which Swift-Hohenberg equation can be seen and to which our results are applicable. To obtain the theorems about the existence of solutions for Swift-Hohenberg equation, stated in Section 1.2, we employ the technique of forcing of solutions via the Conley index for braids. The main ideas of forcing are summarized in Section 1.4, while a brief survey of Conley index theory for braids can be found in Section 1.5.

1.2. The fourth order equation

Whereas the solutions of second order autonomous ODEs can be represented in a phase plane, leading to modest complexity of the dynamics, equations of higher order can exhibit a plethora of distinct behaviors, and the dependence of the dynamics on parameters is extremely complex. The results of this thesis are concerned with the solutions of the nonlinear fourth order equations of the form

$$-\gamma u'''' + \beta u'' + f(u) = 0 \qquad \gamma > 0, \ \ \beta \in \mathbb{R}, \tag{1.2.1}$$

and their generalizations. Equation (1.2.1) describes the stationary solutions of the equation

$$\frac{\partial u}{\partial s} = -\gamma \frac{\partial^4 u}{\partial t^4} + \beta \frac{\partial^2 u}{\partial t^2} + f(u) = 0 \qquad \gamma > 0,\ \ \beta \in \mathbb{R}.$$

We are especially interested in the bi-stable nonlinearity $f(u) = u - u^3$. For this nonlinearity Equation (1.2.1) is known as the *extended Fisher-Kolmogorov equation* for $\beta > 0$, and for $\beta < 0$ the name *Swift-Hohenberg* equation is more appropriate. One of the reasons to investigate Equation (1.2.1) is that, with various nonlinearities $f(u)$, they serve as models in an abundance of applications. A more detailed explanation can be found at Section 1.6.

Using a scaling argument Equation (1.2.1) with $f(u) = u - u^3$ can be rewritten as

$$u'''' + \alpha u'' - u + u^3 = 0, \quad \alpha \in \mathbb{R}, \tag{1.2.2}$$

where $\alpha = -\frac{\beta}{\sqrt{\gamma}}$.

Of particular importance in the study of Equation (1.2.2) is the variational formulation. We explain this concept later on in more detail, see Chapter 2. Equation (1.2.2) occurs as the Euler-Lagrange equation of an action functional involving a second order Lagrangian:

$$\int_I L(u, u', u'')dt,$$

where

$$L(u, v, w) = \frac{1}{2}w^2 - \frac{\alpha}{2}v^2 + \frac{1}{4}(u^2 - 1)^2.$$

The results of this thesis are applicable to a broad class of the second order Lagrangians. In the present work we apply them to Equation (1.2.2) in order to study its periodic solutions.

The Lagrangian action is translation invariant and through Noether's Theorem a conserved quantity can be related to variational structure: solutions of Equation (1.2.2) satisfy the energy equation

$$\mathbb{E}[u] = -u'u''' + \frac{1}{2}(u'')^2 - \frac{\alpha}{2}(u')^2 - \frac{1}{4}(u^2 - 1)^2 = E.$$

The energy equation defines three dimensional energy surfaces M_E that foliate $\mathbb{R}^4$. For the values $E = -\frac{1}{4}$ and $E = 0$ the energy surfaces are singular with singularities at $u = 0$ and $u = \pm 1$ respectively. For all $E > 0$ we have that $M_E \cong S^1 \times \mathbb{R}^2$. There may not be any periodic solutions at a positive energy level, but the existence of certain types of periodic solutions may force additional periodic solutions based on topological invariants. The level $E = 0$ plays the role of organizing center due to the existence of the equilibrium states $u = \pm 1$. Intuitively, homoclinic solutions to $u_{\pm} = \pm 1$ and/or a heteroclinic cycle will, if they exist, lie in this energy level and it is well known that such connecting orbits may be the source of complicated dynamics [8, 10, 12, 13]. This naturally leads us to the study of solutions in this *singular* energy level. The focus on the singular energy level is not new. We will highlight some of the known results and introduce the classification of periodic solutions of Equation (1.2.2).

Classification of periodic solutions

The structure of the set of periodic solutions of Equation (1.2.2) depends very much on the linearization around the constant solutions $u_{\pm} = \pm 1$ and hence on the value of the parameter α. In particular, one can identify two critical values of $\alpha : +\sqrt{8}$ and $-\sqrt{8}$. At these values the linearization around the constant solutions $u_{\pm}$, i.e. the points $P_{\pm} = (\pm 1, 0, 0, 0)$ in (u, u', u'', u''') phase space, changes type, as indicated in Figure 1.1.

In fact, for $\alpha \leq -\sqrt{8}$, the equilibria $u_{\pm}$ are real saddles and there are no periodic solutions on the zero energy level. The set of *all* bounded solutions is very limited, and consists of the three equilibrium points, two monotone antisymmetric heteroclinic loops and (modulo transitions) a one parameter family of single bump periodic solutions, which are even with respect to their extrema and odd with respect to their zeros. These periodic solutions can be parameterized by the energy $E \in \left(-\frac{1}{4}, 0\right)$, see [21].

As α increases beyond $-\sqrt{8}$ the equilibria $u_{\pm}$ become saddle-foci and the set of periodic solutions becomes much richer. There is a plethora of periodic solutions on the energy level zero bifurcating from the heteroclinic loop at $\alpha = -\sqrt{8}$. It has been proved that for $-\sqrt{8} < \alpha \leq 0$ the zero energy level contains a great variety of multi-bump periodic solutions. For detailed results we refer to [10, 12, 13]. For $0 < \alpha < \sqrt{8}$ the results are more tentative and less complete. For $\alpha > \sqrt{8}$ the equilibria change to centers and small periodic oscillations around equilibria $u_{\pm}$ appear.

Figure 1.1 shows the bifurcation diagram, where we graph the norm of the solutions u as function of α. Three branches with very different geometric behavior appear in the bifurcation diagram. It is shown in [20] that at a regular energy level every solution is a concatenation of monotone laps between extrema and the number of the

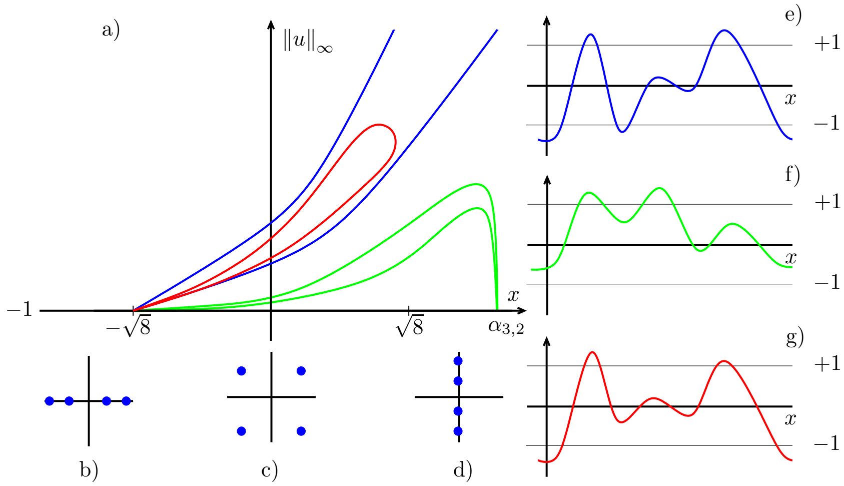

Figure 1.1: Bifurcation diagram a) shows three different types of branches, in the plane $(\alpha, \|u\|_\infty)$, which bifurcate for $\alpha = -\sqrt{8}$. Solutions on the branches that extend beyond the boundary of the diagram are of the first type, see e) for an example; branches that form closed loops consist of solutions of the second type, see f) for an example; branches collapsing on $\|u\|_\infty = 1$ consist of solutions of the third type, see g) for an example. Also depicted is the spectrum of the linearization around P_+ and P_- for b) $\alpha \leq -\sqrt{8}$; c) $\alpha \in (-\sqrt{8}, \sqrt{8})$; d) $\sqrt{8} \leq \alpha$.

monotone laps is finite and even per period. Two essential properties are preserved for the solutions lying on the same branch of the bifurcation diagram:

(1) the number of monotone laps;
(2) the number of crossings of the solution with the u_+ and u_-.

We stress that it is important that the counting of monotone laps and crossings is done with the following conventions. A regular monotone lap is a piece of the solution u such that u' does not change sign i.e. $u' < 0$ or $u' > 0$, and a degenerate monotone lap is an inflexion point. We have to count both non-degenerate and degenerate monotone laps in order to obtain the invariant along the bifurcation branch, see [16]. The number of crossings of the solution u with $u_\pm$ is the number of zero points of the function $u - u_\pm$ counted over one period *without* multiplicity, i.e. every zero point is counted just once even if it is a multiple zero. The zero points of the function are isolated thus this number is well defined, finite and preserved along the continuous branches, see [16].

Now we proceed to make a classification of solution branches. Define the intersection sequence $\sigma = (\sigma_{j_1}\sigma_{j_2}\ldots\sigma_{j_m})$, $j_k \in \{1,2\}$, where σ_1 represents the intersections of a periodic function u with $u_- = -1$ while σ_2 represents the intersections with $u_+ = +1$, counted over the period τ. Due to periodicity of u we can suppose that σ starts with σ_1 if u intersects u_-. We group the same elements together and use the notation with powers instead of repeating the symbols e.g. we write $(\sigma_1^2\sigma_2^2)^2$ instead of $\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1\sigma_2\sigma_2$. We distinguish different types of periodic functions based on their intersections σ.

DEFINITION 1.2.1. We can distinguish the following three classes of periodic functions:

(I) $\sigma = (\sigma_1^2\sigma_2^2)^q$ for some $q \in \mathbb{N}$;
(II) both σ_1 and σ_2 are present in σ but $\sigma \neq (\sigma_1^2\sigma_2^2)^q$ for any $q \in \mathbb{N}$;
(III) $\sigma = (\sigma_1^{2q})$, or $\sigma = (\sigma_2^{2q})$ for some $q \in \mathbb{N}$.

These three classes do not contain all periodic functions. A function which does not attain at least one of the values $+1$ or -1 is not in any of them. To motivate the previous definition a relation to braids can be made.

DEFINITION 1.2.2. A braid β on n strands is a collection of embeddings $\{\beta^\alpha : [0,1] \to \mathbb{R}^3\}_{\alpha=1}^n$ with *disjoint* images such that $\beta^\alpha(1) = \beta^{\tau(\alpha)}(0)$ for some permutation τ and the image of each β^α is transverse to all planes $\{x_1 = \text{const}\}$. Two such braids are said to be of the same topological braid type if they are homotopic in the space of braids. The strands must remain disjoint along the homotopy.

The solutions u and $u_\pm$ can be interpreted as a braid via the map $t \to (t, u(t), u'(t))$. For a more detailed explanation see [9]. The intersection sequence σ defines the topological type. Therefore the sequence σ distinguishes a subset in the space of braids. Later on we will introduce a topological invariant called the Conley index for braid types. This invariant is an extension of the degree. Non-triviality of the invariant for a braid type yields the existence of a solution of (1.2.2) with this braid type. A detailed explanation can be found in Section 1.5.

The bifurcation diagram in Figure 1.1 can be explained by using this invariant. For the first type of solutions the topological invariant is non-trivial for $\alpha \geq 0$ and the bifurcation branch extends to infinity. For solutions of the second type the invariant is trivial and does not provide information about the existence of solutions. However, if there is a non-degenerate (hyperbolic) solution of this type then there has to be

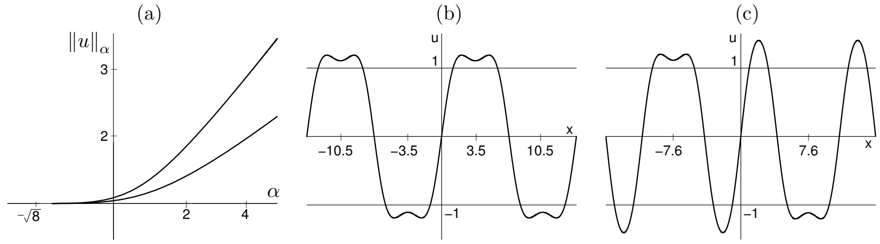

Figure 1.2: Bifurcation diagram for the solutions of the first type and corresponding solutions for $\alpha = 1.5$. The number of monotone laps for the solution pictured at (b) is six and it intersects $u_\pm$ two times. For solution (c) number of monotone laps is ten and crossing number with $u_\pm$ is six.

another one and the bifurcation branches of these solutions form loops. The loop turns for some parameter value α^* for which the two solutions coalesce. For solutions of the third type we cannot define the topological invariant as for the previous types because there is a fixed point corresponding to $u_+ = 1$ at the boundary. We use linearization at this fixed point to overcome the problem, see Chapter 2. For the third type of solutions the topological invariant is non-trivial up to some value of α and then it becomes trivial. This explains the bifurcation diagram in Figure 1.1. In fact topological methods are employed on slightly positive energy levels and the limit process extends the results to the energy level $E = 0$.

Further classification of periodic solutions can be done by counting the monotone laps of solutions.

DEFINITION 1.2.3. Let u be a solution of (1.2.2) with p-monotone laps per period and the intersection sequence σ. Then we say that solution u lies in the class $[\sigma, p]$.

For periodic solutions of Type (I) it is proved in [9] that in the parameter range $\alpha \geq 0$ there exist at least two periodic solutions of the class $[(\sigma_1^2, \sigma_2^2)^q, 2p]$ for any $q \leq p$ at the energy surface $E > 0$, for E sufficiently small. The limit process used in Chapter 2 extends the result for the zero energy level. The existence actually holds for all $\alpha > -\sqrt{8}$, see [18]. In this thesis we obtain the following results for solutions of the second and third type. In Chapter 2 the following theorem, concerning solutions of the third type, is proved.

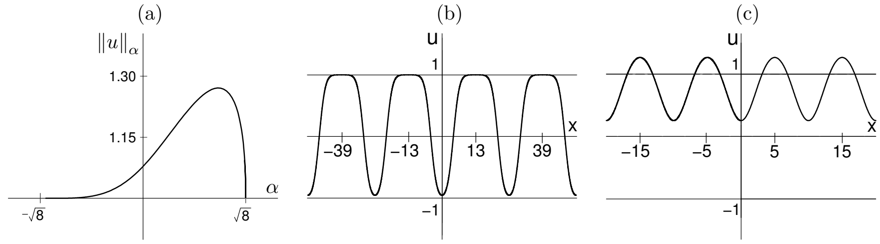

Figure 1.3: Bifurcation diagram for the solutions of the third type and corresponding solutions which are in the class $\left[\sigma_2^2, 2\right]$. Solution (b) corresponds to $\alpha = -1$ and (c) to $\alpha = 1$.

THEOREM 1.2.4. *Let $p, q \in \mathbb{N}$ be relatively prime and $q < p$ and let $\alpha_{p,q} = \sqrt{2}\left(\frac{p}{q} + \frac{q}{p}\right)$. Then for every $\alpha \in [\sqrt{8}, \alpha_{p,q})$ there exists a solution*

$$u_\alpha \in \left[\sigma_1^{2q}, 2p\right]$$

of Equation (1.2.2) with $\mathbb{E}[u_\alpha] = 0$.

Figure 1.3 shows two solutions of the class $[\sigma_2^2, 2]$ for different value of the parameter α, and the bifurcation curve on which these solutions lie. In the previous theorem we proved existence of one solution of the class $[\sigma_1^{2q}, 2p]$. However it seems that there are two different solutions of this class, see Section 1.6. An analogue of Theorem 1.2.4 can be proved for solutions of the class $\left[\sigma_2^{2q}, 2p\right]$. Moreover one should be able to extend the previous theorem to the parameter range $[0, \sqrt{8})$, where the eigenvalues of equilibria $u_\pm = \pm 1$ are saddle-foci, by perturbing the potential $F = \frac{1}{4}(u^2 - 1)^2$ as explained in [9], see Section 1.6 for a more detailed explanation. Solutions of the second type are studied in Chapter 3, where the following result is proved.

THEOREM 1.2.5. *Let $\alpha \in [0, 2]$ and $p, q_1, \ldots q_n \in \mathbb{N}$ such that $q_i > 1$ and $\sum_{i=1}^n q_i \leq 2p$. If $\sum_{i=1}^n q_i < 2p$ suppose that at least one $q_i > 3$. Then there exists a solution*

$$u_\alpha \in \left[\sigma_1^2 \sigma_2^{2q_1} \ldots \sigma_1^2 \sigma_2^{2q_n}, 2p\right]$$

of Equation (1.2.2) with $\mathbb{E}[u_\alpha] = 0$. Moreover, if $\sum_{i=1}^n q_i = 2p$ then there are at least $2^{(p-3)}$ geometrically different solutions; otherwise the lower bound is given by 2^q, where $q = \sum_{i=1}^n q_i - 4$.

Different solutions of the second type are depicted in Figure 1.5, Figure 1.7 and Figure 1.8.

1.3. Twist systems

In this thesis the main application of the theory is the existence of periodic solutions of Equation (1.2.2). However, the results are applicable for a broad class of the second order Lagrangian systems. The second order Lagrangian systems are used as mathematical models for a variety of physical problems. Their applications range from nonlinear elasticity, nonlinear optics to solid mechanics and many other fields of physics. We briefly recall the concept of the second order Lagrangians.

Let $L : \mathbb{R}^3 \to \mathbb{R}$ be a C^2-function of the variables u, v, w. Then the functional

$$J[u] = \int_I L(u, u', u'')dt,$$

defined for any smooth function $u : I \subseteq \mathbb{R} \to \mathbb{R}$, is called the Lagrangian action. The pair (L, dt) is called a second order Lagrangian system. The function u is a stationary point of the Lagrangian system if it satisfies the equation

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u'} + \frac{d^2}{dt^2}\frac{\partial L}{\partial u''} = 0,$$

which is called the Euler-Lagrange equation of the Lagrangian system (L, dt), and is given by $\delta J[u] = 0$ with respect to variations $\delta u \in C_c^\infty(I, \mathbb{R})$.

If we seek closed characteristics, i.e., a periodic solution of Equation (1.2.2), at a given energy level E, we can invoke the following variational principle:

$$\text{Extremize } \{J_E[u] : u \in \Omega_{per}, \tau > 0\}, \tag{1.3.1}$$

where $\Omega_{per} = \cup_{\tau>0} C^2(S^1, \tau)$, the periodic functions with period τ, and

$$J_E[u] = \int_0^\tau (L(u, u', u'') + E)dt.$$

The function $L \in C^2(\mathbb{R}^3, \mathbb{R})$ is assumed to satisfy $\frac{\partial^2 L}{\partial w^2}(u, v, w) \geq \delta > 0$ for all $(u, v, w) \in \mathbb{R}^3$. For the general second order Lagrangian system the (conserved) energy is given by

$$\mathbb{E}[u] = \left(\frac{\partial L}{\partial u'} - \frac{d}{dt}\frac{\partial L}{\partial u''}\right) u' + \frac{\partial L}{\partial u''}u'' - L(u, u', u'').$$

The variations in τ guarantee that any critical point u of (1.3.1) has energy $\mathbb{E}[u] = E$.

An energy value E is called regular if $\frac{\partial L}{\partial u}(u,0,0) \neq 0$ for all u that satisfy $L(u,0,0)+E=0$. The energy manifold $M_E \subset \mathbb{R}^4$ for a regular energy value E is a smooth non-compact manifold without boundary. For a fixed regular energy value E, the extrema of a characteristic are contained in the closed set $\{u : L(u,0,0)+E \geq 0\}$. The connected components I_E of this set are called interval components. Moreover, it follows from [20] that solutions on a regular energy level do not have inflexion points. For a singular energy level the interval component I_E contains critical points and the situation is more complicated. If we work at a singular energy level we avoid the critical points, hence we assume all extrema to be non-degenerate. Under this restriction the behavior on the singular energy level is the same as on the regular one. Therefore, we restrict ourselves to the regular energy levels for the moment.

As mentioned above, the extremal points of every solution (on a regular energy level) are confined to the set I_E. In this thesis we concentrate on the systems for which the solution can be coded by its extremal points. Systems with this property admit a map T defined on a direct sum of every single connected component of I_E, see Section 1.4. The map T satisfies a *twist* property $\partial_y T(x,y) > 0$. A brief survey of twist systems is provided here while detailed explanations are postponed until Section 2.2. A second order Lagrangian system (L, dt) is called a twist system if for any points $u_1, u_2 \in I_E$ there exists a positive number τ and a monotone function u_τ which satisfies $u'(0) = u'(\tau) = 0$, $u(0) = u_1$, $u(\tau) = u_2$ and minimizes

$$\inf_{u \in X_\tau, \tau \in \mathbb{R}^+} \int_0^\tau (L(u,u',u'')+E)dt,$$

where X_τ is a space of all monotone functions connecting u_1 and u_2.

It is shown in [20] that Lagrangian systems $J[u] = \int_I L(u,u',u'')dt$, where $L(u,u',u'') = \frac{1}{2}u''^2 + K(u,u')$ at energy levels E which satisfy

$$\frac{\partial K}{\partial v}v - K(u,v) - E \leq 0 \text{ for all } u \in I_E \text{ and } v \in \mathbb{R}, \tag{1.3.2a}$$

$$\frac{\partial^2 K}{\partial v^2}v^2 - \frac{5}{2}\left\{\frac{\partial K}{\partial v} - K(u,v) - E\right\} \geq 0 \text{ for all } u \in I_E \text{ and } v \in \mathbb{R}, \tag{1.3.2b}$$

are twist systems.

For a twist system, the function

$$S_E(u_1, u_2) = \inf_{u \in X_\tau, \tau \in \mathbb{R}^+} \int_0^\tau (L(u, u', u'') + E)dt \tag{1.3.3}$$

is well defined for $u_1 \neq u_2$ and called the *generating function*. The properties of this function are listed in Chapter 2. We only mention that $\partial_1 \partial_2 S_E(u_1, u_2) > 0$, which embodies the twist property.

The question of finding closed characteristics for a twist system can now be formulated in terms of S_E. Any periodic solution u is a concatenation of monotone laps. Let us take an arbitrary $2p$ periodic sequence $\{u_i\}$ and define u as a concatenation of monotone laps (minimizers $u_\tau(u_i, u_{i+1})$) between the consecutive extremal points u_i solving the Euler-Lagrange equation in between any two extrema. The concatenation u does not have to be a solution on $\mathbb{R}$ because the third derivatives of two monotone laps do not have to match at the extremal point u_i. It was proved in [20] that the third derivatives match if and only if the sequence of extrema $\{u_i\}$ is a critical point of

$$W_{2p} = \sum_{i=0}^{2p-1} S_E(u_i, u_{i+1}). \tag{1.3.4}$$

The situation is thus comparable with the method of broken geodesics.

Parabolic recurrence relation

Critical points of W_{2p} satisfy the equations

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = \partial_2 S_E(u_{i-1}, u_i) + \partial_1 S_E(u_i, u_{i+1}) = 0. \tag{1.3.5}$$

The properties of the function S_E listed in Chapter 2 ensure that $\mathcal{R}_i(s, t, r)$ is well-defined and C^1 on the domains

$$\Omega_i = \{(r, s, t) \in I_E^3 : (-1)^{i+1}(s - r) > 0, (-1)^{i+1}(s - t) > 0\}. \tag{1.3.6}$$

The functions $\mathcal{R}_i$ and domains Ω_i satisfy $\mathcal{R}_i = \mathcal{R}_{i+2}$ and $\Omega_i = \Omega_{i+2}$ for $i \in \mathbb{Z}$. Moreover, $\partial_1 \mathcal{R}_i = \partial_1 \partial_2 S(u_{i-1}, u_i) > 0$ and $\partial_3 \mathcal{R}_i = \partial_1 \partial_2 S(u_i, u_{i+1}) > 0$. The functions $\mathcal{R}_i$ are not defined at the diagonal boundaries of Ω_i. This corresponds to the nature of solutions of the second order twist systems, namely that minima and maxima alternate. However, close to these boundaries $\mathcal{R}$ interpreted as a vector field, points away from the diagonal. Above-mentioned properties imply that $\mathcal{R}_i$ is a parabolic recurrence relation of up-down type, see Chapter 2 for the definition. In this section we restrict to parabolic recurrence relations defined on $\mathbb{R}^{\mathbb{Z}}$.

DEFINITION 1.3.1. A parabolic recurrence relation $\mathcal{R}$ on $\mathbb{R}^{\mathbb{Z}}$ is a sequence of real-valued functions $\mathcal{R} = (\mathcal{R}_i)_{i\in\mathbb{Z}}$ satisfying

(a) [*monotonicity*] $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_i > 0$ for all $i \in \mathbb{Z}$,
(b) [*periodicity*] for some $d \in \mathbb{N}, \mathcal{R}_{i+d} = \mathcal{R}_i$ for all $i \in \mathbb{Z}$.

The relation between solutions of the second order Lagrangian system and zero points of the parabolic recurrence relation $\mathcal{R}$ is summarized in the following theorem of which the proof can be found in [9].

THEOREM 1.3.2. *Let $J[u] = \int L(u, u', u'')dt$ be a second order Lagrangian twist system. Suppose that W_{2p} is the discrete action defined through (1.3.3) and (1.3.4) at a regular energy level E. Then*

(a) *the functions $\mathcal{R}_i = \partial_i W_{2p}$ defined on Ω_i are components of a parabolic recurrence relation $\mathcal{R}$ of up-down type,*
(b) *periodic solutions of $\mathcal{R} = 0$ correspond to periodic solutions on the energy level E.*

In order to find solutions of $\mathcal{R} = 0$ we interpret $\mathcal{R}$ as a vector field and study an invariant set of the gradient flow generated by this vector field. In the case of a gradient vector field invariant sets have special structure and thus information about critical points can be obtained. There is a natural way to define a flow generated by an up-down parabolic recurrence relation on the set

$$\Omega^{2p} = \{\mathbf{u} \in \mathbb{R}^{\mathbb{Z}} : \mathbf{u} \text{ is } 2p \text{ periodic and } (u_{i-1}, u_i, u_{i+1}) \in \Omega_i, \text{ for } i \in \mathbb{Z}\}. \tag{1.3.7}$$

Consider the differential equations

$$\frac{d}{dt}u_i(t) = \mathcal{R}_i(\mathbf{u}(t)), \quad \mathbf{u}(t) \in \Omega^{2p}, \quad t \in \mathbb{R}. \tag{1.3.8}$$

Equation (1.3.8) defines a (local) C^1 flow ψ^t on Ω^{2p}. This flow is not defined at the boundary of Ω^{2p}. We will show in Section 2.2 that close to the boundary of Ω^{2p} the flow points away from it. In the following section we explain how to employ information about known zero points of the parabolic recurrence relation $\mathcal{R}$ to find more zero points of the same parabolic recurrence relation.

1.4. Forcing and ordering on solutions

A classical forcing result is Sharkovskii's ordering of natural numbers. Sharkovskii's ordering states a forcing relation for one dimensional discrete dynamical systems given by iterating the continuous map $f : I \subseteq \mathbb{R} \to \mathbb{R}$. Before we formulate the forcing result let us

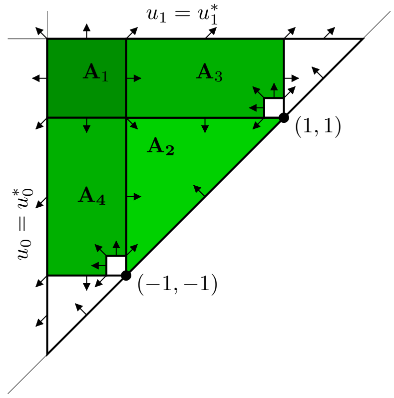

Figure 1.4: The triangle $D = I \times I \setminus \{u_1 > u_0\}$. The arrows denote (schematically) the direction of the gradient ∇W_2. Clearly W_2 has a minimum on A_1 and maximum on A_2. Additionally if the equilibrium points ± 1 are saddle-foci then the direction of ∇W_2 in a small neighborhood of $+1$ (-1) is as depicted.

recall the ordering. Every positive integer n can be uniquely written in a form $2^r p$, where p is an odd number and r is such that 2^r is the highest power of 2 that divides n. Using this description we order the natural numbers in the following way:

$$3 \prec 5 \prec 7 \ldots \prec 2 \cdot 3 \prec 2 \cdot 5 \ldots \prec 2^r \cdot 3 \prec 2^r \cdot 5 \ldots \prec 2^r \prec \ldots \prec 2 \prec 1.$$

PROPOSITION 1.4.1. *Let $f : I \to \mathbb{R}$ be a continuous map, where $I \subseteq \mathbb{R}$ is a bounded interval. Then the dynamical system defined by iterating the map f has the following property. If the system has an orbit with period n then it has at least one orbit of period m, provided that $n \prec m$.*

A twist system at a regular energy level corresponds to a two dimensional discrete system defined on Poincaré sections. To be more precise, there are functions $\rho_\pm : \Sigma_E^\pm \to \Sigma_E^\mp$ where $\Sigma_E^\pm = \{(u, u', u'', u''') \in M_E : u' = 0, (\pm 1)u'' > 0\}$ defined as follows. Let $\mathbf{x} = (x_0, x_1, x_2, x_3) \in \Sigma^+$. Then the solution u with initial data $\frac{d^i}{dt^i}u(0) = x_i$ has a minimum for $t = 0$. The map ρ_+ maps $\mathbf{x} \to \mathbf{y}$ where $y_i = \frac{d^i}{dt^i}u(t_0)$ and t_0 is the first positive time for which $u'(t_0) = 0$. Non-degeneracy of extremal points, at regular energy levels, ensures that u has a maximum at t_0. Roughly speaking, the maps $\rho_\pm$ map a minimum (maximum) of the solution u onto the consecutive maximum (minimum). Due to the fact that $M_E \backsimeq \mathbb{R}^2 \times S^1$, there are invertible projections $\pi_\pm$ of $\Sigma_E^\pm$ to $\mathbb{R}^2$ such that $T_\pm = \pi_\mp \rho_\pm \pi_\pm^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous maps. Therefore a periodic solution u of the twist system with $2p$ extremal points per period corresponds to a p-periodic orbit of the two dimensional discrete system generated by the function $T = T_+ \circ T_- : \mathbb{R}^2 \to \mathbb{R}^2$. Moreover, the map T is a twist map, i.e. $\partial_y T(x,y) > 0$, see [20]. In the case of two dimensional discrete dynamical systems the results are far from being as complete as the one mentioned in the previous theorem, see [3].

In Chapter 3 we investigate a partial order on the pairs $[\sigma, p]$ which is defined by a forcing relation.

DEFINITION 1.4.2. The class of solutions $[\sigma^n, p_n]$ precedes $[\sigma^m, p_m]$ if and only if the existence of a solution in the class $[\sigma^n, p_n]$ forces the existence of a solution in the class $[\sigma^m, p_m]$; we write $[\sigma^n, p_n] \prec [\sigma^m, p_m]$.

Let us mention the implication for the bifurcation diagram of Equation (1.2.2). If Γ_n and Γ_m are continuous curves corresponding to solutions of the class $[\sigma^n, p_n]$ and $[\sigma^m, p_m]$ then Γ_m has to exist at least as long as Γ_n does.

Simple examples of forcing

In order to find a fixed point of the two dimensional discrete system generated by $T : \mathbb{R}^2 \to \mathbb{R}^2$, we focus on finding stationary points of the discrete action W_{2p} defined by (1.3.4) and (1.3.3). We demonstrate the basic ideas for Equation (1.2.2). The constant solutions $u_\pm = \pm 1$ are exploited to find a critical point of

$$W_2(u_0, u_1) = S(u_0, u_1) + S(u_1, u_0),$$

which according to the previous section corresponds to a solution of (1.2.2).

EXAMPLE 1.4.3. Figure 1.4 denotes the direction of the gradient ∇W_2 on the triangle $D = I \times I \cap \{u_1 > u_0\}$ where $I = [u_0^*, u_1^*]$. The system generated by (1.2.2) is dissipative, i.e. we can choose $u_0^* < u_1^*$ in such a way that $\partial_{u_0} W_2(u_0^*, u_1) < 0$ for all $u_1 \in (u_0^*, u_1^*]$ and $\partial_{u_1} W_2(u_0, u_1^*) > 0$ for all $u_0 \in [u_0^*, u_1^*)$. Define $A_1 = \{-1 < u_0 < u_1 < 1\}$ and $A_2 = D \cap \{u_0 < -1, u_1 > 1\}$. The gradient of W_2 points outwards on ∂A_2 and inwards on ∂A_1, see Figure 1.4. Thus W_2 attains a local maximum on A_2 and a local minimum on A_1. This proves the existence of two different periodic solutions of (1.2.2). We note that one of them

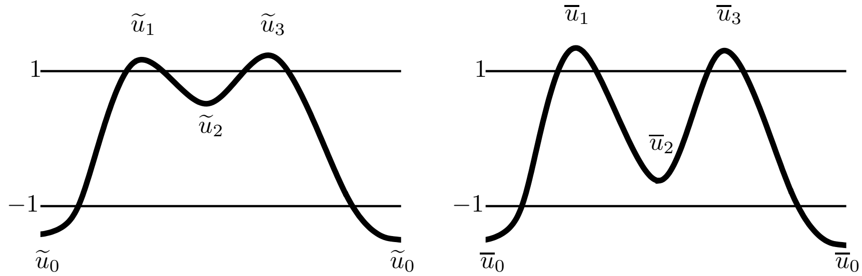

Figure 1.5: Sketch of two periodic solutions $\widetilde{u}, \overline{u} \in \left[\sigma_1^2\sigma_2^4, 4\right]$. Their extremal points are labeled by $\widetilde{u}_i$ and $\overline{u}_i$.

does not intersect u_+ and u_-. The other one is of the class $[\sigma_1^2\sigma_2^2, 2]$. Actually, the gradient ∇W_2 is not defined for $u_0 = u_1$ but by slightly shrinking the set A_1 we obtain the previous result. In the parameter range $\alpha \in [0, \sqrt{8})$, in which case the stationary solutions $u_\pm$ are saddle-foci, the sets A_3 and A_4 depicted in Figure 1.4 are isolating neighborhoods with respect to the gradient flow of W_2, see [20] for more details. The Conley index of A_3 and A_4 is non-trivial. Hence there is a critical point of W_2 in both isolating neighborhoods A_3 and A_4. These critical points correspond to solutions of the class $[\sigma_2^2, 2]$, respectively $[\sigma_1^2, 2]$.

We used the explicitly known solutions $u = \pm 1$ to prove existence of geometrically different ones. However, forcing can be considered in a more general framework. Instead of using explicitly known solution(s) one can ask the following question. When does existence of a solution $\widetilde{u} \in [\sigma, p]$ force existence of some other solution? In Chapter 3 we show that the existence of the solution $\widetilde{u} \in [\sigma_1^2\sigma_2^4, 4]$ forces the existence of solutions in a plethora of different classes. Let us present a simple example.

EXAMPLE 1.4.4. Let $\widetilde{u}$ be a solution of the class $[\sigma_1^2\sigma_2^4, 4]$. Then $\widetilde{u}$ has four non-degenerate extremal points per period with properties $\widetilde{u}_0 < -1 < \widetilde{u}_2 < 1 < \widetilde{u}_1, \widetilde{u}_3$, see Figure 1.5. According to the previous section $\mathcal{R}_1(u_1, \widetilde{u}_2, u_3) < \mathcal{R}_1(\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3) = 0$ for $u_1 < \widetilde{u}_1$ and $u_3 < \widetilde{u}_3$. Hence for $(\widetilde{u}_2, u_0) \in \partial A$, where $A = \{-1 < u_0 < u_1 < 1\} \cap \{u_0 < \widetilde{u}_2\}$, it holds that

$$\partial_1 W_2(\widetilde{u}_2, u_1) = \partial_2 S_0(u_1, \widetilde{u}_2) + \partial_1(\widetilde{u}_2, u_1) = \mathcal{R}_1(u_1, \widetilde{u}_2, u_3) < 0.$$

As before the gradient ∇W_2 points inward on ∂A and W_2 attains a local minimum on A, see Figure 1.6. This maximum corresponds to a solution u of (1.2.2) with two extremal points per period and $-1 < \min\{u(t),\ t \in \mathbb{R}\} < \widetilde{u}_2 < 1$, while $-1 < \max\{u(t),\ t \in \mathbb{R}\} < 1$.

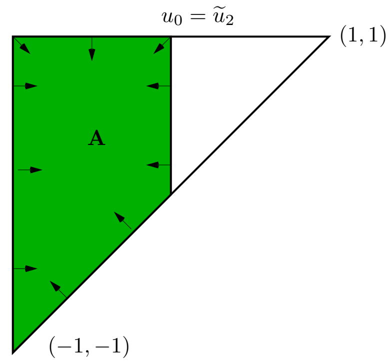

Figure 1.6: The triangle $D = [-1, 1] \times [-1, 1] \setminus \{u_1 > u_0\}$. The arrows denote (schematically) the direction of the gradient ∇W_2. Clearly W_2 has a maximum on A.

We studied W_2 to force the existence of periodic solutions with two extremal points per period. In order to force existence of general periodic solutions we need to study the gradient flow generated by W_{2p} for $p > 1$.

Ordering of solutions of the second type

The existence of a solution $\widetilde{u}$ of Equation (1.2.2) which lies in the class $[\sigma_1^2\sigma_2^4, 4]$ forces existence of solutions in a plethora of different classes. The forcing results summarized in this section are used to prove Theorem 1.2.5. Before we formulate them, we present a brief overview of the known results about solutions of the class $[\sigma_1^2\sigma_2^4, 4]$. In [11] the existence of a solution $\widetilde{u} \in [\sigma_1^2\sigma_2^4, 4]$ is shown for every $\alpha \in [-\sqrt{8}, \varepsilon]$ where $\varepsilon > 0$ is sufficiently small. This solution is a minimizer of the underlying Lagrangian system. Hence its Morse index is zero. Its extremal points $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ (see Figure 1.5 a) are very close to $u_+ = 1$. The numerics suggest that there is another solution with the Morse index one in the same class. This solution has a different shape, see Figure 1.5b. In this thesis a topological invariant, which can be seen as a generalization of the degree theory, is used to prove the existence of solution with a certain braid type. In the parameter range $[0, \sqrt{8}]$ the topological invariant of the braid type corresponding to the solutions of the class $[\sigma_1^2\sigma_2^4, 4]$, is trivial. Therefore the solution $\widetilde{u}$ cannot be the only one in this class. The numerics suggest that these two solutions lie on the same bifurcation branch which forms a loop which turns at some point $\alpha^* > 0$ where these two solutions coalesce, see Figure 1.1.

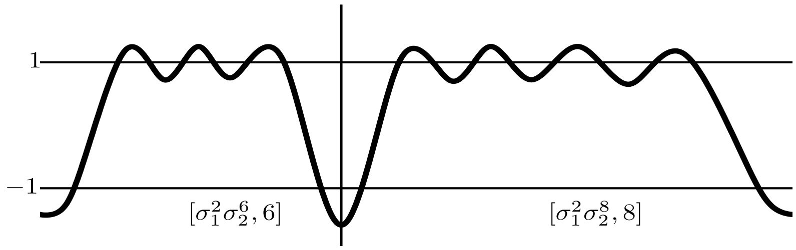

Figure 1.7: Sketch of aperiodic solution $u \in \left[\sigma_1^2\sigma_2^6\sigma_1^2\sigma_2^8, 14\right]$ which consists of two blocks namely $\left[\sigma_1^2\sigma_2^6, 6\right]$ and $\left[\sigma_1^2\sigma_2^8, 8\right]$.

Rigorous numeric is used in [22] to prove the existence of a solution $\widetilde{u} \in [\sigma_1^2\sigma_2^4, 4]$ for $\alpha \in [0, \mu)$ where $\mu > 2$. In the same paper, the Conley index for the discretized braid diagrams is employed to force the existence of solutions which are build up by connecting any finite number of blocks $[\sigma_1^2\sigma_2^{2p}, 2p]$, see Figure 1.7. We formalize this statement in the following proposition.

THEOREM 1.4.5. *Let $\alpha \in [0, \sqrt{8})$ and $q_1, \dots, q_n \in \mathbb{N}$ with $q_i > 1$. Then*

$$\left[\sigma_1^2\sigma_2^4, 4\right] \prec \left[\sigma_1^2\sigma_2^{2q_1} \dots \sigma_1^2\sigma_2^{2q_n}, 2\sum_{i=1}^{n} q_i\right].$$

In this thesis we generalize the structure of the blocks which can be used to build up solutions forced by $\widetilde{u} \in \left[\sigma_1^2\sigma_2^4, 4\right]$. We justify the use of the blocks $\left[\sigma_1^2\sigma_2^{2q}, 2p\right]$ with $3 < q < p$. An example of the solution of the class $\left[\sigma_1^2\sigma_2^8, 10\right]$ is depicted in Figure 1.8. The essential part of our result is that instead of finding just one solution of the class $\left[\sigma_1^2\sigma_2^{2q}, 2p\right]$ we prove the existence of many geometrically different solutions in the same class. Different solutions weave around the solution $\widetilde{u}$ in a different manner and attain different extrema values. More precisely, if we interpret u and $\widetilde{u}$ as a braid then different solutions correspond to different topological braid types. Figure 1.5 shows two solutions of the class $[\sigma_1^2\sigma_2^4, 4]$. The following theorem estimates the number of solutions of the class $\left[\sigma_1^2\sigma_2^{2q}, 2p\right]$.

THEOREM 1.4.6. *Let $\alpha \in [0, \sqrt{8})$ and $p \in \mathbb{N}$ such that $p \geq 2$. Then*

$$\left[\sigma_1^2\sigma_2^4, 4\right] \prec \left[\sigma_1^2\sigma_2^{2p}, 2p\right].$$

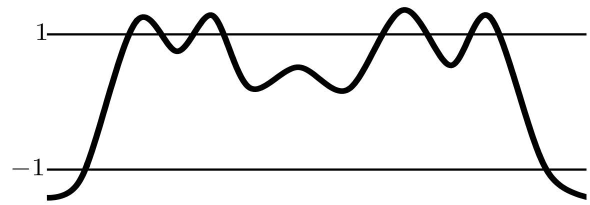

Figure 1.8: Sketch of a periodic solution $u \in \left[\sigma_1^2\sigma_2^8, 10\right]$.

If $q \in \mathbb{N}$ such that $3 < q < p$ then

$$\left[\sigma_1^2\sigma_2^4, 4\right] \prec \left[\sigma_1^2\sigma_2^{2q}, 2p\right].$$

Moreover, there are at least $2^{(p-3)}$ geometrically distinct solutions in the class $\left[\sigma_1^2\sigma_2^{2p}, 2p\right]$ and at least $2^{(q-4)}$ in the class $\left[\sigma_1^2\sigma_2^{2q}, 2p\right]$.

Figure 1.9 shows four geometrically different solutions of the class $\left[\sigma_1^2\sigma_2^{10}, 10\right]$. In the case of the general class $\left[\sigma_1^2\sigma_2^{2p}, 2p\right]$ we construct $2^{(p-3)}$ different solutions in the same manner as we do for the class $\left[\sigma_1^2\sigma_2^{10}, 10\right]$. Figure 1.8 displays a solution of the class $\left[\sigma_1^2\sigma_2^8, 10\right]$.

By concatenation of the building blocks and employing the solution $\widetilde{u} \in \left[\sigma_1^2\sigma_2^4, 4\right]$ which was proved to exist in [22], we obtain Theorem 1.2.5. To prove Theorem 1.4.6 the Conley index theory is employed.

Theorem 1.2.4 is also proved by using forcing of solutions via nontriviality of the Conley index. However, in this case the Conley index cannot be applied directly because the sets which contain the solutions of the class $[\sigma_1^{2q}, 2p]$ are not isolating neighborhoods, see Chapter 2 for a detailed explanation. We present a short overview of Conley index theory in this introduction.

1.5. The Conley index for discretized braids

In the previous sections we introduced the main ideas behind forcing the existence of solutions for Equation (1.2.2). The constant solutions $u_\pm = \pm 1$ were used to force the existence of periodic solutions with one minimum and one maximum per period. This is done by analyzing the discrete action W_2 defined by (1.3.4) and (1.3.3). The stationary points of W_{2p} correspond to the solutions of (1.2.2) with $2p$ extremal points per period. To find fixed points of W_{2p} we introduce the gradient flow Ψ^t generated by the parabolic recurrence relation of up-down

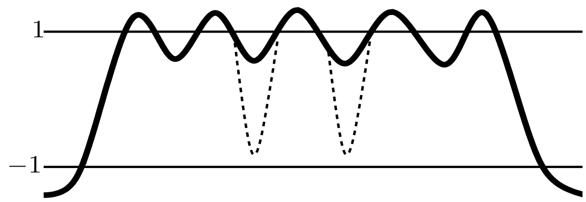

Figure 1.9: Sketch of four different periodic solution $u \in \left[\sigma_1^2\sigma_2^{10}, 10\right]$. The solid black curve represents one solution. The second (third) solution is obtained by replacing the second (third) dip with the dashed curve. The fourth solution is the one for which both dips are replaced by the dashed curves.

type $\mathcal{R}$ where $\mathcal{R} = \nabla W_{2p}$. The flow Ψ^t cannot be defined on the whole space $\mathbb{R}^{2p}$ because $\mathcal{R}$ is only defined on $\Omega^{2p} \subset \mathbb{R}^{2p}$, given by (1.3.7). The space of piecewise linear braid diagrams of up-down type proves to be a good choice of underlying space for the flow Ψ^t. Information about the already known solutions decomposes this space into subsets called braid classes. In the last section of this introduction we review the results obtained in [9] which show that under certain conditions braid class is an isolating neighborhood for the flow Ψ^t. Hence Conley index theory can be applied to study the isolated invariant sets within these braid classes. In the case of a gradient flow, non-triviality of the invariant set implies the existence of a fixed point.

In this thesis we extend the approach developed in [9] to the braid classes which are not isolating neighborhoods, i.e., to those classes for which there is a fixed point of the flow Ψ^t lying on their boundary. Such a non-proper braid class is not an isolating neighborhood. The careful analysis of the flow near the fixed point allows us to take a subset of the non-proper braid class which is an isolating neighborhood for a slightly perturbed flow. Due to the robustness of the index with respect to small perturbations we obtain information about the invariant set within a non-proper braid class.

Parabolic flows on the space of discretized braid diagrams

First we motivate a connection between the solution $u \in [\sigma, 2p]$, or its $2p$-periodic sequence of extrema, and a discretized braid diagram. Let u be a solution of (1.2.2) in the class $[\sigma, 2p]$. Then its extrema sequence $\{u_i\}$ is $2p$-periodic and we can construct a piecewise linear graph by connecting the consecutive points $(i, u_i) \in \mathbb{R}^2$ by straight line segments. The piecewise linear graph, called a strand, is cyclic: one

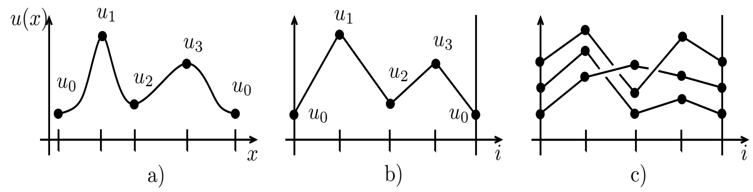

Figure 1.10: (a) A periodic function and (b) its piecewise linear graph (c) a braid consisting of 3 strands.

restricts to $0 \leq i \leq 2p$ and identifies the end points abstractly. A collection of n closed characteristics of period $2p$ then gives rise to a collection of n strands. We place on this diagram a *braid structure* by assigning a crossing type (positive) to every transverse intersection of the graphs: larger slope crosses over smaller slope, see Figure 1.10. In this way we represent periodic extrema sequence in the space of closed, positive, piecewise linear braid diagrams $\mathcal{D}_d^n$ see Definition 2.2.4.

Any PL-braid diagram corresponds to some n-collection $\{\mathbf{u}^k\}_{k=0}^{n-1}$ of $2p$-periodic extrema sequences. The converse to this statement is not true because collection of extrema sequences can have a non-transversal intersection which is not allowed for PL diagrams. The non-transversal intersection mean that there are two sequences $\mathbf{u}^k$ and $\mathbf{u}^l$ such that $u_i^k = u_i^l$ for some i and

$$(u_{i-1}^k - u_{i-1}^l)(u_{i+1}^k - u_{i+1}^l) \geq 0.$$

The middle diagram on the left side in Figure 1.11 shows a non-transversal intersection while the intersections in the other diagrams are transversal. A collection of n extrema sequences for with non-transversal intersection corresponds to a singular PL-braid diagram, see Definition 2.2.6. We denote by $\overline{\mathcal{D}}_d^n$ the space of all PL-diagrams, also the ones with non-transversal intersections, and $\Sigma = \overline{\mathcal{D}}_d^n \setminus \mathcal{D}_d^n$ is the set of singular braid diagrams. The space $\overline{\mathcal{D}}_d^n$ can be interpreted as a closure of $\mathcal{D}_d^n$. The set Σ^- is the subset of Σ which consist of braid diagrams for which two strands are identical. See Chapter 2 for a detailed explanation.

Since for bounded characteristics local minima and maxima occur alternatingly, we require that $(-1)^i(u_{i+1} - u_i) > 0$: the (natural) up-down restriction. Therefore an n-collection of extrema sequences $\{\mathbf{u}^k\}_{k=0}^{n-1}$ can be seen as a point in the space of up-down piecewise linear braid diagrams.

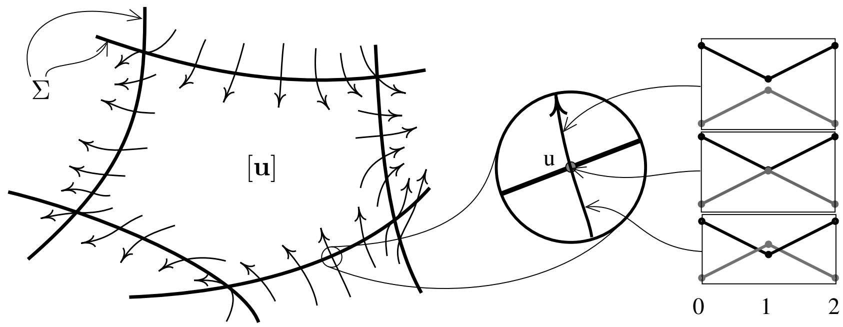

Figure 1.11: A schematic picture of a parabolic flow on a braid class. The boundary of the braid class is contained in the set of singular braids Σ.

DEFINITION 1.5.1. The space $\mathcal{E}^n_{2p}$ of up-down PL-braid diagrams on n strands with period $2p$ is the subset of $\mathcal{D}^n_{2p}$ determined by the relation $(-1)^i(u^k_{i+1} - u^k_i) > 0$, for $k = 1, \ldots, n$ and $i = 0, \ldots, 2p-1$.

As before we denote $\overline{\mathcal{E}}^n_d$ the closure of the space $\mathcal{E}^n_d$, i.e the space containing braid diagrams with non-transversal intersections. The up-down restriction on the space $\mathcal{E}^n_{2p}$ ensures that we can define the flow ψ^t on this space by differential equations

$$\frac{d}{dt}u^k_i = \mathcal{R}_i(u^k_{i-1}, u^k_i, u^k_{i+1}) \tag{1.5.1}$$

However, this flow does not respect the transversality condition. It can happen that a non-transversal intersection is created in finite time. Therefore, we consider the flow on the space $\overline{\mathcal{E}}^n_{2p}$.

The set $\overline{\mathcal{E}}^n_{2p}$ has a boundary in $\overline{\mathcal{D}}^n_{2p}$ which can be characterized as follows:

$$\partial\overline{\mathcal{E}}^n_{2p} = \{\mathbf{u} \in \overline{\mathcal{E}}^n_{2p} : u^k_i = u^k_{i+1} \text{ for at least one } i \text{ and } k\}. \tag{1.5.2}$$

Such braids, called horizontal singularities, are not included in Definition of $\overline{\mathcal{E}}^n_{2p}$ since the recurrence relation (1.3.5) does not induce a well-defined flow on the boundary $\partial\overline{\mathcal{E}}^n_{2p}$.

The flow coming from the parabolic recurrence relation evolves the anchor points of the braid diagram in such a way that the braid class can change, but only so as to decrease complexity: the word metric $|\mathbf{u}|_{word}$ of the braid diagram $\mathbf{u}$ may not increase with time. By $|\mathbf{u}|_{word}$ we mean the number of pairwise strand crossings in the braid diagram

$\mathbf{u}$. It is proved in [9] that the flow Ψ^t defined on the space $\overline{\mathcal{E}}_{2p}^n$ by Equations (1.5.1) is transversal to the set $\Sigma \backslash \Sigma^-$ and points from the class with bigger word metric to the one with smaller word metric, see Figure 1.11. Hence the orbit can not leave the braid class and return to it later. Moreover, the boundary $\partial \overline{\mathcal{E}}_{2p}^n$ is a repeller for the flow Ψ^t. Lemma 2.2.8 in Chapter 2 formalizes what we said above.

If v is a closed characteristic of a second order Lagrangian system, then the strand $\mathbf{v}$ corresponding to its extrema sequence $\{v_i\}$ is a fixed point for the parabolic flow Ψ^t generated by (1.5.1). Let $[\mathbf{u}]_{\mathcal{E}}$ be a braid class and suppose that $\mathbf{v} \notin \mathrm{cl}[\mathbf{u}]_{\mathcal{E}}$, then the intersection number of $\Psi^t(\mathbf{u}^k)$ and $\mathbf{v}$, for all strands $\mathbf{u}^k \in [\mathbf{u}]_{\mathcal{E}}$, is non-increasing. To use the fact that the crossing number is non-increasing, we will evolve certain components of a braid diagram while fixing the remaining components. This motivates working with a class of *relative* braid diagrams.

Let $\mathbf{u} \in \overline{\mathcal{E}}_{2p}^n$ and $\mathbf{v} \in \overline{\mathcal{E}}_{2p}^m$, then the union $\mathbf{u} \cup \mathbf{v} \in \overline{\mathcal{E}}_{2p}^{m+n}$ is naturally defined as the unordered union of the strands. For given $\mathbf{v} \in \mathcal{E}_{2p}^m$, we define

$$\mathcal{E}_{2p}^n \text{ rel } \mathbf{v} := \{\mathbf{u} \in \mathcal{E}_{2p}^n : \mathbf{u} \cup \mathbf{v} \in \mathcal{E}_{2p}^{n+m}\}.$$

The path components of $\mathcal{E}_{2p}^n$ rel $\mathbf{v}$, denoted $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$, define relative discrete braid classes. The braid $\mathbf{v}$ is fixed and it is called the skeleton. The set of singular braids $\Sigma_{\mathcal{E}}$ rel $\mathbf{v}$ are those braids $\mathbf{u}$ such that $\mathbf{u} \cup \mathbf{v} \in \Sigma_{\mathcal{E}}$. The collapsed singular braids are denoted by $\Sigma_{\mathcal{E}}^-$ rel $\mathbf{v}$. As before, the set $(\mathcal{E}_{2p}^n \text{ rel } \mathbf{v}) \cup (\Sigma_{\mathcal{E}} \text{ rel } \mathbf{v})$ is denoted by $\overline{\mathcal{E}}_{2p}^n$ rel $\mathbf{v}$.

Two relative braid classes $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ and $[\mathbf{u}' \text{ rel } \mathbf{v}']_{\mathcal{E}}$ in $\mathcal{E}_{2p}^n$ rel $\mathbf{v}$ and $\mathcal{E}_{2p}^n$ rel $\mathbf{v}'$ are equivalent if they lie in the same path component in

$$\mathbf{E} = \{(\mathbf{u}, \mathbf{v}) \in \mathcal{E}_{2p}^n \times \mathcal{E}_{2p}^m : \ \mathbf{u} \cup \mathbf{v} \in \mathcal{E}_{2p}^{n+m}\}.$$

We use the notation $[\mathbf{u} \text{ rel } [\mathbf{v}]]$, for the path component in $\mathbf{E}$, and $\mathbf{u}$ rel $[\mathbf{v}]$ for the fibers of $[\mathbf{u} \text{ rel } [\mathbf{v}]]$, see Figure 1.12. Before we apply the Conley index theory to relative braid classes we present a brief survey of the general Conley index theory. Finally, we present two important types of braid classes.

DEFINITION 1.5.2. A discrete relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ is proper if it is impossible to find a continuous path of braid diagrams $\mathbf{u}(t)$ for $t \in [0, 1]$ such that $\mathbf{u}(0) = \mathbf{u}$, $\mathbf{u}(t)$ rel $\mathbf{v}$ defines a braid for all $t \in [0, 1)$, and $\mathbf{u}(1)$ rel $\mathbf{v}$ is a diagram where an entire component of the closed braid has collapsed onto itself or onto another component of $\mathbf{u}$ or $\mathbf{v}$.

DEFINITION 1.5.3. A discrete relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ is called bounded if the set $[\mathbf{u} \text{ rel } \mathbf{v}]$ is bounded.

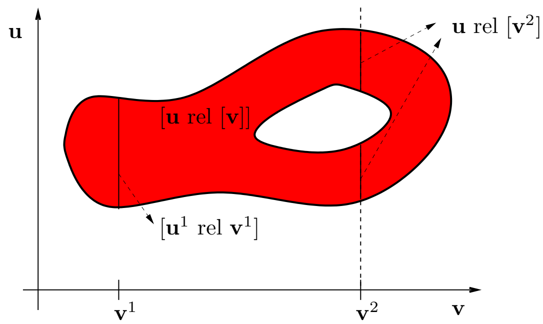

Figure 1.12: The path component $[\mathbf{u} \text{ rel } \mathbf{v}]$ and three different braid classes which are equivalent because they lie in the same path component. The fiber $\mathbf{u}$ rel $[\mathbf{v}^2]$ is a union of two braid class.

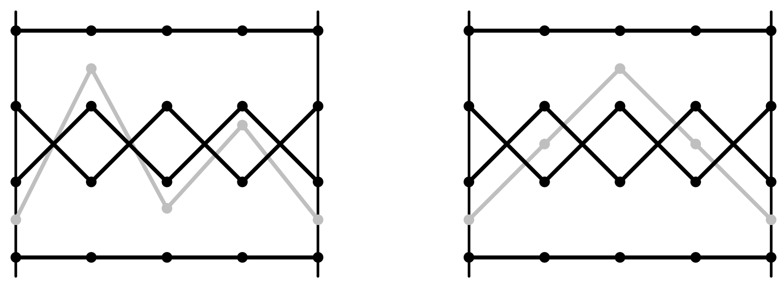

Figure 1.13: Non-proper [left] and proper [right] relative braid classes. Both classes are bounded.

Figure 1.13 shows examples of both proper and non-proper braid classes.

Conley index theory

The Conley index is a topological tool for studying the maximal invariant set within an isolating neighborhood. The Conley index is an invariant and thus robust with respect to perturbation of the isolating neighborhood as well as the flow. The price for robustness is that the information about the invariant set is rather coarse. It is possible to define the index in a very general setting. However, for the purpose of this thesis we just do so for a locally compact metric space X and a continuous flow $\Psi^t(x) : \mathbb{R} \times X \to X$. A compact set $N \subset X$ is an isolating neighborhood for the flow Ψ^t on X if the maximal invariant set

$$\text{Inv}(N) := \{x \in N \ : \ \text{cl}\{\Psi^t(x)\}_{t\in\mathbb{R}} \subset N\} \subset \text{int}(N)$$

is contained in the interior of N. The invariant set Inv(N) is then called a compact isolated invariant set for Ψ^t.

DEFINITION 1.5.4. A *pointed space* (Y, y_0) is a topological space Y with a distinguished point $y_0 \in Y$. Given a pair (N, L) of spaces with $L \subset N$,

$$N/L := (N \setminus L) \cup [L]$$

where $[L]$ denotes the equivalence class of points in L, with the equivalence relation $x \sim y$ if and only if $x = y$ or $x, y \in L$. We use N/L to denote the pointed space $(N/L, [L])$.

DEFINITION 1.5.5. Let $S = \text{Inv}(N)$ be an isolated invariant set. A pair of compact sets (N, L), where $L \subset N$, is called an *index pair* for S if:

(1) $S = \text{Inv}(\text{cl}(N \setminus L))$ and $N \setminus L$ is a neighborhood of S.
(2) L is positively invariant in N i.e. given $x \in L$ and $\Psi^t(x) \in N$ for all $t \in [0, t_0]$ then $\Psi^t(x) \in L$ for all $t \in [0, t_0]$.
(3) L is an exit set for N i.e. given $x \in N$ and $t_1 > 0$ such that $\Psi^{t_1}(x) \notin N$ then there exists $t_0 \in [0, t_1]$ with properties $\Psi^t(x) \in N$ for all $t \in [0, t_0]$ and $\Psi^{t_0}(x) \in L$.

The following proposition guarantees the existence of an index pair for every isolated invariant set and states a relation between different index pairs, see [14] for more details.

PROPOSITION 1.5.6. *Given an isolated invariant set S, there exists an index pair. Let (N, L) and (N', L') be index pairs for S. Then the pointed spaces*

$$[N/L] \simeq [N'/L]$$

have the same homotopy type.

DEFINITION 1.5.7. The homotopy Conley index of the isolated invariant set S is given by the homotopy type of pointed space

$$h(S) \simeq [N/L].$$

The Conley index of the invariant set is defined via its index pair (N, L). Hence the notation $h(N)$ is also used to denote the index $h(S)$. The crucial property of the index is given by the following proposition, see [14] for more details..

PROPOSITION 1.5.8. *If the Conley index $h(N)$ is non-trivial then*

$$\mathrm{Inv}(N) \neq \varnothing.$$

To compute the homotopy type is extremely difficult in general. The standard way to get around this problem is by using the homological Conley index defined by

$$H_*(h(N)) := H_*(N/L) \backsimeq H_*(N, L),$$

where $H_*(N/L)$ is the homology of the pointed space N/L and $H_*(N, L)$ is the relative homology.

REMARK 1.5.9. It is not true that $H_*(N/L) \backsimeq H_*(N, L)$ for any index pair. However, one can always find index pairs for which this isomorphism holds. For all index pairs used in this thesis the isomorphism is satisfied.

The characteristic polynomial is defined as

$$CP_t(N) := \sum_{k \geq 0} \beta_k t^k,$$

where $\beta_k = \mathrm{dim} H_k(N, L)$. As explained later, $N = \mathrm{cl}([\mathbf{u} \text{ rel } \mathbf{v}])$ is an isolating neighborhood for ever proper bounded relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$. In [9] the following estimate is proved:

$$\#\mathrm{Fix}([\mathbf{u} \text{ rel } \mathbf{v}]) \geq |CP_t|,$$

where $\mathrm{Fix}([\mathbf{u} \text{ rel } \mathbf{v}])$ is the number of the fixed points within $[\mathbf{u} \text{ rel } \mathbf{v}]$ and $|CP_t|$ is the number of distinct nonzero monomials in the characteristic polynomial CP_t.

The Conley index for braids

We demonstrate an application of the Conley index to the braid diagrams on an easy example. Let the braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ be given by the representant depicted in Figure 1.14. Suppose that the black skeleton strands $\mathbf{v}$ are fixed points of the flow Ψ^t generated by a parabolic recurrence relation. The anchor points u_0 and u_1 of the free strand $\mathbf{u}$ cannot cross any anchor points of the skeleton strands without changing the number of intersections with the skeleton and hence leaving the braid class. If u_0 moves out of the braid class then the intersection number increases, while for u_1 it decreases. According to Lemma 2.2.8 the direction of the flow is transversal to the boundary of the braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ and $N = \mathrm{cl}([\mathbf{u} \text{ rel } \mathbf{w}])$ is an isolating neighborhood. Figure 1.14 shows the sketch of the flow at the boundary of $[\mathbf{u} \text{ rel } \mathbf{v}]$. The exit set

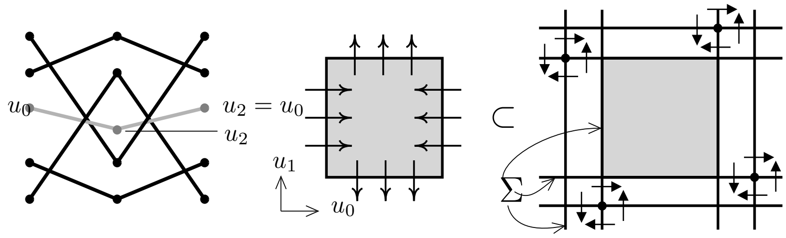

Figure 1.14: A representant of the relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}] \subset \mathcal{D}_2^1$ rel $\mathbf{v}$ with four skeleton strands (black) and one free strand (gray). A sketch of the flow Ψ^t at the boundary is shown in the middle. At the right we present a bigger portion of the space $\mathcal{D}_2^1$ rel $\mathbf{v}$ to show the direction of the flow in the neighborhood of fixed points corresponding to the skeleton $\mathbf{v}$. The braid classes adjacent to these fixed points are not proper.

$N^- \subset \partial N$ consists of the points for which u_1 coincides with an anchor point of some skeletal strand. The homotopy index $h(N) \simeq S^1$ and there exists a fixed point in $[\mathbf{u} \text{ rel } \mathbf{v}]$. Notice that the result holds for any flow Ψ^t generated by a parabolic recurrence relation as long as $\Psi^t(\mathbf{v}) = \mathbf{v}$ for all t.

In the same way as above the Conley index theory can be applied to arbitrary proper and bounded braid classes. The following theorem summarizes the results obtained in [9].

THEOREM 1.5.10. *Suppose* $[\mathbf{u} \text{ rel } \mathbf{v}]$ *is a bounded proper relative braid class and* Ψ^t *is a parabolic flow fixing* $\mathbf{v}$. *Then the following are true:*

(a) $N := \text{cl}([\mathbf{u} \text{ rel } \mathbf{v}])$ *is an isolating neighborhood for the flow* Ψ^t, *which thus yields a well-defined Conley index* $h(\mathbf{u} \text{ rel } \mathbf{v}) = h(N)$;

(b) *The index* $h(\mathbf{u} \text{ rel } \mathbf{v})$ *is independent of the choice of parabolic flow* Ψ^t *so long as* $\Psi^t(\mathbf{v}) = \mathbf{v}$;

One can also define an invariant for $[\mathbf{u} \text{ rel } [\mathbf{v}]]$ as follows. Given a fiber $\mathbf{u}$ rel $[\mathbf{v}] \in [\mathbf{u} \text{ rel } [\mathbf{v}]]$ define

$$\mathbf{H}(\mathbf{u} \text{ rel } [\mathbf{v}]) := \bigvee h(\mathbf{u}(i) \text{ rel } \mathbf{v})$$

where $\vee$ is a topological wedge and the classes $[\mathbf{u}(i) \text{ rel } \mathbf{v}]$ are the components of the fiber $\mathbf{u}$ rel $[\mathbf{v}]$, see Figure 1.12. In [9] it is proved that $\mathbf{H}$ is an invariant of $[\mathbf{u} \text{ rel } [\mathbf{v}]]$.

For any bounded proper relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ we can define its index intrinsically, independent of any notions of parabolic flows via the index pair (N, N^-). The set $N = \text{cl}[\mathbf{u} \text{ rel } \mathbf{v}] \subset \overline{\mathcal{D}}_{2p}^1$ rel $\mathbf{v}$. To define

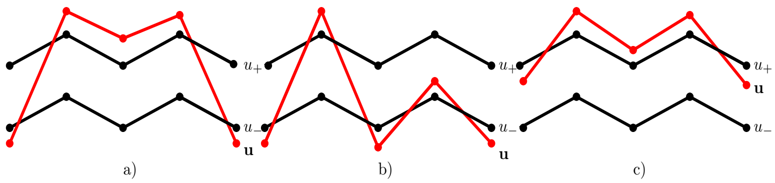

Figure 1.15: Representants of the three different relative braid classes. A fixed point in the relative braid class defined by its representant a) corresponds to the solution of the type I, b) type II and c) type III. Braid classes (a) and (b) are proper but (c) is not.

$N^- \subset \partial N \subset \Sigma$, consider $\mathbf{w} \in \partial N$ and let V be a small neighborhood of $\mathbf{w}$ for which the subset $V \setminus \Sigma$ has a finite number of connected components $\{V_j\}$ where $V_0 = V \cap N$. We define N^- as the set of $\mathbf{w}$ for which the word metric is locally maximal on V_0, namely

$$N^- := \{\mathbf{w} \in \partial N : |V_0|_{word} > |V_j|_{word} \quad \forall j > 0\}. \tag{1.5.3}$$

Let us now relate the three types of solutions in Figure 1.1 to braid classes and put them in the context of the definitions presented in this section. The three types of solutions are distinguished according to their intersections with the constant solutions $u_\pm = \pm 1$. The most straightforward way of relating a solution to a relative braid class would be to take the two constant strands $\pm\mathbf{1}$ as a skeleton and define the relative braid class by the free strand $\mathbf{u}$ which intersects the constant strands $\pm\mathbf{1}$ in the same manner as the solution u intersects $u_\pm$. However, the flow Ψ^t is well defined only for the braids with up-down restriction. Hence instead of taking the constant strands we have to use the skeleton $\mathbf{v} = \mathbf{u}_+ \cup \mathbf{u}_-$, where the strands $\mathbf{u}_\pm$ correspond to the solutions of Equation (1.2.2) which oscillate around $u_\pm$ with a small amplitude (on a slightly positive energy level) and the free strand $\mathbf{u}$ intersects the skeleton strands in the same manner as u intersects $u_\pm$. Figure 1.15 shows the three different braid classes which correspond to the three different types of solutions. The first two braid classes are proper and the third one is not. All these braid classes are obviously unbounded. It is shown in [9] how to use the properties of Equation (1.2.2) to find extra fixed strands which make the class bounded. See Chapter 2 for more details.

According to [9] the Conley index for any braid class corresponding to a solution of the first type is non-trivial and there is a fixed point in this class. This fixed point corresponds to a solution of Equation (1.2.2) of the first type. Thus there are many different solutions of the first type and their bifurcation branches exist for all $\alpha \geq 0$, see Figure 1.1.

For the second braid class the Conley index is trivial and thus does not provide information about fixed points. However, if we know that there exists a non-degenerate (hyperbolic) solution of the second type then it corresponds to a fixed point in the braid class with a trivial Conley index. Hence there must be another fixed point in this class which corresponds to a different solution of the same type and the bifurcation curves form loops, see Figure 1.1.

In the third case, the braid class is not proper (not an isolating neighborhood), since the free strand can collapse on a skeletal strand $\mathbf{u}_+$. Using the information about the flow Ψ^t near the strand $\mathbf{u}_+$, we perturb the parabolic recurrence relation on a neighborhood of the boundary of the non-proper braid class $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ and construct some new fixed strands which will make the class proper, without changing the invariant set inside the class. In Figure 1.16 we schematically demonstrate the direction of the flow Ψ^t and its perturbation on the boundary of the non-proper braid class and on the boundary of a new proper braid class $[\mathbf{u} \text{ rel } \overline{\mathbf{v}}]_{\mathcal{E}}$. The skeleton $\overline{\mathbf{v}}$ is created by adding extra strands which are fixed points of the perturbed vector field. We show that the Conley index $h([\mathbf{u} \text{ rel } \overline{\mathbf{v}}]_{\mathcal{E}})$ is non-trivial. Thus there is a fixed point of the flow Ψ^t within the class $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$. This is the essence behind the proof of Theorem 1.2.4. For a detailed treatment of the non-proper braid classes we refer the reader to Chapter 2.

1.6. Reflections

At different places in this introduction we already mentioned several interesting unresolved issues. This last section surveys them in more detail. We want to stress that although the main application of the theory developed in this thesis is the Swift-Hohenberg equation, it can be applied to a broad class of second order Lagrangian systems, see Section 1.3.

Different nonlinearities

We present extensions to equations

$$u'''' + \alpha u'' + f(u) = 0, \quad \alpha \in \mathbb{R},$$

for different nonlinearities $f(u)$. In this thesis we concentrate on the nonlinearity $f(u) = -u + u^3$, which corresponds to the double well potential $F(u) = \frac{1}{4}(u^2 - 1)^2$, depicted in Figure 1.17. However, the

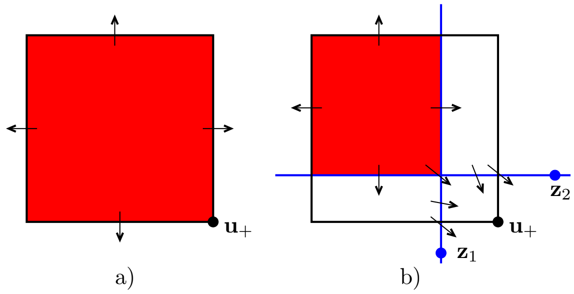

Figure 1.16: Figure (a) schematically shows the behavior of the vector field $\mathcal{R}$ on the boundary of the non-proper braid class $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ corresponding to the third type of solution. The strand $\mathbf{u}_+$ is a fixed point for the flow Ψ^t and some trajectories approach this point as $t \to -\infty$. Figure (b) shows the behavior of the perturbed vector field on the boundary of the proper braid class $[\mathbf{u} \text{ rel } \overline{\mathbf{v}}]_{\mathcal{E}}$, where $\overline{\mathbf{v}} = \mathbf{v} \cup \mathbf{z}_1 \cup \mathbf{z}_2$ and $\mathbf{z}_1, \mathbf{z}_2$ are fixed strands for this perturbed vector field.

previous equation has applications also for different nonlinearities. For example, in the study of a strut on a nonlinear elastic foundation and in the study of shallow waves [5], this equation arises with the nonlinearity $f(u) = u - u^2$. The homoclinic orbits of this equation have been extensively studied [1, 5, 6, 17]. The results obtained in Chapter 2 are applicable also to this nonlinearity. Basically Chapter 2 applies to any positive number of wells and Chapter 3 to two or more wells, see Figure 1.17c for a potential with the three wells. Potentials do not need to be either positive or negative.

Solutions of the third type

Theorem 1.2.4 states the existence of a solution of the class $[\sigma_1^{2q}, 2p]$ for $\alpha \in [\sqrt{8}, \alpha_{p,q})$. The Swift-Hohenberg equation is a twist system for $\alpha \geq 0$. Hence it should be possible to extend the result to $\alpha \in [0, \sqrt{8}]$. The major problem of this extension is that equilibria $u_\pm = \pm 1$ are saddle-foci and they do not perturb to periodic solutions for $E > 0$. Figure 1.17 shows a perturbation of the potential $F = \frac{1}{4}(u^2-1)^2$ of the Swift-Hohenberg equation, for a detailed explanation about perturbation F_ε see [9]. For every perturbation F_ε the small oscillations around $u_\pm$ are present on the positive energy levels. By applying Conley index theory for braid diagrams we can prove the existence of a solution u^ε of the perturbed system. Then by taking the limit $u = \lim_{\varepsilon \to 0} u^\varepsilon$ we get a solution of the original equation. The limit process is very similar

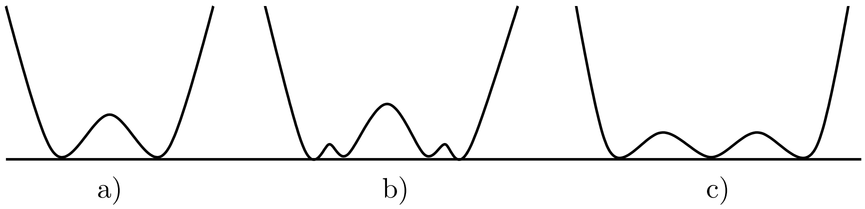

Figure 1.17: Figure a) Potential $F(u) = \frac{1}{4}(u^2 - 1)^2$; b) small perturbation of $F(u)$; c) potential with three wells.

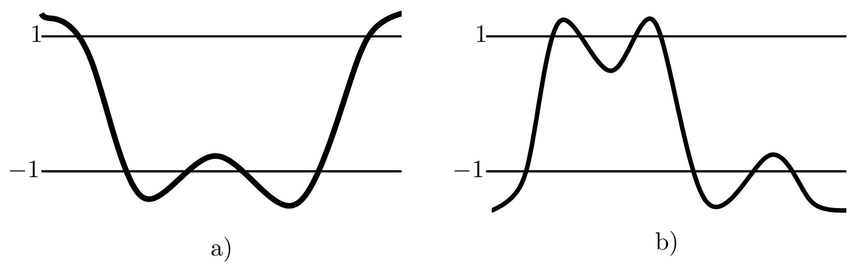

Figure 1.18: Sketch of periodic solutions; a) of the class $\left[\sigma_2^2\sigma_1^4, 4\right]$; b) of the class $\left[\sigma_1^4\sigma_2^4, 4\right]$.

to the one carried out in Chapter 2. However, the perturbation of the potential is not present in Chapter 2. In order to extend the result of Theorem 1.2.4 to the parameter range $\alpha \in [0, \sqrt{8}]$ one has to check that $u \not\equiv 1$ in the limit $\varepsilon \to 0$.

The Conley index computed in Lemma 2.5.6 implies the existence of two different solutions of the Swift-Hohenberg equation on the positive energy levels. We believe that by a careful analysis of the limit process one can show that these solutions do not collapse on each other. Hence there should be at least two different solutions in every class $[\sigma_1^{2q}, 2p]$. For comparison, the study of the bifurcation diagram for Equation (1.2.2) in [19] proves the existence of two solutions of the same class in a special case.

THEOREM 1.6.1. *For $p \in \mathbb{N}$ there exists two families of even periodic solutions of the class $[\sigma_1^2, 2p]$ for $\alpha \in (-\sqrt{8}, \alpha_n)$ where $\alpha_p = \sqrt{2}\left(1 + \frac{1}{p}\right)$.*

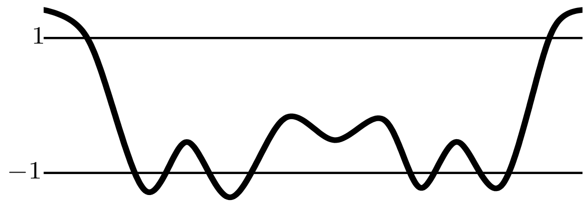

Figure 1.19: Sketch of a periodic solutions of the class $\left[\sigma_2^2\sigma_1^8, 10\right]$.

Solutions of the second type

Theorem 1.4.6 shows that the class $\left[\sigma_1^2\sigma_2^4, 4\right]$ precedes classes of the form $\left[\sigma_1^2\sigma_2^{2q}, 2p\right]$. The symmetry of the Swift-Hohenberg equation furnishes analogous results for the class $\left[\sigma_2^2\sigma_1^4, 4\right]$, which can be formulated as follows:

$$\left[\sigma_2^2\sigma_1^4, 4\right] \prec \left[\sigma_2^2\sigma_1^{2q}, 2p\right].$$

The number of solutions of the class $\left[\sigma_2^2\sigma_1^{2q}, 2p\right]$ is the same as in Theorem 1.4.6. Figure 1.18a shows a solution of the class $\left[\sigma_2^2\sigma_1^4, 4\right]$ and Figure 1.19 depicts a solution of the class $\left[\sigma_2^2\sigma_1^8, 10\right]$.

By combining these results one should be able to prove the following theorem.

THEOREM 1.6.2. *Let* $p, q_1, q_2 \in \mathbb{N}$ *such that* $p > 3$, $q_1 + q_2 \le p$ *and* $\max\{q_1, q_2\} > 3$ *if* $q_1 + q_2 < p$. *Then*

$$\left[\sigma_2^4\sigma_1^4, 6\right] \prec \left[\sigma_2^{2q_1}\sigma_1^{2q_2}, 2p\right].$$

A solution of the class $\left[\sigma_2^4\sigma_1^4, 6\right]$ is depicted in Figure 1.18b, and Figure 1.20 shows a solution of the class $\left[\sigma_2^8\sigma_1^6, 12\right]$. By similar arguments as in Chapter 3 one can estimate the number of solutions of the class $\left[\sigma_2^{2q_1}\sigma_1^{2q_2}, 2p\right]$ and by connecting the building blocks we can obtain solutions of the class

$$\left[\sigma_2^{2q_1}\sigma_1^{2q_2}\ldots\sigma_2^{2q_{n-1}}\sigma_1^{2q_n}, 2p\right],$$

for every $p, q_1, \ldots q_n \in \mathbb{N}$ such that $\sum_{i=1}^n q_i \le p$ and $\max_{i\in\{1,\ldots n\}} q_i > 3$ if $\sum_{i=1}^n q_i < p$.

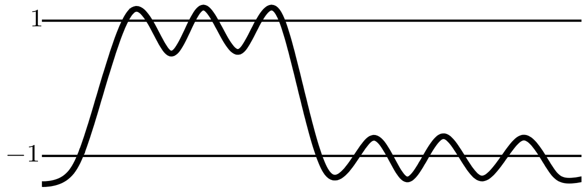

Figure 1.20: Sketch of a periodic solutions of the class $\left[\sigma_1^8\sigma_2^6, 12\right]$.

CHAPTER 2

The Conley index for non-proper braid classes and application to Swift-Hohenberg equation

2.1. Introduction

Whereas the solutions of second order autonomous ODEs can be represented in a phase plane, leading to modest complexity of the dynamics, equations of higher order can exhibit a plethora of distinct behaviors, and the dependence of the dynamics on parameters is extremely complex. One of the challenges is to obtain global results for general families of equations that have additional structure. This chapter revolves around the periodic solutions of the fourth order equation

$$u'''' + \alpha u'' - u + u^3 = 0, \tag{2.1.1}$$

and it generalizations. As explained in Chapter 1 Equation (2.1.1) has a variational formulation via second order Lagrangian system:

$$\int_I L(u, u', u'')dt, \tag{2.1.2}$$

where

$$L(u, v, w) = \frac{1}{2}w^2 - \frac{\alpha}{2}v^2 + \frac{1}{4}(u^2 - 1)^2. \tag{2.1.3}$$

Related to this variational structure (through Noether's theorem) is a conserved quantity: solutions of Equation (2.1.1) satisfy the energy equation

$$\mathbb{E}[u] = -u'u''' + \frac{1}{2}(u'')^2 - \frac{\alpha}{2}(u')^2 - \frac{1}{4}(u^2 - 1)^2 = E. \tag{2.1.4}$$

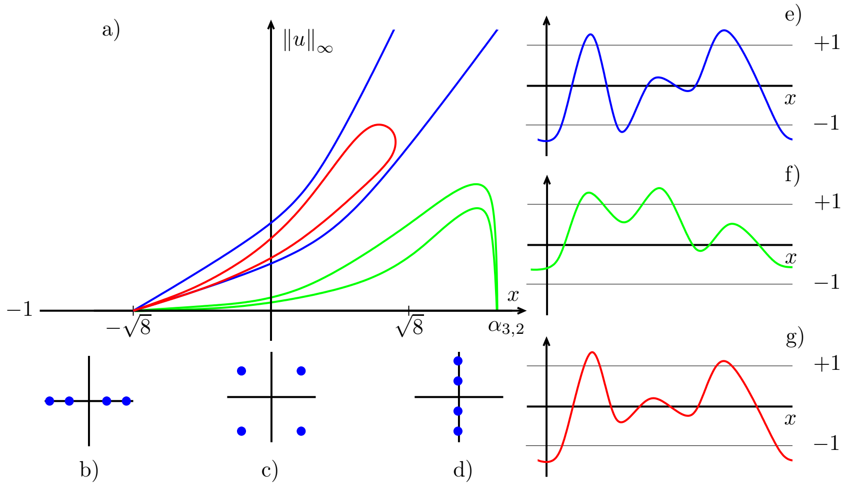

Figure 2.1: Bifurcation diagram a) shows three different types of branches, in the plane $(\alpha, \|u\|_\infty)$, which bifurcate for $\alpha = -\sqrt{8}$. Solutions on the branches that extend beyond the boundary of the diagram are of the first type, see e) for an example; branches that form closed loops consist of solutions of the second type, see f) for an example; branches collapsing on $\|u\|_\infty = 1$ consist of solutions of the third type, see g) for an example. Also depicted is the spectrum of the linearization around P_+ and P_- for b) $\alpha \leq -\sqrt{8}$; c) $\alpha \in (-\sqrt{8}, \sqrt{8})$; d) $\sqrt{8} \leq \alpha$.

In the case $\alpha < 0$ the Lagrangian defined by (2.1.3) is referred to as the eFK-Lagrangian (see e.g. [10, 12, 13]), while for $\alpha \geq 0$ it is usually referred to as the Swift-Hohenberg Lagrangian [19]. Equation (2.1.1) appears in the description of special phase transitions, as well as in the Swift-Hohenberg model for Rayleigh-Bénard convection.

In Chapter 1 we introduced a classification of periodic solutions of Equation (2.1.1). We distinguished three main classes of periodic solutions. In this chapter we are interested in solutions of the third type. For this solutions we can not define the isolating neighborhood as for the other types of solutions because there is a fixed point at the boundary. We use linearization at this fixed point to overcome the problem. The Conley index for the third type of solutions is non-trivial up to some value of α and then it becomes trivial. This explains the bifurcation diagram in Figure 2.2.

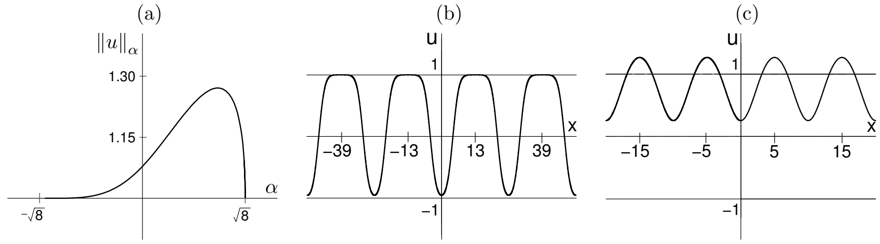

Figure 2.2: Bifurcation diagram for the solutions of the third type and corresponding solutions which are in the class $\mathbf{u}_{1,1}$. Solution (b) corresponds to $\alpha = -1$ and (c) to $\alpha = 1$.

In the present chapter we use slightly different notation for classes of solutions of the third type. The next definition introduces the notation used in this chapter.

DEFINITION 2.1.1. We say that a periodic solution is of class $\mathbf{u}_{p,q}$ if it is a solution of the third type with $2p$ monotone laps per period and intersects $2q$ times the constant solution $u_+ = 1$.

Solutions of the third type come as a family of countably many distinct periodic solutions which bifurcate from the heteroclinic loop at $\alpha = -\sqrt{8}$. However, this family does not extend to infinity (as the first type) in parameter space nor do they lie on loops (as the second type). Instead, numerical results indicate that these periodic solutions bifurcate from the constant solution u_+ as α tends to a critical value $\alpha_{p,q}$ (see Figure 2.2a) of the form

$$\alpha_{p,q} = \sqrt{2}\left(\frac{p}{q} + \frac{q}{p}\right), \quad p, q \in \mathbb{N}, \quad (p \geq q). \tag{2.1.5}$$

For $q = 1$ and $p \in \mathbb{N}$ it was analytically shown in [19] that there exists a family of solutions in the class $\mathbf{u}_{p,1}$ for $\alpha \in (-\sqrt{8}, \alpha_{p,1})$. Moreover, for $p \geq 2$ these solutions come in pairs. Numerically computed graphs of two solutions of class $\mathbf{u}_{1,1}$ are shown in Figure 2.2.

The shooting technique used in [19] to prove existence of a solution of class $\mathbf{u}_{1,1}$ depends very strongly on the particular equation. The method which we develop here generalizes this result in two ways. The application of our method to Equation (2.1.1) proves the existence of solutions of the class $\mathbf{u}_{p,q}$ for every relatively prime $p, q \in \mathbb{N}$. The other aspect is that this technique is not limited to this specific equation. It can be applied to a huge variety of equations which have a variational formulation (2.1.2). The main idea is to use already known solutions to force existence of another one. This idea goes back to [9] where it was shown that a solution of Euler-Lagrange equation of Lagrangian system with a *twist property* corresponds to a fixed point of the flow Ψ^t generated by a parabolic recurrence relation which is defined on the space of braids. The space of braids is not connected; its connected components are called braid classes. For a more detailed explanation see the next section.

The braid classes used in [9] are isolating neighborhoods for the flow Ψ^t. Therefore the Conley index can be employed to show the existence of a fixed point within the class. However, in our case the braid class fails to be an isolating neighborhood because there is a fixed point on its boundary. This type of braid classes is called non-proper. In this chapter, inspired by techniques in [2], we show how to use local information near this fixed point to define a modified braid class which is an isolating neighborhood, and the modifications do not change the invariant set inside the braid class. This allows us to prove the existence of a solution which corresponds to a fixed point in a non-proper braid class. It is enough to know one solution (and its linearisation) in order to force the existence of different solutions.

By applying this result to Equation (2.1.1) we show the existence of different solutions of the third type. Namely for any $p \geq q$ we prove that there is a solution $u \in \mathbf{u}_{p,q}$, for $\alpha \in [\sqrt{8}, \alpha_{p,q})$. On the other hand we cannot use this approach to extend the result for $\alpha \geq \alpha_{p,q}$ because the local behavior in the fixed point on the boundary of the braid class changes character for this parameter value. Indeed, numerics suggest that the branch of the solutions in the class $\mathbf{u}_{p,q}$ bifurcates from the constant solution $u_+ = 1$ at $\alpha = \alpha_{p,q}$.

We note that the solution found by topological methods is on a slightly positive energy level E. Hence a limit process for $E \to 0$ is needed to prove the following theorem.

THEOREM 2.1.2. *Let $p, q \in \mathbb{N}$ be relatively prime and $q < p$. Then there exists a solution $u^\alpha \in \mathbf{u}_{p,q}$ of Equation (2.1.1) with $\mathbb{E}[u^\alpha] = 0$ for every $\alpha \in [\sqrt{8}, \alpha_{p,q})$.*

REMARK 2.1.3. One should be able to extend the previous theorem to the parameter range $[0, \sqrt{8}]$, where the eigenvalues of equilibria $u_\pm = \pm 1$ are saddle-foci, by perturbing the potential $F = \frac{1}{4}(u^2 - 1)^2$ as explained in [9].

By using topological methods we have thus found a plethora of different solutions of the third type.

2.2. Reduction to the finite dimensional problem

In this section we give a brief survey of the reduction of the problem of finding periodic solutions for Equation (2.1.1) to the problem of finding fixed points of a vector field generated by a parabolic recurrence relation. We present this approach in the context of general second order Lagrangians.

If we seek closed characteristics i.e., a periodic solution of Equation (2.1.1) at a given energy level E we can invoke the following variational principle:

$$\text{Extremise } \{J_E[u] : u \in \Omega_{per}, \tau > 0\}, \tag{2.2.1}$$

where $\Omega_{per} = \cup_{\tau>0} C^2(S^1, \tau)$, the periodic functions with period τ, and

$$J_E[u] = \int_0^\tau (L(u, u', u'') + E)dt. \tag{2.2.2}$$

The function $L \in C^2(\mathbb{R}^3, \mathbb{R})$ is assumed to satisfy $\frac{\partial^2 L}{\partial w^2}(u, v, w) \geq \delta > 0$ for all $(u, v, w) \in \mathbb{R}^3$. For the general second order Lagrangian system the (conserved) energy is given by

$$\mathbb{E}[u] = \left(\frac{\partial L}{\partial u'} - \frac{d}{dt}\frac{\partial L}{\partial u''}\right) u' + \frac{\partial L}{\partial u''} u'' - L(u, u', u''). \tag{2.2.3}$$

It follows from [20] that the variations in τ guarantee that any critical point u of (2.2.1) has energy $\mathbb{E}[u] = E$.

An energy value E is called regular if $\frac{\partial L}{\partial u}(u, 0, 0) \neq 0$ for all u that satisfy $L(u, 0, 0) + E = 0$. The energy manifold $M_E \subset \mathbb{R}^4$ for a regular energy value E is a smooth non-compact manifold without boundary. For a fixed regular energy value E, the extrema of a characteristic are contained in the closed set $\{u : L(u, 0, 0) + E \geq 0\}$. The connected components I_E of this set are called interval components. Moreover, it follows from [20] that solutions on a regular energy level do not have inflexion points. For a singular energy level the interval component I_E contains critical points and the situation is more complicated.

First, we restrict to the regular energy levels. It was shown in [20] that for Lagrangian systems $J[u] = \int_I L(u, u', u'')dt$, where $L(u, u', u'') = \frac{1}{2}u''^2 + K(u, u')$ at energy levels E which satisfy

$$\frac{\partial K}{\partial v}v - K(u,v) - E \leq 0 \text{ for all } u \in I_E \text{ and } v \in \mathbb{R}, \tag{2.2.4a}$$

$$\frac{\partial^2 K}{\partial v^2}v^2 - \frac{5}{2}\left\{\frac{\partial K}{\partial v} - K(u,v) - E\right\} \geq 0 \text{ for all } u \in I_E \text{ and } v \in \mathbb{R}, \tag{2.2.4b}$$

there is a unique pair (τ, u_τ) minimizing

$$\inf_{u \in X_\tau, \tau \in \mathbb{R}^+} \int_0^\tau (L(u, u', u'') + E)dt,$$

where $X_\tau = X_\tau(u_1, u_2) = \{u \in C^2([0,\tau]) : u(0) = u_1, u(\tau) = u_2, u'(0) = u'(\tau) = 0, u|_{(0,\tau)} > 0 \text{ if } u_1 < u_2 \text{ and } u|_{(0,\tau)} < 0 \text{ if } u_1 > u_2\}$ for $(u_1, u_2) \in I_E \times I_E \setminus \Delta$ and $\Delta = \{(u_1, u_2) \in I_E \times I_E : u_1 = u_2\}$. Moreover, the function defined by

$$S_E(u_1, u_2) = \inf_{u \in X_\tau, \tau \in \mathbb{R}^+} \int_0^\tau (L(u, u', u'') + E)dt, \tag{2.2.5}$$

for $(u_1, u_2) \in I_E \times I_E \setminus \Delta$ and $S_E|_\Delta = 0$ has the following properties:

(a) $S_E \in C^2(I_E \times I_E \setminus \Delta)$.
(b) $\partial_1\partial_2 S_E(u_1, u_2) > 0$ for all $u_1 \neq u_2 \in I_E$.
(c) $\lim_{u_1 \nearrow u_2} -\partial_1 S_E(u_1, u_2) = \lim_{u_2 \searrow u_1} \partial_2 S_E(u_1, u_2) =$
$= \lim_{u_1 \searrow u_2} \partial_1 S_E(u_1, u_2) = \lim_{u_2 \nearrow u_1} -\partial_2 S_E(u_1, u_2) = +\infty$.

We call the function S_E a *generating function* and the Lagrangian system possessing such a generating function is called a *twist system*. The second order Lagrangian system associated to Equation (2.1.1) is a twist system for $\alpha \geq 0$. For more examples see [20].

The question of finding closed characteristics for a twist system can now be formulated in terms of S_E. Any periodic solution u is a concatenation of monotone laps. Let us take an arbitrary $2p$ periodic sequence $\{u_i\}$ and define u as a concatenation of monotone laps (minimizers $u_\tau(u_i, u_{i+1})$) between the consecutive extremal points u_i solving the Euler-Lagrange equation in between any two extrema. The concatenation u does not have to be a solution on $\mathbb{R}$ because the third derivatives of two monotone laps do not have to match at the extremal point u_i. It was proved in [20] that the third derivatives match if and only if the extrema sequence $\{u_i\}$ is a critical point of discrete action

$$W_{2p} = \sum_{i=0}^{2p-1} S_E(u_i, u_{i+1}). \tag{2.2.6}$$

Critical points of W_{2p} satisfy equations

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = \partial_2 S_E(u_{i-1}, u_i) + \partial_1 S_E(u_i, u_{i+1}) = 0, \tag{2.2.7}$$

where $\mathcal{R}_i(s,t,r)$ is, according to property (a), well-defined and C^1 on the following domain

$$\Omega_i = \{(r,s,t) \in I_E^3 : (-1)^{i+1}(s-r) > 0, (-1)^{i+1}(s-t) > 0\}. \tag{2.2.8}$$

The functions $\mathcal{R}_i$ and domains Ω_i satisfy $\mathcal{R}_i = \mathcal{R}_{i+2}$ and $\Omega_i = \Omega_{i+2}$ for $i \in \mathbb{Z}$. Property (b) implies that $\partial_1 \mathcal{R}_i = \partial_1\partial_2 S(u_{i-1}, u_i) > 0$, and $\partial_3 \mathcal{R}_i = \partial_1 \partial_2 S(u_i, u_{i+1}) > 0$.

Property (c) provides information about the behavior of $\mathcal{R}_i$ at the diagonal boundaries of Ω_i, namely,

$$\lim_{s \searrow r} \mathcal{R}_i(r,s,t) = \lim_{s \searrow t} \mathcal{R}_i(r,s,t) = +\infty, \tag{2.2.9}$$

$$\lim_{s \nearrow r} \mathcal{R}_i(r,s,t) = \lim_{s \nearrow t} \mathcal{R}_i(r,s,t) = -\infty. \tag{2.2.10}$$

Above-mentioned properties of $\mathcal{R}_i$ give us that $\mathcal{R}_i$ is parabolic recurrence relation of up-down type as defined below. First, we define parabolic recurrence relations.

DEFINITION 2.2.1. A parabolic recurrence relation $\mathcal{R}$ on $\mathbb{R}^{\mathbb{Z}}$ is a sequence of real-valued functions $\mathcal{R} = (\mathcal{R}_i)_{i\in\mathbb{Z}}$ satisfying

(A1): [monotonicity] $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_i > 0$ for all $i \in \mathbb{Z}$

(A2): [periodicity] for some $d \in \mathbb{N}, \mathcal{R}_{i+d} = \mathcal{R}_i$ for all $i \in \mathbb{Z}$.

We see that our $\mathcal{R}$ is not a parabolic recurrence relation in the strict sense because it is not defined on whole space $\mathbb{R}^{\mathbb{Z}}$. It is not defined for any sequence satisfying $u_i = u_{i+1}$ for some $i \in \mathbb{Z}$. This corresponds to the nature of solutions of Equation (2.1.1), namely that minima and maxima alternate.

DEFINITION 2.2.2. A parabolic recurrence relation $\mathcal{R}$ defined on domain given by (2.2.8) is said to be of up-down type if (2.2.9) and (2.2.10) are satisfied.

These results can be summarized in terms of parabolic recurrence relation as follows.

THEOREM 2.2.3. *Let $J[u] = \int L(u, u', u'')dt$ be a second order Lagrangian twist system. Suppose that W_{2p} is the discrete action defined through (2.2.5) and (2.2.6) at the regular energy level E. Then*

(a) *the functions $\mathcal{R}_i = \partial_i W_{2p}$ defined on Ω_i are components of a parabolic recurrence relation $\mathcal{R}$ of up-down type,*

(b) *solutions of $\mathcal{R} = 0$ correspond to periodic solutions on the energy level E.*

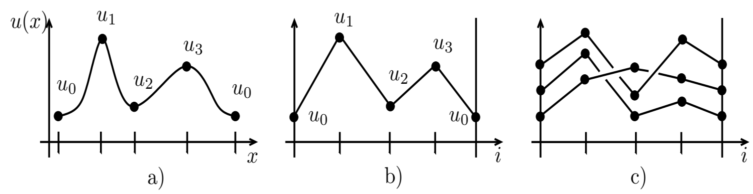

Figure 2.3: (a) A periodic function and (b) its piecewise linear graph (c) a braid consisting of 3 strands.

In order to find solutions of $\mathcal{R} = 0$ we will employ the Conley index. Conley index theory gives information about the invariant set of a flow inside an isolating neighborhood for this flow. In the case of a gradient vector field invariant sets have special structure and thus information about critical points can be obtained. There is a natural way to define a flow generated by an up-down parabolic recurrence relation on the set

$$\Omega^{2p} = \{\mathbf{u} \in \mathbb{R}^{\mathbb{Z}} : \mathbf{u} \text{ is } 2p \text{ periodic and } (u_{i-1}, u_i, u_{i+1}) \in \Omega^i, \text{ for } i \in \mathbb{Z}\}. \tag{2.2.11}$$

Consider the differential equations

$$\frac{d}{dt} u_i(t) = \mathcal{R}_i(\mathbf{u}(t)), \quad \mathbf{u}(t) \in \Omega^{2p}, \quad t \in \mathbb{R}. \tag{2.2.12}$$

Equation (2.2.12) defines a (local) C^1 flow ψ^t on Ω^{2p}. This flow is not defined at the boundary of Ω^{2p}, but conditions (2.2.9) and (2.2.10) give us information about the flow close to this boundary.

Thus, finding a periodic solution within the class $\mathbf{u}_{p,q}$ can be reduced to constructing an appropriate isolating neighborhood for the flow Ψ^t and calculating its (nontrivial) Conley index. We will use the concept of up-down discretized braid diagrams to construct this isolating neighborhood. For any $2p$-periodic extrema sequence we can construct a piecewise linear graph by connecting the consecutive points $(i, u_i) \in \mathbb{R}^2$ by straight line segments. The piecewise linear graph, called a strand, is cyclic: one restricts to $0 \leq i \leq 2p$ and identifies the end points abstractly. A collection of n closed characteristics of period $2p$ then gives rise to a collection of n strands. We place on this diagram a *braid structure* by assigning a crossing type (positive) to every transverse intersection of the graphs: larger slope crosses over smaller slope, see Figure 2.3. We represent periodic sequence of extrema in the space of closed, positive, piecewise linear braid diagrams. We briefly recall some basic facts from (discrete) braid theory (for more details see [9]).

DEFINITION 2.2.4. Denote by $\mathcal{D}_d^n$ the space of all closed piecewise linear braid diagrams (PL-braid diagrams) on n strands with period d. That is, the space of all (unordered) collections $\beta = \{\beta^k\}_{k=1}^n$ of continuous maps $\beta^k : [0,1] \to \mathbb{R}$ such that

(a) β^k is affine linear on $[\frac{i}{d}, \frac{i+1}{d}]$ for all k and for all $i = 0, \ldots, d-1$;
(b) $\beta^k(0) = \beta^{\tau(k)}(1)$ for some permutation τ;
(c) for any s such that $\beta^k(s) = \beta^l(s)$ with $k \neq l$, the crossing is transversal: for ϵ sufficiently small

$$(\beta^k(s-\epsilon) - \beta^l(s-\epsilon))(\beta^k(s+\epsilon) - \beta^l(s+\epsilon)) < 0.$$

Any PL-braid diagram corresponds to some n-collection $\{\mathbf{u}^k\}_{k=0}^{n-1}$ of $2p$-periodic extrema sequences via the relation

$$u_i^k = \beta^k\,(i \bmod 2p)\,, \tag{2.2.13}$$

where u_i^k is i-th entry of k-th extrema sequence. The converse to this statement is not true because condition (c) of Definition 2.2.4 is not satisfied for arbitrary collection of extrema sequences. A collection of n extrema sequences for which this condition is violated corresponds to a singular PL-braid diagram (see Definition 2.2.6). We switch between the notation u_i^k of the anchor points and β^k of the piecewise linear braid diagrams throughout this section, using β only if necessary.

Since for bounded characteristics local minima and maxima occur alternatively, we require that $(-1)^i(u_{i\pm1} - u_i) > 0$: the (natural) up-down restriction. Therefore an n-collection of extrema sequences $\{\mathbf{u}^k\}_{k=0}^{n-1}$ can be seen as a point in the space of up-down piecewise linear braid diagrams.

DEFINITION 2.2.5. The space $\mathcal{E}_{2p}^n$ of up-down PL-braid diagrams on n strands with period $2p$ is the subset of $\mathcal{D}_{2p}^n$ determined by the relation $(-1)^i(u_{i+1}^k - u_i^k) > 0$, for $k = 1, \ldots, n$ and $i = 0, \ldots, 2p-1$.

The up-down restriction on the space $\mathcal{E}_{2p}^n$ ensures that we can define the flow on this space by differential equations

$$\frac{d}{dt}u_i^k = \mathcal{R}_i(u_{i-1}^k, u_i^k, u_{i+1}^k) \tag{2.2.14}$$

However, this flow does not respect condition (c) of Definition 2.2.4. It can happen that this condition gets violated in finite time. Therefore, we introduce a concept of singular braid diagrams which act as gates between the path components of $\mathcal{D}_{2p}^n$ ($\mathcal{E}_{2p}^n$).

DEFINITION 2.2.6. Denote by $\overline{\mathcal{D}}_d^n$ the space of all PL-braid diagrams which satisfy properties (a) and (b) of Definition 2.2.4 (strong closure). Denote by $\Sigma = \overline{\mathcal{D}}_d^n \setminus \mathcal{D}_d^n$ the set of singular braid diagrams and by $\Sigma^- = \{\beta \in \Sigma : \beta^k(s) = \beta^l(s)$ for all s and some $k \neq l\}$ the set of collapsed singularities. The path component in $\mathcal{D}_d^n$ is called a braid class.

DEFINITION 2.2.7. Let $\overline{\mathcal{E}}_{2p}^n$ be the subset of all braid diagrams in $\overline{\mathcal{D}}_{2p}^n$ satisfying $(-1)^i(u_{i+1}^k - u_i^k) > 0$. As before the singular braid diagrams are defined as $\Sigma_{\mathcal{E}} = \overline{\mathcal{E}}_{2p}^n \setminus \mathcal{E}_{2p}^n$. The path components in $\mathcal{E}_{2p}^n$ comprise the up-down braid types $[\mathbf{u}]_{\mathcal{E}}$, where $\mathbf{u} = (u_i^k)$. Again $\Sigma_{\mathcal{E}}^- = \{\beta \in \Sigma_{\mathcal{E}} : \beta^k(s) = \beta^l(s)$ for all s and some $k \neq l\}$ is the set of collapsed singularities.

The set $\overline{\mathcal{E}}_{2p}^n$ has a boundary in $\overline{\mathcal{D}}_{2p}^n$ which can be characterized as follows:

$$\partial\overline{\mathcal{E}}_{2p}^n = \{\mathbf{u} \in \overline{\mathcal{E}}_{2p}^n : u_i^k = u_{i+1}^k \text{ for at least one } i \text{ and } k\}. \tag{2.2.15}$$

Such braids, called horizontal singularities, are not included in Definition of $\overline{\mathcal{E}}_{2p}^n$ since the recurrence relation (2.2.7) does not induce a well-defined flow on the boundary $\partial\overline{\mathcal{E}}_{2p}^n$.

The flow coming from the parabolic recurrence relation evolves the anchor points of the braid diagram in such a way that the braid class can change, but only so as to decrease complexity: the word metric $|\mathbf{u}|_{word}$ of the braid diagram $\mathbf{u}$ may not increase with time. By the $|\mathbf{u}|_{word}$ we mean the number of pairwise strand crossings in the braid diagram $\mathbf{u}$. The following results, proved in [9], shows that the crossing number acts as a discrete Lyapunov function for any parabolic flow on $\mathcal{E}_{2p}^n$ and the boundary $\partial\overline{\mathcal{E}}_{2p}^n$ is a repeller.

LEMMA 2.2.8. *Let Ψ^t be a parabolic flow of up-down type on $\overline{\mathcal{E}}_{2p}^n$.*

(a) *For each point $\mathbf{u} \in \Sigma_{\mathcal{E}} - \Sigma_{\mathcal{E}}^-$, the local orbit $\{\Psi^t(\mathbf{u}) : t \in [-\epsilon, \epsilon]\}$ intersects $\Sigma_{\mathcal{E}}$ uniquely at $\mathbf{u}$ for all ϵ sufficiently small.*
(b) *For any such $\mathbf{u}$, the word metric of the braid diagram $\Psi^t(\mathbf{u})$ for $t > 0$ is strictly less than that of the diagram $\Psi^t(\mathbf{u})$, $t < 0$.*
(c) *The flow blows up in a neighborhood of $\partial\overline{\mathcal{E}}_{2p}^n$ in such a manner that the vector field points into $\overline{\mathcal{E}}_{2p}^n$.*

If v is a closed characteristic of a second order Lagrangian system, then the strand $\mathbf{v}$ corresponding to its extrema sequence $\{v_i\}$ is a fixed point for parabolic flow Ψ^t generated by (2.2.14). Let $[\mathbf{u}]_{\mathcal{E}}$ be a braid class and suppose that $\mathbf{v} \notin \mathrm{cl}[\mathbf{u}]_{\mathcal{E}}$ then the intersection number of

$\Psi^t(\mathbf{u}^k)$ and $\mathbf{v}$, for all strands $\mathbf{u}^k \in [\mathbf{u}]_{\mathcal{E}}$, is non-increasing. To use the fact that crossing number is non-increasing we will evolve certain components of a braid diagram while fixing the remaining components. This motivates working with a class of *relative* braid diagrams.

Let $\mathbf{u} \in \overline{\mathcal{E}}_{2p}^n$ and $\mathbf{v} \in \overline{\mathcal{E}}_{2p}^m$, the union $\mathbf{u} \cup \mathbf{v} \in \overline{\mathcal{E}}_{2p}^{m+n}$ is naturally defined as the unordered union of the strands. For given $\mathbf{v} \in \mathcal{E}_{2p}^m$, we define

$$\mathcal{E}_{2p}^n \text{ rel } \mathbf{v} := \{\mathbf{u} \in \mathcal{E}_{2p}^n : \mathbf{u} \cup \mathbf{v} \in \mathcal{E}_{2p}^{n+m}\}.$$

The path components of $\mathcal{E}_{2p}^n$ rel $\mathbf{v}$, denoted $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$, define relative discrete braid classes. The braid $\mathbf{v}$ is fixed and it is called skeleton. The set of singular braids $\Sigma_{\mathcal{E}}$ rel $\mathbf{v}$ are those braids $\mathbf{u}$ such that $\mathbf{u} \cup \mathbf{v} \in \Sigma_{\mathcal{E}}$. The collapsed singular braids are denoted by $\Sigma_{\mathcal{E}}^-$ rel $\mathbf{v}$. As before, the set $(\mathcal{E}_{2p}^n \text{ rel } \mathbf{v}) \cup (\Sigma_{\mathcal{E}} \text{ rel } \mathbf{v})$ is denoted $\overline{\mathcal{E}}_{2p}^n$ rel $\mathbf{v}$.

Two relative braid classes $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ and $[\mathbf{u}' \text{ rel } \mathbf{v}']_{\mathcal{E}}$ in $\mathcal{E}_{2p}^n$ rel $\mathbf{v}$ and $\mathcal{E}_{2p}^n$ rel $\mathbf{v}'$ are equivalent if they lay in the same path component in

$$\mathbf{E} = \{(\mathbf{u}, \mathbf{v}) \in \mathcal{E}_{2p}^n \times \mathcal{E}_{2p}^m : \ \mathbf{u} \cup \mathbf{v} \in \mathcal{E}_{2p}^{n+m}\}.$$

We use the notation $[\mathbf{u} \text{ rel } [\mathbf{v}]]$ for the path component in $\mathbf{E}$. In general a relative braid class is not an isolating neighborhood. However, it was shown in [9] that any proper bounded relative braid class is an isolating neighborhood.

DEFINITION 2.2.9. A relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}} \in \mathcal{E}_{2p}^1 \text{ rel } \mathbf{v}$ is called bounded if the set $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ is bounded.

DEFINITION 2.2.10. A discretized relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ is proper if it is impossible to find a continuous path of PL-braid diagrams $\mathbf{u}(t)$ for $t \in [0, 1]$ such that $\mathbf{u}(0) = \mathbf{u}$, $\mathbf{u}(t) \in \mathcal{E}_{2p}^1$ rel $\mathbf{w}$ defines a braid for all $t \in [0, 1)$, and $\mathbf{u}(1)$ is a diagram where an entire component of the closed braid has collapsed onto itself or onto another component of $\mathbf{u}$ or $\mathbf{v}$.

We can see that the braid classes in Figure 2.4a and Figure 2.4b are proper because it is impossible to find a continuous path of PL-braid diagrams which stays in the same braid class and makes the free strand to collapse on some other strand. On the other hand, the braid class displayed in Figure 2.4c is non-proper.

THEOREM 2.2.11. *Suppose* $[\mathbf{u} \text{ rel } \mathbf{v}]$ *is a bounded proper relative braid class and* Ψ^t *is a parabolic flow fixing* $\mathbf{v}$. *Then*

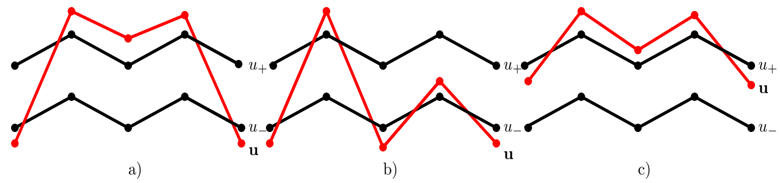

Figure 2.4: Representants of the three different relative braid classes. A fixed point in the relative braid class defined by its representant a) corresponds to the solution of the type I, b) type II and c) type III. Braid classes (a) and (b) are proper but (c) is not.

(a) $N := \mathrm{cl}[\mathbf{u}\,\mathrm{rel}\,\mathbf{v}]$ *is an isolating neighborhood for the flow* Ψ^t*, which thus yields a well-defined Conley index*

$$h(\mathbf{u}\,\mathrm{rel}\,\mathbf{v}) = h(N).$$

(b) *The index* $h(\mathbf{u}\,\mathrm{rel}\,\mathbf{v})$ *is independent of the choice of parabolic flow* Ψ^t *so long as* $\Psi^t(\mathbf{v}) = \mathbf{v}$*.*

(c) *The index* $\mathbf{H}(\mathbf{u}\,\mathrm{rel}\,[\mathbf{v}])$ *is an invariant of* $[\mathbf{u}\,\mathrm{rel}\,[\mathbf{v}]]$*.*

For the flow generated by a parabolic recurrence relation with an up-down restriction

$$\mathrm{cl}_{\overline{\mathcal{E}},\epsilon}[\mathbf{u}\,\mathrm{rel}\,\mathbf{v}]_{\mathcal{E}} := \{\mathbf{u} \in \mathrm{cl}_{\overline{\mathcal{E}}}[\mathbf{u}\,\mathrm{rel}\,\mathbf{v}]_{\mathcal{E}} \;:\; (-1)^i(u^k_{i+1} - u^k_i) \geq \epsilon \quad \forall i, k\}$$

is an isolating neighborhood for $\epsilon > 0$ *sufficiently small. Moreover* $h([\mathbf{u}\,\mathrm{rel}\,\mathbf{v}]_{\mathcal{E}}) = h(\mathbf{u}\,\mathrm{rel}\,\overline{\mathbf{v}})$ *where the skeleton* $\overline{\mathbf{v}} = \mathbf{v} \cup \mathbf{v}^+ \cup \mathbf{v}^-$ *and*

$$v^+_i = \max_{k,i} v^k_i + 1 + (-1)^{i+1}, \quad v^-_i = \min_{k,i} v^k_i - 1 + (-1)^{i+1}.$$

Non-triviality of the index $h(\mathbf{u}\,\mathrm{rel}\,\mathbf{v})$ implies the existence of a fixed point $\mathbf{u}$ within the braid class $[\mathbf{u}\,\mathrm{rel}\,\mathbf{v}]$. Moreover non-triviality of $\mathbf{H}(\mathbf{u}\,\mathrm{rel}\,[\mathbf{v}])$ implies that for every fiber $\mathbf{v}$ there exists a braid class $[\mathbf{u}\,\mathrm{rel}\,\mathbf{v}] \subset \mathbf{u}\,\mathrm{rel}\,[\mathbf{v}]$ which contains a fixed point.

Let us now relate the three types of solutions in Figure 2.1 to braid classes and put them in the context of the definitions presented in this section. The three types of solutions are distinguished according to their intersections with the constant solutions $u_\pm = \pm 1$. The most straightforward way of relating a solution to a relative braid class would be to take the two constant strands $\pm\mathbf{1}$ as a skeleton and define the relative braid class by the free strand $\mathbf{u}$ which intersects the constant strands $\pm\mathbf{1}$ in the same manner as the solution u intersects $u_\pm$. However, the flow Ψ^t is well defined only for the braids with up-down restriction. Hence instead of taking the constant strands we have to use

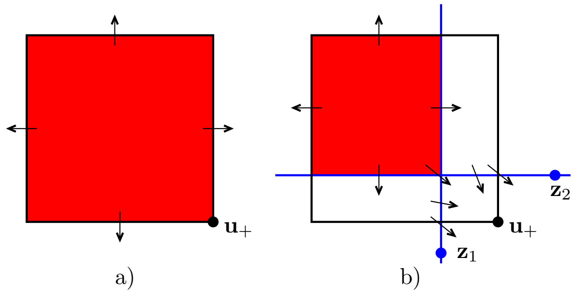

Figure 2.5: Figure (a) schematically shows behavior of the vector field $\mathcal{R}$ on the boundary of the non-proper braid class $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$ corresponding to the third type of solution. The strand $\mathbf{u}_+$ is a fixed point for the flow Ψ^t and some trajectories approach this point as $t \to -\infty$. Figure (b) shows the behavior of the perturbed vector field on the boundary of the proper braid class $[\mathbf{u} \text{ rel } \overline{\mathbf{v}}]_{\mathcal{E}}$, where $\overline{\mathbf{v}} = \mathbf{v} \cup \mathbf{z}_1 \cup \mathbf{z}_2$ and $\mathbf{z}_1, \mathbf{z}_2$ are fixed strands for this perturbed vector field.

the skeleton $\mathbf{v} = \mathbf{u}_+ \cup \mathbf{u}_-$, where the strands $\mathbf{u}_\pm$ correspond to the solutions of Equation (2.1.1) which oscillate around $u_\pm$ with a small amplitude (on a slightly positive energy level) and the free strand $\mathbf{u}$ intersects the skeleton strands in the same manner as u intersects $u_\pm$. Figure 2.4 shows the three different braid classes which correspond to the three different types of solutions. The first two braid classes are proper and the third one is not. All these braid classes are obviously unbounded. It was shown in [9] how to use the properties of Equation (2.1.1) to find extra fixed strands which make the class bounded. We will give more details in Section 2.5.

According to [9] the Conley index for any braid class corresponding to a solution of the first type is non trivial. Conley index theory guarantees the existence of a fixed point in this class. A fixed point in this braid class corresponds to the solution of Equation (2.1.1) of the first type. Thus there are many different solutions of the first type and their bifurcation branches exist for all $\alpha \geq 0$ as we can see in Figure 1.2. For the second braid class the Conley index is trivial and thus does not provide information about fixed points. However, if we know that there exists a non-degenerate (hyperbolic) solution of the second type then it corresponds to a fixed point in the braid class with a trivial Conley index. Hence there must be another fixed point in this class which corresponds to a different solution of the same type. This explains that the bifurcation curves form loops in Figure 2.6.

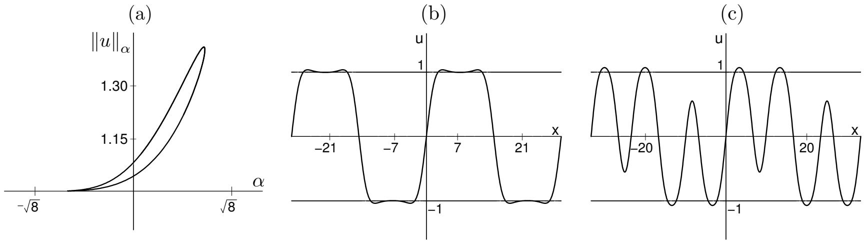

Figure 2.6: Bifurcation diagram for the solutions of the second type and corresponding solutions. Both solutions in (b) and (c) lie on the same bifurcation branch and have six monotone laps and crosses $u_\pm$ four times. These solutions corresponds to the parameter value $\alpha = -\frac{1}{10}$.

In the third case, the braid class is not proper (not an isolating neighborhood), since the free strand can collapse on a skeletal strand $\mathbf{u}_+$. Using the information about the flow Ψ^t near the strand $\mathbf{u}_+$, we will perturb the parabolic recurrence relation on a neighborhood of the boundary of the non-proper braid class $[\mathbf{u} \text{ rel } \mathbf{v}]_\mathcal{E}$ and construct some new fixed strands which will make the class proper, without changing the invariant set inside the class. In Figure 2.5 we schematically demonstrate the behavior of the vector field $\mathcal{R}$ and its perturbation on the boundary of the non-proper braid class and boundary of a new proper braid class $[\mathbf{u} \text{ rel } \overline{\mathbf{v}}]_\mathcal{E}$ created by adding extra strands which are fixed points of this perturbed vector field. We will show that the Conley index $h([\mathbf{u} \text{ rel } \overline{\mathbf{v}}]_\mathcal{E})$ is non trivial. Thus there is a fixed point of the flow Ψ^t within the class $[\mathbf{u} \text{ rel } \mathbf{v}]_\mathcal{E}$.

Throughout this thesis we concentrate on braid classes with one free strand i.e., $[\mathbf{u} \text{ rel } \mathbf{v}] \subset \mathcal{E}^1_{2p} \text{ rel } \mathbf{v}$. In this case the free strand $\mathbf{u}$ of a non-proper braid class can collapse only onto some skeleton strand, because there are no other free strands. Hence the set $\Sigma^-_\mathcal{E}$ consists of isolated points which are fixed points for the flow Ψ^t. In the next section we study linearisation of W_{2p} at the fixed points and introduce notion of a rotational number. Proofs of the results listed in this section can be found in [2].

2.3. Linearisation of W_{2p}

Assume that $\mathbf{x} \in \mathcal{E}^1_{2p}$ is a critical point of $W_{2p} = \sum_{i=0}^{2p-1} S_E(u_i, u_{i+1})$. Define $P_i = (x_i, y_i)$, where $y_i = \partial_1 S_E(x_i, x_{i+1})$. It was shown in [2] that we can define the differentiable functions F_i on some neighborhood of

P_i by the relation

$$(x', y') = F_i(x, y) \Leftrightarrow y = \partial_1 S_E(x, x') \text{ and } y' = -\partial_2 S_E(x, x').$$

It holds that $P_{i+1} = F_i(P_i)$ because $\mathbf{x} \in \mathcal{E}^1_{2p}$ is a critical point of W_{2p}.

We define the rotation number as follows. Take a vector $u_0 \in T_{P_0}\mathbb{R}^2$ such that $u_0 \neq 0$, and define $u_i \in T_{P_i}\mathbb{R}^2$ by

$$u_i = dF_i(P_{i-1})u_{i-1}, \quad \text{for all } i.$$

Identify the tangent spaces $T_{P_i}\mathbb{R}^2$ with $\mathbb{R}^2$ in the obvious way, and let the vector u_i have components (ξ_i, η_i). For each integer i we define θ_i to be the angle between u_{i-1} and u_i, oriented in the clockwise sense. This angel is only defined up to a multiple of 2π, so we have to specify which multiple we mean. For this we use the following rule:

$$\text{if} \quad \xi_{i-1}\xi_i \geqslant 0, \text{ then } -\pi < \theta_i \leq \pi, \tag{2.3.1a}$$

$$\text{if} \quad \xi_{i-1}\xi_i < 0, \text{ then } \ 0 < \theta_i < 2\pi. \tag{2.3.1b}$$

Then we define the rotation number of the orbit $\mathbf{x}$, $\tau(\mathbf{x})$, to be

$$\tau(\mathbf{x}) = \lim_{n\to\infty} (2n)^{-1} \sum_{i=-2pn}^{+2pn} \theta_i/2\pi. \tag{2.3.2}$$

Roughly speaking, $2\pi\tau(\mathbf{x})$ is the average angle about which $dF(P_0)$ rotates the vector u_0, where $F = F_{2p-1} \circ \ldots \circ F_0$. Or, alternatively, $2\tau(\mathbf{x})$ is the average number of times the sequence ξ_n changes sign, in interval of the length $2p$. This holds due to the choice done in (2.3.1a) and (2.3.1b)

If we differentiate ∇W_{2p} at the point $\mathbf{x}$ we get the following expression for the i-th component of the linearization L

$$(L\xi)_i = \alpha_i\xi_{i-1} + \beta_i\xi_i + \alpha_{i+1}\xi_{i+1}, \tag{2.3.3}$$

where

$$\alpha_i = \partial_1\partial_2 S_E(x_{i-1}, x_i) > 0, \tag{2.3.4a}$$

$$\beta_i = \partial_2^2 S_E(x_{i-1}, x_i) + \partial_1^2 S_E(x_i, x_{i+1}), \tag{2.3.4b}$$

The fact that $\alpha_i > 0$ follows from the monotonicity property $\partial_1\partial_2 S_E > 0$ of the generating function. Thus L is a Jacobi matrix, and the following is known (see [23]).

PROPOSITION 2.3.1. *The spectrum of L is given by*

$$sp(L) = \{\lambda_0 > \lambda_1 \geqslant \lambda_2 > \lambda_3 \geqslant \ldots \geqslant \lambda_{2p-1}\}.$$

In particular, for all i we have $\lambda_{2i} > \lambda_{2i+1}$.

Let us summarize the results obtain for the linearization L in [2]. We use a symbol $[a]$ for the lower integer part of the number a.

LEMMA 2.3.2. *Let w_i be an eigenvector of L corresponding to the eigenvalue λ_i and $1 \leq k \leq l \leq 2p$ be given. Then any nonzero linear combination of $w_k, w_{k+1}, \ldots, w_l$ has at least $2[(k+1)/2]$ and at most $2[(l+1)/2]$ sign changes.*

LEMMA 2.3.3. *The linearization L has at least $2[\tau(\mathbf{x})]+1$ positive eigenvalues.*

It follows from Equation (2.3.3) that if $L\xi = 0$ then all ξ_i can be computed from (ξ_0, ξ_1) and

$$\begin{pmatrix} \xi_{2p} \\ \xi_{2p+1} \end{pmatrix} = M(\mathbf{x}) \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}, \tag{2.3.5}$$

with

$$M(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ \frac{-\alpha_{2p-1}}{\alpha_{2p}} & \frac{-\beta_{2p-1}}{\alpha_{2p}} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ \frac{-\alpha_0}{\alpha_1} & \frac{-\beta_0}{\alpha_1} \end{pmatrix}, \tag{2.3.6}$$

where α_i, β_i are given by (2.3.4a), (2.3.4b). One can see that $\det(M(\mathbf{x})) = \frac{\alpha_{2p-1}}{\alpha_{2p}} \frac{\alpha_{2p-2}}{\alpha_{2p-1}} \cdots \frac{\alpha_0}{\alpha_1} = \frac{\alpha_0}{\alpha_{2p}} = 1$.

REMARK 2.3.4. The matrix $M(\mathbf{x})$ is conjugate to the matrix

$$dF(P_0) = dF_{2p-1}(P_{2p-1}) \circ \ldots \circ dF_0(P_0),$$

see Lemma 3.1. in [2].

2.4. The invariant set of a non-proper braid class

The closure of a proper braid class is an isolating neighborhood and Conley index theory provides information about qualitative properties of the invariant set within the braid class. Therefore, if we show that the invariant set inside a non-proper braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ is identical to the invariant set in the closure of some proper braid class $[\mathbf{u} \text{ rel } \overline{\mathbf{v}}]$, then we can use the Conley index to study qualitative properties of the invariant set $\text{Inv}([\mathbf{u} \text{ rel } \mathbf{v}])$.

The basic idea behind creating a corresponding proper braid class is to add new skeleton strands which will prevent the free strand from collapsing onto the skeleton, see Figure 2.7. However, these new skeletal strands have to be fixed points of the underlying flow because that is a requirement of the relative braid class construction. Thus we will perturb the parabolic recurrence $\mathcal{R}$ in such a way that we can find the new skeletal strands, which are fixed points for the flow generated by the perturbed parabolic recurrence relation $\mathcal{N}$. Then we introduce a new braid class $[\mathbf{u} \text{ rel } \overline{\mathbf{v}}]$ by adding these strands to the skeleton. Finally we will show that the invariant set of the flow Ψ^t inside the braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ is the same as invariant set of the flow, generated by $\mathcal{N}$, in the closure of $[\mathbf{u} \text{ rel } \overline{\mathbf{v}}]$.

As we mentioned the braid classes assumed in this thesis have only one free strand. To avoid technical difficulties we will confine ourselves to the non-proper braid class $[\mathbf{u} \text{ rel } \mathbf{v}] \subset \mathcal{E}^1_{2p} \text{ rel } \mathbf{v}$ for which skeleton is given by

$$\mathbf{v} = \mathbf{v}^1 \cup \ldots \cup \mathbf{v}^n,$$

and the free strand $\mathbf{u}$ can collapse only on the skeleton strand $\mathbf{v}^1$. In this case we define

$$\sigma(\mathbf{v}) := \min(|v^1_i - v^j_i| > 0 : i \in \{0, \ldots, 2p-1\}, j \in \{2, \ldots, n\}). \quad (2.4.1)$$

The arguments are easily extended to the case when the free strand can collapse on several skeletal strands and the result can be formulated as follows.

THEOREM 2.4.1. *Let* $[\mathbf{u} \text{ rel } \mathbf{v}] \subset \mathcal{E}^1_{2p} \text{ rel } \mathbf{v}$ *be a non-proper bounded braid class for which skeleton is given by* $\mathbf{v} = \mathbf{v}^1 \cup \ldots \cup \mathbf{v}^n$. *Suppose that the free strand* $\mathbf{u}$ *can collapse only on the skeleton strand* $\mathbf{v}^1$ *and the intersection number* $I(\mathbf{u}, \mathbf{v}^1) \neq 2\tau(\mathbf{v}^1)$. *Then there exists an augmentation* $\overline{\mathbf{v}}$ *of the skeleton* $\mathbf{v}$ *and a flow* Φ^t *such that the braid class* $[\mathbf{u} \text{ rel } \overline{\mathbf{v}}]$ *is proper and*

$$\mathrm{Inv}_\Psi(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}]) = \mathrm{Inv}_\Phi(\mathrm{cl}[\mathbf{u} \text{ rel } \overline{\mathbf{v}}]).$$

COROLLARY 2.4.2. *Under the conditions stated in Theorem 2.4.1 the set* $S = \mathrm{Inv}_\Psi(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}])$ *is an isolated invariant set and Conley index*

$$h(S) = h(\mathbf{u} \text{ rel } \overline{\mathbf{v}}).$$

The augmented skeleton $\overline{\mathbf{v}}$ *is shown at Figure 2.7.*

REMARK 2.4.3. The prove of Theorem 2.4.1 follows from Lemma 2.4.13 and Remark 2.4.14.

Perturbation of the parabolic recurrence relation

The parabolic recurrence relation $\mathcal{R}$, generated by Equation (2.1.1) is two periodic. However, we will deal with a more general setting, namely that $\mathcal{R}$ is $2p$-periodic i.e. $\mathcal{R}_{i+2p} = \mathcal{R}_i$ for all $i \in \mathbb{Z}$. Every component $\mathcal{R}_i$ depends only on (u_{i-1}, u_i, u_{i+1}), and we use the notation $\mathbf{u}_i = (u_{i-1}, u_i, u_{i+1})$.

Throughout this section we use a smooth bump function $\omega^\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ which satisfies

$$\omega^\varepsilon(x_1, x_2, x_3) = \begin{cases} 1, & \text{for } \|x\| \leq \frac{\varepsilon}{2}, \\ 0, & \text{for } \|x\| \geqslant \varepsilon, \end{cases}$$

where $\|\mathbf{x}\| = \|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Moreover we suppose that $\left|\frac{\partial \omega^\varepsilon}{\partial x_i}\right| < \frac{A}{\varepsilon}$ and $\left|\frac{\partial^2 \omega^\varepsilon}{\partial u_i \partial u_j}\right| < \frac{B}{\varepsilon^2}$, for $1 \leq i, j \leq 3$, for some $A, B > 0$ (independent of ε).

Now we introduce a perturbation of the vector field $\mathcal{R}$ which is linear near $\mathbf{v}^1$. Due to a technical reason which will become clear later, we do not replace the vector field $\mathcal{R}$ by its linearization at $\mathbf{v}^1$ but with a linear function which is sufficiently close to this linearization.

DEFINITION 2.4.4. Let $\varepsilon > 0$ and $\alpha, \beta \in \mathbb{R}^{2p}$ such that

$$\left\|\alpha - \partial_1 \mathcal{R}(\mathbf{v}^1)\right\| < \varepsilon \text{ and } \left\|\beta - \partial_2 \mathcal{R}(\mathbf{v}^1)\right\| < \varepsilon,$$

where $\partial_i \mathcal{R}(\mathbf{v}^1) = (\partial_i \mathcal{R}_0(\mathbf{v}^1), \ldots, \partial_i \mathcal{R}_{2p-1}(\mathbf{v}^1))$. Then

$$\mathcal{N}_i^{\varepsilon\alpha\beta}(\mathbf{u}_i) = \omega^\varepsilon(\mathbf{u}_i - \mathbf{v}_i^1)\mathcal{L}_i^{\varepsilon\alpha\beta}(\mathbf{u}_i) + (1 - \omega^\varepsilon(\mathbf{u}_i - \mathbf{v}_i^1))\mathcal{R}_i(\mathbf{u}_i), \quad (2.4.2)$$

where

$$\mathcal{L}_i^{\varepsilon\alpha\beta}(\mathbf{x}_i) = \alpha_{i-1}(x_{i-1} - v_{i-1}^1) + \beta_i(x_i - v_i^1) + \alpha_{i+1}(x_{i+1} - v_{i+1}^1).$$

REMARK 2.4.5. If there is no ambiguity in choosing (α, β) or results do not depend on their values just on the distance form $\partial_i \mathcal{R}(\mathbf{v}^1)$ then we use the notation $\mathcal{N}^\varepsilon$.

Following two lemmas summarize properties of $\mathcal{N}^\varepsilon$.

LEMMA 2.4.6. *There exists an* $\varepsilon_0 > 0$ *such that* $\mathcal{N}^\varepsilon$ *is a parabolic recurrence relation of up-down type for* $0 < \varepsilon < \varepsilon_0$ *.*

PROOF. Every $\mathcal{N}_i^\varepsilon$ is well defined on the set Ω_i and $\mathcal{N}_i^\varepsilon(\mathbf{u}_i) = \mathcal{R}(\mathbf{u}_i)$ if $\mathbf{u}_i \notin B_i(\varepsilon)$, where

$$B_i(\varepsilon) := \{\mathbf{v} \in \mathbb{R}^3 : \left\|\mathbf{v} - \mathbf{v}_i^1\right\| \leq \varepsilon\}. \quad (2.4.3)$$

Thus $\mathcal{N}_i$ has all required properties on the complement of the set $B_i(\varepsilon)$. The up-down restriction for the braid $\mathbf{v}^1$ implies that

$$\sigma := \min(|v_i^1 - v_{i-1}^1|,\ i \in \{0, \ldots, 2p-1\}) > 0. \quad (2.4.4)$$

If we choose $\varepsilon < \frac{\sigma}{3}$ then sufficiently small neighborhood of $\partial\Omega_i$ is in the complement of $B_i(\varepsilon)$ and $\mathcal{N}^\varepsilon$ is of up-down type because the limits (2.2.9) and (2.2.10) for $\mathcal{N}_i^\varepsilon$ are the same as for $\mathcal{R}_i$.

To prove the monotonicity condition $\partial_1\mathcal{N}_i^\varepsilon(\mathbf{u}_i) > 0$ on all of Ω_i we will show the existence of a universal constant $C_i > 0$, such that, for $\varepsilon < \frac{\sigma}{3}$,

$$\partial_1\mathcal{N}_i^\varepsilon(\mathbf{u}_i) \geqslant \partial_1\mathcal{R}_i(\mathbf{v}_i^1) - \varepsilon C_i \quad \text{for } \mathbf{u}_i \in B_i(\varepsilon). \tag{2.4.5}$$

Monotonicity of $\mathcal{R}$ ($\partial_1\mathcal{R}_i > 0$) combined with inequality (2.4.5) implies that $\partial_1\mathcal{N}_i^\varepsilon(\mathbf{u}_i) > 0$ for $\mathbf{u}_i \in B^i(\varepsilon)$, where $0 < \varepsilon < \delta_i$, and $\delta_i = \min\left\{\frac{\sigma}{3}, \frac{\partial_1\mathcal{R}_i(\mathbf{v}_i^1)}{C_i}\right\}$.

In order to prove inequality (2.4.5) we use that $\mathcal{R}_i(\mathbf{v}_i^1) = 0$ to estimate

$$\begin{aligned}|\mathcal{L}_i^\varepsilon(\mathbf{u}_i) - \mathcal{R}_i(\mathbf{u}_i)| \leq {} & |(\alpha_i - \partial_1\mathcal{R}_i(\mathbf{v}_i^1))(u_{i-1} - v_{i-1}^1) + \\ & + (\beta_i - \partial_2\mathcal{R}_i(\mathbf{v}_i^1))(u_i - v_i^1) + \\ & + (\alpha_{i+1} - \partial_3\mathcal{R}_i(\mathbf{v}_i^1))(u_{i+1} - v_{i+1}^1)| + \\ & + \frac{1}{2}\left\|d^2\mathcal{R}_i(\mathbf{v}_i)(\mathbf{u}_i - \mathbf{v}_i^1, \mathbf{u}_i - \mathbf{v}_i^1)\right\|,\end{aligned}$$

where $\mathbf{v}_i = (1-t)\mathbf{u}_i + t\mathbf{v}_i^1$, for some $t \in (0,1)$. So

$$|\mathcal{L}_i^\varepsilon(\mathbf{u}_i) - \mathcal{R}_i(\mathbf{u}_i)| \leq 3\varepsilon\left\|\mathbf{u}_i - \mathbf{v}_i^1\right\| + \frac{D_i}{2}\left\|\mathbf{u}_i - \mathbf{v}_i^1\right\|^2, \tag{2.4.7}$$

for $\mathbf{u}_i \in B_i(\varepsilon)$ where $D_i = \max_{\mathbf{v}\in B_i(\varepsilon)}\left\|d^2\mathcal{R}_i(\mathbf{v})\right\|$. In the same manner we can show that

$$\left|\frac{\partial_1\mathcal{L}_i^\varepsilon(\mathbf{u}_i)}{\partial u_i} - \frac{\partial_1\mathcal{R}_i(\mathbf{u}_i)}{\partial u_i}\right| < \varepsilon + D_i\left\|\mathbf{u}_i - \mathbf{v}_i^1\right\|, \tag{2.4.8}$$

for every $\mathbf{u}_i \in B_i(\varepsilon)$. We can write

$$\mathcal{N}_i^\varepsilon = \mathcal{L}_i^\varepsilon + (1-\omega^\varepsilon)(\mathcal{R}_i - \mathcal{L}_i^\varepsilon).$$

Using the estimate $|\partial_1\omega^\varepsilon| < \frac{A}{\varepsilon}$, for some $A > 0$ we get

$$\begin{aligned}\partial_1\mathcal{N}_i^\varepsilon(\mathbf{u}_i) = \partial_1\mathcal{N}_i^\varepsilon(\mathbf{u}_i) &= \partial_1\mathcal{L}_i^\varepsilon(\mathbf{u}_i) - \partial_1\omega^\varepsilon(\mathbf{u}_i)(\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i)) \\ &+ (1-\omega^\varepsilon(\mathbf{u}_i))\partial_1(\mathcal{R}(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i)) \geqslant \\ &\geqslant \alpha_i - \frac{A}{\varepsilon}|\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i)| - |\partial_1(\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i))| \geqslant \\ &\geqslant \partial_1\mathcal{R}_i(\mathbf{v}_i^1) - \varepsilon - \frac{A}{\varepsilon}|\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i)| - |\partial_1(\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i))|.\end{aligned}$$

So

$$\partial_1\mathcal{N}_i^\varepsilon(\mathbf{u}_i) \geqslant \partial_1\mathcal{R}_i(\mathbf{v}_i^1) - \varepsilon - \frac{A}{\varepsilon}\varepsilon^2\left(3 + \frac{D_i}{2}\right) - \varepsilon\left(1 + \frac{D_i}{2}\right),$$

for $\mathbf{u}_i \in B_i(\varepsilon)$ and $\varepsilon < \frac{\sigma}{3}$.

The last inequality guarantees the existence of the universal constant C_i in (2.4.5). Positivity of $\partial_3\mathcal{N}_i$ can be shown in the same way. Therefore, the monotonicity condition for parabolic recurrence relation is satisfied. □

REMARK 2.4.7. Inequality (2.4.7) implies that for ε_0 small enough there exists a constant C^{ε_0} such that $\|\mathcal{N}^\varepsilon(\mathbf{u}) - \mathcal{R}(\mathbf{u})\| < \varepsilon C^{\varepsilon_0}$, for all $\mathbf{u} \in \mathcal{E}^1_{2p}$ and $0 < \varepsilon < \varepsilon_0$.

LEMMA 2.4.8. *There exists an $\varepsilon_0 > 0$ and a positive constants K^{ε_0} such that $\mathcal{N}^\varepsilon$ can be written in the form*

$$\mathcal{N}^\varepsilon(\mathbf{u}) = \mathcal{L}^\varepsilon(\mathbf{u}) + P^\varepsilon(\mathbf{u}), \tag{2.4.10}$$

where

$$\|P^\varepsilon(\mathbf{u})\| \leq K^{\varepsilon_0}\|\mathbf{u} - \mathbf{v}\|^2, \tag{2.4.11}$$

for all $\mathbf{u} \in \mathcal{E}^1_{2p}$ such that $\|\mathbf{u} - \mathbf{v}^1\| < \varepsilon_0$ and $0 < \varepsilon < \varepsilon_0$.

PROOF. We will show that there exists P^ε_i and $K^{\varepsilon_0}_i$ such that (2.4.10) holds for every component $\mathcal{N}^\varepsilon_i$ and (2.4.11) holds for every P^ε_i. Then the lemma holds for $P^\varepsilon = (P^\varepsilon_1, \ldots, P^\varepsilon_{2p})^T$ and $K^{\varepsilon_0} = \sqrt{2p}\max_{i\in\{0,\ldots,2p-1\}} K^{\varepsilon_0}_i$.

Due to the usual estimate on the remainder of the Taylor series it is enough to show that, for every $k, l \in \{0, \ldots, 2p-1\}$, there exists a constant $K^{\varepsilon_0}_{i,k,l}$ with the property

$$\left|\frac{\partial^2\mathcal{N}^\varepsilon_i}{\partial u_k \partial u_l}(\mathbf{u}_i)\right| \leq K^{\varepsilon_0}_{i,k,l}, \tag{2.4.12}$$

for $\mathbf{u}_i \in B_i(\varepsilon_0)$. One can compute that

$$\begin{aligned}
\frac{\partial^2\mathcal{N}^\varepsilon_i}{\partial u_k \partial u_l}(\mathbf{u}_i) = {} & \omega^\varepsilon(\mathbf{u}_i)\frac{\partial^2\mathcal{L}^\varepsilon_i}{\partial u_k \partial u_l}(\mathbf{u}_i) + (1-\omega^\varepsilon(\mathbf{u}_i))\frac{\partial^2\mathcal{R}_i}{\partial u_k \partial u_l}(\mathbf{u}_i) \\
& + \frac{\partial\omega^\varepsilon}{\partial u_l}(\mathbf{u}_i)\left(\frac{\partial\mathcal{L}^\varepsilon_i}{\partial u_k}(\mathbf{u}_i) - \frac{\partial\mathcal{R}_i}{\partial u_k}(\mathbf{u}_i)\right) + \\
& + \frac{\partial\omega^\varepsilon}{\partial u_k}(\mathbf{u}_i)\left(\frac{\partial\mathcal{L}^\varepsilon_i}{\partial u_l}(\mathbf{u}_i) - \frac{\partial\mathcal{R}_i}{\partial u_l}(\mathbf{u}_i)\right) \\
& + \frac{\partial^2\omega^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i)\left(\mathcal{L}^\varepsilon_i(\mathbf{u}_i) - \mathcal{R}_i(\mathbf{u}_i)\right).
\end{aligned}$$

Using the estimates $\left|\frac{\partial \omega^\varepsilon}{\partial u_l}\right| < \frac{A}{\varepsilon}$ and $\left|\frac{\partial^2 \omega^\varepsilon}{\partial u_l \partial u_l}\right| < \frac{B}{\varepsilon^2}$, the bounds (2.4.7), (2.4.8) imply

$$\left|\frac{\partial^2 \mathcal{N}_i^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i)\right| \leq D_i + 2\frac{A}{\varepsilon}\varepsilon D_i + \frac{B}{\varepsilon^2}\varepsilon^2 \frac{D_i}{2},$$

for $\mathbf{u}_i \in B_i(\varepsilon_0)$, where $D_i = \max_{\mathbf{v}\in B_i(\varepsilon_0)} \left\|d^2\mathcal{R}_i(\mathbf{v})\right\|$. □

Construction of the proper braid class

In the previous subsection we defined the perturbation $\mathcal{N}^\varepsilon$ of the parabolic recurrence relation $\mathcal{R}$. Now we will show that for every $\varepsilon > 0$ we can associate the braid class $[\mathbf{u} \operatorname{rel} \mathbf{v}]$ with a proper braid class $[\mathbf{u} \operatorname{rel} \mathbf{v}^\varepsilon]$ where $\mathcal{N}^\varepsilon(\mathbf{v}^\varepsilon) = 0$ and $0 < \varepsilon < \varepsilon_0$. Before we define the skeleton $\mathbf{v}^\varepsilon$ we will show how to employ local information about the parabolic recurrence relation $\mathcal{R}$ near $\mathbf{v}^1 \in \mathcal{E}^1_{2p}$ to construct a fixed point for $\mathcal{N}^\varepsilon$.

LEMMA 2.4.9. *Let $\mathbf{v}^1 \in \mathcal{E}^1_{2p}$ have a positive rotation number $\tau(\mathbf{v}^1)$ and assume that the matrix $M(\mathbf{v}^1)$ given by (2.3.6) is conjugate to the matrix*

$$\begin{pmatrix} \cos 2\pi\tau(\mathbf{v}^1) & -\sin 2\pi\tau(\mathbf{v}^1) \\ \sin 2\pi\tau(\mathbf{v}^1) & \cos 2\pi\tau(\mathbf{v}^1) \end{pmatrix}.$$

Then for every $\varepsilon_0 > 0$ there exist $\varepsilon > 0$, $p', q' \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}^{2p}$ such that $p', q', 2p$ are relatively prime, $0 \leq \left|\frac{q'}{p'} - \tau(\mathbf{v}^1)\right| < \varepsilon < \varepsilon_0$ and $\mathcal{N}^{\varepsilon\alpha\beta}$ possesses a $2pp'$-periodic zero $\mathbf{c}^\varepsilon$ with up-down restriction satisfying:

(a) $|c_i^\varepsilon - v_i^1| < \varepsilon/4$, *for all* i,
(b) *the sequence* $(c_0^\varepsilon - v_0^1, \ldots, c^\varepsilon_{2pp'-1} - v^1_{2pp'-1})$ *changes the sign* $2q'$ *times and intersects the zero sequence transversally,*
(c) $c^\varepsilon_{2ip} = c^\varepsilon_{2jp}$ *only if* $|i-j| = kp'$ *for some* $k \in \mathbb{N}$.

PROOF. To prove the lemma we need to construct a parabolic recurrence relation $\mathcal{N}^{\varepsilon\alpha\beta}$ and a point $\mathbf{c}^\varepsilon$ satisfying conditions (a), (b) and (c) such that $\mathcal{N}^{\varepsilon\alpha\beta}(\mathbf{c}^\varepsilon) = 0$. However, $\mathcal{N}^{\varepsilon\alpha\beta} = \mathcal{L}^{\varepsilon\alpha\beta}$ near $\mathbf{v}^1$. Therefore we will construct a zero point of $\mathcal{L}^{\varepsilon\alpha\beta}$.

As we mentioned in Section 2.3, finding a $2pp'$-periodic zero point of $\mathcal{L}^{\varepsilon\alpha\beta}$ is equivalent to solving the equation

$$\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = M^{p'}_{\alpha,\beta} \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}, \tag{2.4.14}$$

where

$$M_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ \frac{-\alpha_{2p-1}}{\alpha_{2p}} & \frac{-\beta_{2p-1}}{\alpha_{2p}} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ \frac{-\alpha_0}{\alpha_1} & \frac{-\beta_0}{\alpha_1} \end{pmatrix}.$$

Vector $\alpha \in \mathbb{R}^{2p}$ and α_{2p} is not defined. To keep the notation from Section 2.3 we set $\alpha_{2p} = \alpha_0$. Then $\det(M_{\alpha\beta}) = 1$ for arbitrary (α, β).

First suppose that $\tau(\mathbf{v}^1) = \frac{q'}{p'}$, for some $p', q' \in \mathbb{N}$ such that $p', q', 2p$ are relatively prime, then for $\alpha = \partial_1 \mathcal{R}(\mathbf{v}^1)$ and $\beta = \partial_2 \mathcal{R}(\mathbf{v}^1)$ the matrix $M_{\alpha\beta} = M(\mathbf{v}^1)$ is conjugate to a rotation about the angle $2\pi\frac{q'}{p'}$. Thus Equation (2.4.14) is satisfied for arbitrary values (ξ_0, ξ_1) and there is a $2pp'$-periodic zero point of the linearization $\mathcal{L}^{\varepsilon\alpha\beta}$ corresponding to (ξ_0, ξ_1) for arbitrary $\varepsilon < \varepsilon_0$.

The fact that the matrix $M(\mathbf{v}^1)$ is conjugate to a rotation about the angle $2\pi\frac{q'}{p'}$ and $p', q', 2p$ are relatively prime allow us to choose (ξ_0, ξ_1) in such a way that the sequence $\{\xi_i\}$ is not constant and $\xi_{2ip} = \xi_{2jp}$ only if $|i - j| = kp'$ for some $k \in \mathbb{N}$. We can suppose that $|\xi_i| < \frac{\varepsilon}{4}$ for all $i \in \mathbb{Z}$ otherwise we take the sequence $\{c\xi_i\}$ where $c = \frac{\varepsilon}{4}\max(\xi_i, i = 0, \ldots, 2pp' - 1)$. According to Section 2.3 the sequence $(\xi_0, \xi_1, \ldots, \xi_{2pp'-1})$ changes sign $2q'$ times.

To prove that the nonconstant sequence $\{\xi_i\}$ intersects zero transversally suppose that $\xi_i = 0$. Then equation $(\mathcal{L}^{\varepsilon\alpha\beta}\xi)_i = 0$ implies that $\xi_{i+1} = -\frac{\alpha_i}{\alpha_{i+1}}\xi_{i-1}$ where $\alpha_i = \partial_1 \mathcal{R}_i(\mathbf{v}^1)$ is positive for all i. Thus $\xi_{i-1}\xi_{i+1} \leq 0$. Hence, for a non transversal intersection it holds that $\xi_{i-1} = \xi_i = \xi_{i+1} = 0$, and the sequence $\{\xi_i\}$ is constant zero sequence which is a contradiction.

We conclude that in the case $\tau(\mathbf{v}^1) = \frac{q'}{p'}$ for some $p', q' \in \mathbb{N}$, where $p', q', 2p$ are relatively prime, we can define

$$c_i^{\varepsilon} = v_i^1 + \xi_i, \text{ for } i \in \mathbb{Z}. \tag{2.4.15}$$

Consider now the case that $\tau(\mathbf{v}^1)$ is not rational or p', q' and $2p$ are not relative prime. We will show that for arbitrary $\varepsilon_0 > 0$ there exists an $\varepsilon < \varepsilon_0$ and $\alpha, \beta \in \mathbb{R}^{2p}$ such that $\|\alpha - \partial_1 \mathcal{R}(\mathbf{v}^1)\| < \varepsilon$, $\|\beta - \partial_3 \mathcal{R}(\mathbf{v}^1)\| < \varepsilon$ and the matrix $M_{\alpha\beta}$ is conjugate to

$$\begin{pmatrix} \cos 2\pi\frac{q'}{p'} & -\sin 2\pi\frac{q'}{p'} \\ \sin 2\pi\frac{q'}{p'} & \cos 2\pi\frac{q'}{p'} \end{pmatrix}$$

for some $p', q' \in \mathbb{N}$, where $p', q', 2p$ are relatively prime and

$$\left|\frac{q'}{p'} - \tau(\mathbf{v}^1)\right| < \varepsilon.$$

Then following the previous construction we get a zero point of $\mathcal{N}^{\varepsilon\alpha\beta}$ with properties (a), (b) and (c).

For $\alpha = \partial_1\mathcal{R}(\mathbf{v}^1)$, $\beta = \partial_3\mathcal{R}(\mathbf{v}^1)$ the matrix $M_{\alpha\beta}$ is conjugate to a rotation matrix about the angle $2\pi\tau(\mathbf{v}^1)$. It is enough to show that by arbitrary small perturbation in (α, β) we can slightly change the eigenvalues of the matrix $M_{\alpha\beta}$ to make the rotation angle $\theta = \frac{q'}{p'}$ rational and such that $p', q', 2p$ are relatively prime. For every $(\alpha, \beta) \in \mathbb{R}^{2p}$ it holds that $\det(M_{\alpha\beta}) = 1$. The equation for the eigenvalues of $M_{\alpha\beta}$ is given by

$$\lambda^2 - \text{Trace}(M_{\alpha\beta})\lambda + 1 = 0.$$

Since $\text{Trace}(M_{\alpha\beta})$ is a rational function of (α, β), it suffices to show that $\text{Trace}(M_{\alpha\beta})$ is not a constant function of (α, β). One can compute that $\text{Trace}(M_{\alpha\beta}) = 2(-1)^p$ for $\alpha = (1, \ldots, 1), \beta = (0, \ldots, 0)$ and $\text{Trace}(M_{\alpha\beta}) = (-1)^p$ for $\alpha = (1, \ldots, 1), \beta = (1, 1, 0, \ldots, 0)$. This proves that the rational function $\text{Trace}(M_{\alpha\beta})$ is not constant. Therefore by arbitrary small perturbation of (α, β) we can continuously change the eigenvalues of the matrix $M_{\alpha\beta}$. □

Using the zero point $\mathbf{c}^\varepsilon$ of $\mathcal{N}^\varepsilon$ we define $\mathbf{v}^\varepsilon$ as

$$\mathbf{v}^\varepsilon := \mathbf{v} \cup \mathbf{z}^\varepsilon, \tag{2.4.16}$$

where

$$(z^\varepsilon)_i^k := c^\varepsilon_{2pk+i},$$

for $k \in \{0, \ldots, p'-1\}$ and $i \in \{0, \ldots, 2p\}$.

LEMMA 2.4.10. *For $\varepsilon < \sigma(\mathbf{v})$, where $\sigma(\mathbf{v})$ is given by (2.4.1), it holds that $\mathbf{v}^\varepsilon \in \mathcal{E}_{2p}^{n+p'}$ and $\mathcal{N}^\varepsilon(\mathbf{v}^\varepsilon) = 0$.*

PROOF. The only condition which needs to be checked is a transversality condition. We start by proving that strands in $\mathbf{z}^\varepsilon$ intersect transversally. For $p' = 1$ it is trivial. Lets suppose that $(z^\varepsilon)_i^k = (z^\varepsilon)_i^l$, for some i and $0 \le l < k < p'$, where $p' > 1$. Then the equation

$$\mathcal{L}_i^{\varepsilon\alpha\beta}((z^\varepsilon)_{i-1}^l - (z^\varepsilon)_{i-1}^k, (z^\varepsilon)_i^l - (z^\varepsilon)_i^k, (z^\varepsilon)_{i+1}^l - (z^\varepsilon)_{i+1}^k) = 0 \text{ for all } i, k, \tag{2.4.17}$$

implies

$$-\frac{\alpha_i}{\alpha_{i+1}}((z^\varepsilon)_{i-1}^l - (z^\varepsilon)_{i-1}^k) = ((z^\varepsilon)_{i+1}^l - (z^\varepsilon)_{i+1}^k).$$

Thus if $(z^\varepsilon)_{i-1}^l \neq (z^\varepsilon)_{i-1}^k$ then the transversality condition

$$((z^\varepsilon)_{i-1}^l - (z^\varepsilon)_{i-1}^k)((z^\varepsilon)_{i+1}^l - (z^\varepsilon)_{i+1}^k) < 0$$

is satisfied. If $(z^\varepsilon)^l_{i-1} = (z^\varepsilon)^k_{i-1}$, applying (2.4.17) we get $(z^\varepsilon)^l_i = (z^\varepsilon)^k_i$ for all i. Therefore $c^\varepsilon_{2kp} = c^\varepsilon_{2lp}$, where $0 < |k-l| < p'$ and we have a contradiction with the condition (c) of the fixed point $\mathbf{c}^\varepsilon$ in Lemma 2.4.9.

We are left with showing that $\mathbf{z}^\varepsilon$ transversally intersects strands in $\mathbf{v}$. Condition (b) of the fixed point $\mathbf{c}^\varepsilon$ implies that $\mathbf{z}^\varepsilon$ transversally intersects the strand $\mathbf{v}^1$. Condition (a) of the fixed point $\mathbf{c}^\varepsilon$ implies that $|(z^\varepsilon)^k_i - v^1_i| < \frac{\varepsilon}{4}$ for all k, i. If the anchor point v^m_i, $m \neq 1$ satisfies $(z^\varepsilon)^l_i = v^m_i$, for some l, then $|v^m_i - v^1_i| < \frac{\varepsilon}{4} < \sigma(\mathbf{v})$ and it follows, from the definition of $\sigma(\mathbf{v})$, that $v^m_i = v^1_i$. All intersections of the strands $\mathbf{v}^1$ and $\mathbf{v}^m$ are transversal. Hence $(v^m_{i-1} - v^1_{i-1})(v^m_{i+1} - v^1_{i+1}) < 0$. The previous inequality combined with $|v^m_{i\pm1} - v^1_{i\pm1}| \geq \sigma(\mathbf{v}) > |(z^\varepsilon)^l_{i\pm1} - v^1_{i\pm1}|$ implies that $(v^m_{i-1} - (z^\varepsilon)^l_{i-1})(v^m_{i+1} - (z^\varepsilon)^l_{i+1}) < 0$.

According to the definition $\mathcal{N}^\varepsilon(\mathbf{v}^k) = \mathcal{R}(\mathbf{v}^k) = 0$ for all k. The previous lemma implies that $\mathcal{N}^\varepsilon(\mathbf{z}^\varepsilon) = 0$, and therefore $\mathcal{N}^\varepsilon(\mathbf{v}^\varepsilon) = 0$. □

Now we associate the improper braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ with a proper one, namely $[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$, as follows.

DEFINITION 2.4.11. The braid class $[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon] \subset \mathcal{E}^1_{2p} \text{ rel } \mathbf{v}^\varepsilon$ is given by its representant $\mathbf{u}$ which satisfies the following properties

(a) $\mathbf{u} \in [\mathbf{u} \text{ rel } \mathbf{v}]$,
(b) $|u_i - v^1_i| \geq \frac{\varepsilon}{2}$ for $i \in \{0, \ldots 2p-1\}$.

LEMMA 2.4.12. *For $\varepsilon < \sigma(\mathbf{v})$ the relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$ is well defined. Moreover if the braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ is bounded and $I(\mathbf{u}, \mathbf{v}^1) \neq 2\tau(\mathbf{v}^1)$, then $[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$ is bounded and proper for $\varepsilon < |I(u, \mathbf{v}^1) - 2\tau(\mathbf{v}^1)|$.*

PROOF. Every up-down braid $\mathbf{u} \in [\mathbf{u} \text{ rel } \mathbf{v}]$ which satisfies $|u_i - v^1_i| \geq \frac{\varepsilon}{2}$ for all i does not have a common anchor point with strands $\mathbf{z}^\varepsilon$. Thus the representant $\mathbf{u}$ lies in $\mathcal{E}^1_{2p} \text{ rel } \mathbf{v}^\varepsilon$.

First we will show that the braid class $[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$ is uniquely defined. Let $\mathbf{u}^1$ and $\mathbf{u}^2$ be arbitrary braids in $[\mathbf{u} \text{ rel } \mathbf{v}]$ which satisfy $|u^{1,2}_i - v^1_i| \geq \frac{\varepsilon}{2}$. Let $\mathbf{u}(t) \in \mathcal{E}^1_{2p} \text{ rel } \mathbf{v}$ be the path between them, which, without loss of generality, evolves just one anchor point at the time. It means that there is a division of interval $[0,1]$, given by $0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = 1$, such that only anchor point u_{i_j} evolves for $t \in (t_j, t_{j+1}]$ where $i_j \in \{0, \ldots, 2p\}$. The path $\mathbf{u}(t)$ does not have to be in $\mathcal{E}^1_{2p} \text{ rel } \mathbf{v}^\varepsilon$, because non-transversal crossing with some strand in $\mathbf{z}^\varepsilon$ can occur. We will modify the path $\mathbf{u}(t)$ in order to avoid this. If $|u_{i_j}(t_{j+1}) - v^1_{i_j}| < \frac{\varepsilon}{2}$ then we perturb the function $\widetilde{u}_{i_j}(t) : (t_j, t_{j+1}] \to \mathbb{R}$ as follows

$$\widetilde{u}_{i_j}(t) = \begin{cases} u_{i_j}(t_j)(1-\bar{t}) + (v_{i_j} + \varepsilon/2)\bar{t}, & \text{if } u_{i_j}(t_{j+1}) \geq v_{i_j}, \\ u_{i_j}(t_j)(1-\bar{t}) + (v_{i_j} - \varepsilon/2)\bar{t}, & \text{otherwise,} \end{cases}$$

where $\bar{t} = \frac{t-t_j}{t_{j+1}-t_j}$. We set $\widetilde{u}_{i_j}(t) = \widetilde{u}_{i_j}(t_{j+1})$, for all $t > t_{j+1}$, until the original path moves u_{i_j} again. The fact that $u_{i_j}(1) \notin (v_i^1 - \varepsilon/2, v_i^1 + \varepsilon/2)$ implies that there is a j' such that $\mathbf{u}(t)$ evolves the point u_{i_j} for $t \in (t_{j'}, t_{j'+1}]$. Then we define $\widetilde{u}_{i_j}(t) : (t_{j'}, t_{j'+1}] \to \mathbb{R}$ as a linear function connecting $\widetilde{u}_{i_j}(t_{j+1})$ with $u_{i_j}(t_{j'+1})$. We repeat the previous procedure for any anchor point ending up closer than $\frac{\varepsilon}{2}$ from $\mathbf{v}^1$. This perturbation does not create non-transversal intersection with $\mathbf{v}^1, \mathbf{v}^2, \ldots \mathbf{v}^n$. Along the perturbed path only one anchor point u_i can be in the interval $(v_i^1 - \varepsilon/2, v_i^1 + \varepsilon/2)$ at a time. If u_i passes trough this interval then $u_{i-1} < v_i^1 - \varepsilon/2 < v_i^1 + \varepsilon/2 < u_{i+1}$ or $u_{i+1} < v_i^1 - \varepsilon/2 < v_i^1 + \varepsilon/2 < u_{i-1}$. Hence a non-transversal crossing with strands $\mathbf{z}^\varepsilon$ is not possible because all their anchor points are within distance $\frac{\varepsilon}{4}$ of $\mathbf{v}^1$.

If $\mathbf{u} \in [\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$ then $\mathbf{u} \in [\mathbf{u} \text{ rel } \mathbf{v}]$. Thus the braid class $[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$ is bounded. To prove the properness we have to show that the free strand $\mathbf{u}$ cannot collapse on the skeleton strands $\mathbf{z}^\varepsilon$ and $\mathbf{v}^1$. If $\mathbf{z}^\varepsilon$ consists only of one strand then Lemma 2.4.9 implies that for the crossing number we have $I(\mathbf{v}^1, \mathbf{z}^\varepsilon) = 2q'$ where $|2q' - 2\tau(\mathbf{v}^1)| < \varepsilon$. We will show that $\mathbf{u}$ cannot collapse on $\mathbf{z}^\varepsilon$ by contradiction. If $\mathbf{u}$ can collapse on $\mathbf{z}^\varepsilon$ then $I(\mathbf{u}, \mathbf{v}^1) = I(\mathbf{z}^\varepsilon, \mathbf{v}^1) = 2q'$ and $|I(\mathbf{u}, \mathbf{v}^1) - 2\tau(\mathbf{v}^1)| < \varepsilon$. This contradicts the assertion of the lemma. It holds that $I(\mathbf{u}, \mathbf{v}^1) = I(\mathbf{u}, \mathbf{z}^\varepsilon)$, hence a similar contradiction proves that $\mathbf{u}$ cannot collapse on $\mathbf{v}^1$. If $\mathbf{z}^\varepsilon$ contains of $p' > 1$ strands then it follows from Lemma 2.4.9 that $(z^\varepsilon)_0^i \neq (z^\varepsilon)_{2p}^i$ for any i. This ensures that $\mathbf{u}$ cannot collapse on any of the strands of $\mathbf{z}^\varepsilon$. We will show that $\mathbf{u}$ cannot collapse on $\mathbf{v}^1$ by contradiction. First of all $\sum_{\mathbf{z} \in \mathbf{z}^\varepsilon} I(\mathbf{u}, \mathbf{z}) = \sum_{\mathbf{z} \in \mathbf{z}^\varepsilon} I(\mathbf{u}, \mathbf{v}^1) = p' I(\mathbf{u}, \mathbf{v}^1)$. However if $\mathbf{u}$ collapse on $\mathbf{v}^1$ then $\sum_{\mathbf{z} \in \mathbf{z}^\varepsilon} I(\mathbf{u}, \mathbf{z}) = \sum_{\mathbf{z} \in \mathbf{z}^\varepsilon} I(\mathbf{v}^1, \mathbf{z}) = 2q'$. The previous two equalities imply that $I(\mathbf{u}, \mathbf{v}^1) = 2\frac{q'}{p'}$. By applying Lemma 2.4.9 one infers that $|I(\mathbf{u}, \mathbf{v}^1) - 2\tau(\mathbf{v}^1)| < \varepsilon$, a contradiction. □

Invariant set of $[\mathbf{u} \text{ rel } \mathbf{v}]$.

The following lemma establishes a connection between the invariant set

$$\mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}]) := \{\mathbf{u} : \mathrm{cl}(\Psi^t(\mathbf{u})) \subset [\mathbf{u} \text{ rel } \mathbf{v}]\}$$

and the invariant set $\mathrm{Inv}_{\Phi^\varepsilon}(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon])$ where the flow Φ^ε is generated by the parabolic recurrence relation $\mathcal{N}^\varepsilon$.

LEMMA 2.4.13. *If $I(\mathbf{u}, \mathbf{v}^1) < 2\tau(\mathbf{v}^1)$ then there exists an $\varepsilon_0 > 0$ such that*

$$\mathrm{Inv}_\Psi(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}]) = \mathrm{Inv}_{\Phi^\varepsilon}(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon])$$

for $0 < \varepsilon < \varepsilon_0$.

REMARK 2.4.14. Similar arguments prove the assertion of the previous lemma also in the case that $I(\mathbf{u}, \mathbf{v}^1) > 2\tau(\mathbf{v}^1)$.

PROOF. We will start with proving the inclusion

$$\mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}]) \subset \mathrm{Inv}_{\Phi^\varepsilon}(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]).$$

The sets $\mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}])$ and $\partial[\mathbf{u} \text{ rel } \mathbf{v}]$ are compact and disjoint. Thus there exists an $\varepsilon_1 < \sigma(\mathbf{v})$ such that their distance

$$\rho(\mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}]), \partial[\mathbf{u} \text{ rel } \mathbf{v}]) > \varepsilon_1. \tag{2.4.18}$$

Moreover, for $\mathbf{z} \in \mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}])$ with $|z_i - v_i^1| < \epsilon_1$, as we show below, it holds that

$$|z_{i\pm1} - v_{i\pm1}^1| > \varepsilon_1, \tag{2.4.19}$$

$$(z_{i-1} - v_{i-1}^1)(z_{i+1} - v_{i+1}^1) < 0. \tag{2.4.20}$$

We will prove (2.4.19) by contradiction. Consider

$$\mathbf{s} = (z_0, \ldots, z_{i-2}, v_{i-1}^1, v_i^1, z_{i+1}, \ldots, z_{2p-1}) \in \partial[\mathbf{u} \text{ rel } \mathbf{v}].$$

If $|z_{i-1} - v_{i-1}^1| \le \epsilon_1$ or $|z_{i+1} - v_{i+1}^1| \le \epsilon_1$ then by using the fact that $\mathbf{z} \in \mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}])$ we can estimate the distance

$$\rho(\mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}]), \partial[\mathbf{u} \text{ rel } \mathbf{v}]) \le \rho(\mathbf{z}, \mathbf{s}) \le \varepsilon_1.$$

This contradicts Inequality (2.4.18). If we suppose that

$$(z_{i-1} - v_{i-1}^1)(z_{i+1} - v_{i+1}^1) \ge 0$$

then we obtain a similar contradiction for

$$\mathbf{s} = (z_0, \ldots, z_{i-1}, v_i^1, z_{i+1}, \ldots, z_{2p-1}) \in \partial[\mathbf{u} \text{ rel } \mathbf{v}].$$

Therefore (2.4.20) holds as well.

First we will now prove that $\mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}]) \subset [\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$ for $\varepsilon < \varepsilon_1$. Let $\mathbf{z} \in \mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}])$. If $|z_i - v_i| \ge \frac{\varepsilon}{2}$ for all i then $\mathbf{z} \in [\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$. In case that some $|z_i - v_i| < \frac{\varepsilon}{2}$, it follows from (2.4.19) and (2.4.20) that we can move z_i out of interval $(v_i - \frac{\varepsilon}{2}, v_i + \frac{\varepsilon}{2})$ without changing intersection number with the skeletal strands $\mathbf{v}^\varepsilon$. Thus $\mathbf{z} \in [\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$.

If $\mathbf{z} \in \mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}])$ then it follows form (2.4.18) that $\Psi^t(\mathbf{z})$ stays away from the boundary $\partial[\mathbf{u} \text{ rel } \mathbf{v}]$. This implies that $\Psi_t(\mathbf{z}) = \Phi_t^\varepsilon(\mathbf{z})$ for $t \in \mathbb{R}$ and $\Phi_t^\varepsilon(\mathbf{z}) \in [\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$. Therefore $\mathbf{z} \in \mathrm{Inv}_{\Phi^\varepsilon}(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon])$. In particular, $\mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}]) \subset \mathrm{Inv}_{\Phi^\varepsilon}(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon])$ for $\varepsilon < \varepsilon_1$.

We are left with proving the opposite inclusion. Suppose that

$$\mathbf{z} \in \mathrm{Inv}_{\Phi^\varepsilon}(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon])$$

and

$$\rho(\Phi_t^\varepsilon(\mathbf{z}), \partial[\mathbf{u} \text{ rel } \mathbf{v}]) > \varepsilon \quad \text{for all } t,$$

then $\Phi_t^\varepsilon(\mathbf{z}) = \Psi_t(\mathbf{z})$ for $t \in \mathbb{R}$ and $\mathbf{z} \in \mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}])$. Therefore it is enough to prove that there exists $\varepsilon_2 > 0$ such that

$$\rho(\mathrm{Inv}_{\Phi^\varepsilon}(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]), \partial[\mathbf{u} \text{ rel } \mathbf{v}]) > \varepsilon \quad \text{for all } 0 < \varepsilon < \varepsilon_2. \tag{2.4.21}$$

We will show that for every braid $\mathbf{y} \in \partial[\mathbf{u} \text{ rel } \mathbf{v}]$ there exists an $\varepsilon_\mathbf{y}$ such that for $\mathbf{x} \in B_{\varepsilon_\mathbf{y}}(\mathbf{y}) = \{\mathbf{x} \in \mathcal{E}_{2p}^1 : \|\mathbf{x} - \mathbf{y}\| < \varepsilon_\mathbf{y}\}$ it holds that $\mathbf{x} \notin \mathrm{Inv}_{\Phi^\varepsilon}(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon])$. The compact set $\partial[\mathbf{u} \text{ rel } \mathbf{v}]$ can by covered by a finite covering $U = \{B_{\varepsilon_{\mathbf{y}_i}}(\mathbf{y}_i)\}$. Hence (2.4.21) holds for $\varepsilon_2 := \min \varepsilon_{\mathbf{y}_i}$.

Let us start with the boundary point $\mathbf{v}^1$. Identify $\mathcal{E}_{2p}^1$ and $\mathbb{R}^{2p}$ via

$$\mathbf{u} \to (u_0 - v_0^1, \ldots, u_{2p-1} - v_{2p-1}^1) \in \mathbb{R}^{2p},$$

so that $\mathbf{v}^1$ becomes origin.

By following the ideas in the proof of Lemma 7.2 in [2] one can see that the linear part $\mathcal{L}^\varepsilon$ of $\mathcal{N}^\varepsilon$ at $\mathbf{v}^1$ can be written as $\mathcal{L}^\varepsilon = \mathcal{L}_+^\varepsilon + \mathcal{L}_-^\varepsilon$, where $\mathcal{L}_+^\varepsilon$ and $\mathcal{L}_+^\varepsilon\mathcal{L}_-^\varepsilon = \mathcal{L}_-^\varepsilon\mathcal{L}_+^\varepsilon = 0$ and

$$(\mathbf{x}, \mathcal{L}_+^\varepsilon \mathbf{x}) \geqslant 0, \quad (\mathbf{x}, \mathcal{L}_-^\varepsilon \mathbf{x}) \leq 0,$$

holds for all nonzero $\mathbf{x} \in \mathbb{R}^{2p}$.

Let $\{w_0, \ldots, w_{2p-1}\}$ and $\{\lambda_0 > \lambda_1 \geq \lambda_2, \ldots, \lambda_{2p-1}\}$ be the eigenvectors and values of $\mathcal{L}^0$, where $\mathcal{L}^0$ is linearization of $\mathcal{R}$ at $\mathbf{v}^1$. The null space of $\mathcal{L}_+^0$ is spanned by $\{w_m, w_{m+1}, \ldots, w_{2p-1}\}$, where $m > I(\mathbf{u}, \mathbf{v}^1)$. Indeed, $I(\mathbf{u}, \mathbf{v}^1) < 2\tau(\mathbf{v}^1)$ implies that $\mathcal{L}^0$ must have at least $I(\mathbf{u}, \mathbf{v}^1)+1$ positive eigenvalues, see Lemma 2.3.3 and [2].

Hence Lemma 2.3.2 implies that if $\mathbf{x} \neq 0$ and $\mathcal{L}_+^0 \mathbf{x} = 0$ then $\mathbf{x}$ has at least $I(\mathbf{u}, \mathbf{v}^1) + 2$ sign changes and therefore $\mathbf{x}$ does not lie in $\mathrm{cl}([\mathbf{u} \text{ rel } \mathbf{v}])$. Thus there is a constant $K^0 > 0$ such that

$$(\mathbf{x}, \mathcal{L}_+^0 \mathbf{x}) \geq K^0 \|\mathbf{x}\|^2$$

holds for all $\mathbf{x} \in \mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}]$. This also implies that $\|\mathcal{L}_+^0 \mathbf{x}\| \geq K^0 \|\mathbf{x}\|$.

It follows from the continuous dependence of the eigenvalues of $\mathcal{L}^\varepsilon$ on ε that there exists a constant $K > 0$ such that

$$(\mathbf{x}, \mathcal{L}_+^\varepsilon \mathbf{x}) \geq K \|\mathbf{x}\|^2,$$

for ε small enough.

Consider the function $G^\varepsilon(\mathbf{x}) = \frac{1}{2}(\mathbf{x}, \mathcal{L}_+^\varepsilon \mathbf{x})$. Lemma 2.4.8 implies that close to $\mathbf{x} = 0$ the flow Φ_t^ε is given by

$$\mathbf{x}'(t) = \mathcal{L}^\varepsilon(\mathbf{x}(t)) + P^\varepsilon(\mathbf{x}(t)), \tag{2.4.22}$$

where

$$|P^\varepsilon(\mathbf{x}(t))| < K_1 \|\mathbf{x}(t)\|^2, \tag{2.4.23}$$

for some $K_1 > 0$ and ε sufficiently small. So

$$\frac{d}{dt} G^\varepsilon(\mathbf{x}) = (\mathcal{L}_+^\varepsilon \mathbf{x}, \mathcal{L}_+^\varepsilon \mathbf{x}) + o(\|\mathbf{x}\|^2) \geq (K^2 + o(1)) \|\mathbf{x}\|^2 > 0,$$

for all $\mathbf{x} \in [\mathbf{u} \,\mathrm{rel}\, \mathbf{v}]$ close to origin and $G^\varepsilon(\mathbf{0}) = 0$. Now, let $\varepsilon_3 > 0$ be so small that $\frac{d}{dt} G^\varepsilon > 0$, whenever $G^\varepsilon(\mathbf{x}) < \varepsilon_3$. Define

$$U = \{\mathbf{x} : G^\varepsilon(\mathbf{x}) \ < \varepsilon_3\}.$$

We choose ε_4 so small that the ball with radius ε_4 is a subset of U. Let us track points in $[\mathbf{u} \,\mathrm{rel}\, \mathbf{v}^\varepsilon] \cap B_{\varepsilon_4}(\mathbf{v}^1)$ back in time for the flow Φ_t^ε. If orbit $\Phi_t^\varepsilon(\mathbf{x})$ stays in $[\mathbf{u} \,\mathrm{rel}\, \mathbf{v}^\varepsilon]$, for $t < 0$, then $\Phi_t^\varepsilon(\mathbf{x})$ stays in U and $\frac{d}{dt} G^\varepsilon(\Phi_\varepsilon^t(\mathbf{x})) > 0$ for all $t < 0$. It follows that $\Phi_\varepsilon^t(\mathbf{x}) \to \mathbf{0}$ as $t \to -\infty$. Hence, $\sum_{\mathbf{z} \in \mathbf{z}^\varepsilon} I(\Phi_t^\varepsilon(\mathbf{x}), \mathbf{z}) \to 2q'$ as $t \to -\infty$. On the other hand if $\Phi_t^\varepsilon(\mathbf{x}) \in [\mathbf{u} \,\mathrm{rel}\, \mathbf{v}^\varepsilon]$ then $\sum_{\mathbf{z} \in \mathbf{z}^\varepsilon} I(\Phi_t^\varepsilon(\mathbf{x}), \mathbf{z}) = p' I(\mathbf{u}, \mathbf{v}^1)$ and $p' I(\mathbf{u}, \mathbf{v}^1) < 2q'$. Therefore $\Phi_t^\varepsilon(\mathbf{x})$ leaves the class $[\mathbf{u} \,\mathrm{rel}\, \mathbf{v}^\varepsilon]$, for some $t_0 < 0$ and $\mathbf{x} \notin \mathrm{Inv}_{\Phi_\varepsilon}(\mathrm{cl}[\mathbf{u} \,\mathrm{rel}\, \mathbf{v}^\varepsilon])$ for any $0 < \varepsilon < \varepsilon_4$ and all $\mathbf{x} \in [\mathbf{u} \,\mathrm{rel}\, \mathbf{v}^\varepsilon] \cap B_{\varepsilon_4}(\mathbf{v}^1)$.

Now, suppose that $\mathbf{y} \in \partial[\mathbf{u} \,\mathrm{rel}\, \mathbf{v}]$ and $\mathbf{y} \neq \mathbf{v}^1$. The flow Ψ^t is transversal to the set $\partial[\mathbf{u} \,\mathrm{rel}\, \mathbf{v}] \setminus \{\mathbf{v}^1\}$. We can suppose that it points out of the set $[\mathbf{u} \,\mathrm{rel}\, \mathbf{v}]$ at $\mathbf{y}$. Otherwise we get the same result for the reversed time direction. According to Lemma 2.2.8 the flow Ψ cannot enter the class $[\mathbf{u} \,\mathrm{rel}\, \mathbf{v}]$ after leaving it. This combined with the transversality of the flow implies that there exists $\varepsilon_5 > 0$ such that for every $\mathbf{x} \in \mathrm{cl}(B_{\varepsilon_5}(\mathbf{y})) \cap [\mathbf{u} \,\mathrm{rel}\, \mathbf{v}]$ there is a constant $T_\mathbf{x}$ with properties $\Psi_{T_\mathbf{x}}(\mathbf{x}) \in \partial[\mathbf{u} \,\mathrm{rel}\, \mathbf{v}]$ and $\Psi_{T_\mathbf{x}+1}(\mathbf{x}) \notin \mathrm{cl}([\mathbf{u} \,\mathrm{rel}\, \mathbf{v}])$. Moreover $T = \sup_{\mathbf{x} \in C} T_\mathbf{x}$ is finite and

$$\delta = \min_{\mathbf{x} \in C} \rho(\Psi_{T_\mathbf{x}+1}(\mathbf{x}), \partial[\mathbf{u} \,\mathrm{rel}\, \mathbf{v}]) > 0.$$

One can estimate

$$\|\Phi_t^\varepsilon(\mathbf{x}) - \Psi_t(\mathbf{x})\| \leq \int_0^t \|\mathcal{N}^\varepsilon(\Phi_s^\varepsilon(\mathbf{x})) - \mathcal{R}(\Psi_s(\mathbf{x}))\| \, ds \leq$$

$$\leq \int_0^t \|\mathcal{N}^\varepsilon(\Phi_s^\varepsilon(\mathbf{x})) - \mathcal{R}(\Phi_s^\varepsilon(\mathbf{x}))\| \, ds + \int_0^t \|\mathcal{R}(\Phi_s(\mathbf{x})) - \mathcal{R}(\Psi_s(\mathbf{x}))\| \, ds.$$

For some constant $C > 0$ and $t \in [0, T+1]$ we get

$$\|\Phi_t^\varepsilon(\mathbf{x}) - \Psi_t(\mathbf{x})\| \le K\varepsilon(T+1) + C\int_0^t \|\Phi_s^\varepsilon(\mathbf{x}) - \Psi_s(\mathbf{x})\|\, ds,$$

where K is a positive constant (see Remark 2.4.7) and the estimate on the second term follows from C^1-regularity of $\mathcal{R}$. Gronwall's theorem implies that

$$\|\Phi_t^\varepsilon(\mathbf{x}) - \Psi_t(\mathbf{x})\| \le K\varepsilon(T+1)e^{Rt},$$

for $t \in [0, T+1]$ and $0 < \varepsilon < \varepsilon_5$. So for

$$\varepsilon < \varepsilon_{\mathbf{y}} := \min\left\{\varepsilon_5, \frac{\delta}{2K(T+1)}e^{-R(T+1)}\right\}$$

we have that

$$\left\|\Phi_{T_x+1}^\varepsilon(\mathbf{x}) - \Psi_{T_x+1}(\mathbf{x})\right\| \le \frac{1}{2}\delta.$$

Therefore $\Phi_{T_{\mathbf{x}}+1}^\varepsilon \notin [\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$, for $\mathbf{x} \in B_{\varepsilon_{\mathbf{y}}}(\mathbf{y})$ and

$$B_{\varepsilon_{\mathbf{y}}}(\mathbf{y}) \cap \mathrm{Inv}_{\Phi^\varepsilon}(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]) = \varnothing$$

for $\varepsilon < \varepsilon_{\mathbf{y}}$. This concludes (2.4.21) which implies that

$$\mathrm{Inv}_{\Phi^\varepsilon}(\mathrm{cl}[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]) \subset \mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}])$$

and thus finishes the proof of the lemma. □

2.5. Application to the fourth order differential equation

In this section we prove existence of solutions of Equation (2.1.1) on the zero energy level. We concentrate on solutions of the third type i.e., those which intersect constant solution $u_+ = +1$ but do not intersect $u_- = -1$. As we mentioned in Section 2.1 these solutions can be further classified by the number of monotone loops ($2p$) and number of intersections ($2q$) with u_+. To prove Theorem 2.1.2 we will show existence of solution $u^\alpha \in \mathbf{u}_{p,q}$ for $\alpha \in (\sqrt{8}, \alpha_{p,q})$ where $\alpha_{p,q}$ is given by (2.1.5).

One obstacle to applying the machinery developed in the previous section is that strands $\mathbf{u}_\pm$ corresponding to the discretization of the constant solutions $u_\pm = \pm 1$ do not obey the up-down restriction. Hence we cannot not include them in the skeleton $\mathbf{v}$ and define the braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ by taking the free strand $\mathbf{u}$ which intersects $2q$ times the strand $\mathbf{u}_+$ but does not intersect the strand $\mathbf{u}_-$.

To overcome this problem we have to use a more elaborate approach. First we will show that for small positive energy E there exist two solutions of (2.1.1) such that one oscillates around u_+ while the other one around u_-. Then we will define the braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$. The strands associated to the small oscillations around $u_\pm$ are included in $\mathbf{v}$ and the free strand is braided with them in the way mentioned above. We will employ the result from the previous section to prove existence of a fixed point within the braid class. This provides a solution u^E of (2.1.1), for small positive E, such that $\mathbb{E}[u^E] = E$. Finally we will use a limit process $E \to 0$ for solutions u^E to find a solution $u \in \mathbf{u}_{p,q}$ at the zero energy level.

Small oscillations

Here we will show the existence of a solution, which oscillates around the constant solution, on every small positive energy level, and associate it with a strand. The rotation number of this strand will be computed as well.

LEMMA 2.5.1. *For every $\alpha > \sqrt{8}$ and sufficiently small $E > 0$ there exists a periodic solution u_+^E of Equation (2.1.1) with two extrema per period such that $\min u_+^E < 1 < \max u_+^E$, and $\mathbb{E}[u_+^E] = E$. Moreover $u_+^E \to +1$ as $E \to 0$.*

PROOF. The transformation $u(t) = 1 + \epsilon w(t)$ transforms Equation (2.1.1) into

$$w'''' + \alpha w'' + 2w + 3\epsilon w^2 + \epsilon^2 w^3 = 0, \tag{2.5.1}$$

with the energy functional given by

$$\mathbb{E}_\epsilon[w] = -w'w''' + \frac{1}{2}(w'')^2 - \frac{\alpha}{2}(w')^2 - F_\epsilon(w),$$

where $F_\epsilon(w) = w^2 + \epsilon w^3 + \frac{1}{4}\epsilon^2 w^4$. If $\epsilon = 0$ then (2.5.1) becomes linear:

$$w'''' + \alpha w'' + 2w = 0. \tag{2.5.2}$$

The eigenvalues of the last equation are given by

$$\lambda_i^2 = \frac{1}{2}[\alpha - (-1)^i\sqrt{\alpha^2 - 8}].$$

Thus $w_0(t) = -\cos(\lambda_1 t)$ is its solution with two extrema per period and energy $\mathbb{E}_0[w_0] = \frac{\lambda_1^4}{2} - 1 > 0$ for $\alpha > \sqrt{8}$.

We now turn to the nonlinear problem. Equation (2.5.1) contains only even derivatives for every $\epsilon \geq 0$. This implies that every solution satisfying

$$w'(0) = w'(T) = w'''(0) = w'''(T) = 0,$$

for some $T \in \mathbb{R}^+$ is $2T$-periodic. Define $G : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$G(A, T, \epsilon) = \begin{pmatrix} w'_{\epsilon,A}(T) \\ w'''_{\epsilon,A}(T) \end{pmatrix},$$

where $w_{\epsilon,A}$ is the solution of (2.5.1) with initial data

$$w_{\epsilon,A}(0) = A, \quad w'_{\epsilon,A}(0) = 0,$$

$$w''_{\epsilon,A}(0) = \sqrt{2(F_\epsilon(A) + \mathbb{E}_0[w_0])}, \quad w'''_{\epsilon,A}(0) = 0.$$

If $G(w_{\epsilon,A}, T, \epsilon) = (0, 0)^T$ then $w_{\epsilon,A}$ is a $2T$-periodic solution of (2.5.1). The condition $w''_{\epsilon,A}(0) = \sqrt{2(F_\epsilon(A) + \mathbb{E}_0[w_0])}$ implies that $\mathbb{E}_\epsilon[w_{\epsilon,A}] = \mathbb{E}_0[w_0]$.

To prove the existence of periodic solutions of (2.5.1) for $\epsilon > 0$ we will employ the implicit function theorem for the function G. For $\epsilon = 0$ we have $w_{0,-1} = w_0$ and

$$G(-1, \frac{\pi}{\lambda_1}, 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We can express

$$w_{\epsilon,A}(t) = C(A)\cos(\lambda_1 t) + D(A)\cos(\lambda_2 t) + g(\epsilon, A, t), \tag{2.5.3}$$

where $g = o(\epsilon)$ and $C(A)$, $D(A)$ satisfy

$$C(A) + D(A) = A$$

$$-\lambda_1^2 C(A) - \lambda_2^2 D(A) = \sqrt{2(F_\epsilon(A) + \mathbb{E}_0[w_0])}.$$

Using (2.5.3) one can compute that

$$\det\left(\frac{\partial G}{\partial A}\frac{\partial G}{\partial T}\right)_{(-1,\frac{\pi}{\lambda_1},0)} = \det\begin{pmatrix} \partial_A w'_{\epsilon,A}(T) & w''_{\epsilon,A}(T) \\ \partial_A w'''_{\epsilon,A}(T) & w''''_{\epsilon,A}(T) \end{pmatrix}_{(-1,\frac{\pi}{\lambda_1},0)} =$$

$$= \frac{\lambda_1^4}{\lambda_1^2 - \lambda_2^2}\sin\frac{\lambda_2}{\lambda_1}\pi \neq 0.$$

Therefore by the implicit function theorem there exist continuous functions $A : (-\delta, \delta) \to \mathbb{R}$ and $T : (-\delta, \delta) \to \mathbb{R}$ for some $\delta > 0$ such that $A(0) = -1$, $T(0) = \frac{\pi}{\lambda_1}$ and

$$G(A(\epsilon), T(\epsilon), \epsilon) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

for $\epsilon \in [0, \delta)$.

The periodic solutions $w_\epsilon(t) := w_{A(\epsilon),\epsilon}(t)$ converge to $w_0 = -\cos \lambda_1 t$ as $\epsilon \to 0$ in C^3 norm. Thus w_ϵ has two extrema per period (one negative, one positive) for ϵ small enough.

Let $\epsilon(E) = \sqrt{\frac{E}{E_0[w_0]}}$. Then the solution $u_+^E(t) = 1 + \epsilon(E) w_{\epsilon(E)}(t)$ of Equation (2.1.1) has energy

$$\mathbb{E}[u_+^E(t)] = \epsilon(E)^2 \mathbb{E}_{\epsilon(E)}[w_{\epsilon(E)}(t)] = E.$$

□

REMARK 2.5.2. An analogous construction can by carried out to construct u_-^E, with the similar properties as u_+^E, and $u_-^E \to -1$ as $E \to 0$.

We have to keep in mind that every solution u_+^E of Equation (2.1.1) is a solution for some value of parameter α although we do not indicate it in the notation. We can associate the solution u_+^E with a braid $\mathbf{u}_+^E \in \mathcal{E}_2^1$ via its sequence of extrema. The following lemma estimates the rotation number $\tau(\mathbf{u}_+^E)$.

LEMMA 2.5.3. *Let $\mathbf{u}_+^E \in \mathcal{E}_2^1$ be a braid corresponding to the solution u_+^E for $\alpha > \sqrt{8}$. Then for every $\varepsilon > 0$ there exists $E_0 > 0$ such that*

$$\left| \tau(\mathbf{u}_+^E) - \frac{\lambda_2}{\lambda_1} \right| < \epsilon \quad \textit{for all } 0 < E < E_0, \tag{2.5.4}$$

where $\lambda_i^2 = \frac{1}{2}[\alpha - (-1)^i \sqrt{\alpha^2 - 8}]$. Moreover, the matrix $M(\mathbf{u}_+^E)$ is conjugate to the matrix

$$\begin{pmatrix} \cos(2\pi\tau(\mathbf{u}_+^E)) & -\sin(2\pi\tau(\mathbf{u}_+^E)) \\ \sin(2\pi\tau(\mathbf{u}_+^E)) & \cos(2\pi\tau(\mathbf{u}_+^E)) \end{pmatrix}. \tag{2.5.5}$$

PROOF. As we mentioned in Section 2.3 the twist maps $F_i^E(x, y)$ corresponding to the generating function S_E for Lagrangian system with Euler-Lagrange equation given by (2.1.1) can be defined as follows. Let u_i be a solution of Equation (2.1.1) with the initial value conditions

$$u_i(0) = x, \quad u_i'(0) = 0,$$
$$u_i''(0) = (-1)^i \sqrt{2E + (x^2 - 1)^2}, \quad u_i'''(0) = y.$$

Let $t_0 > 0$ be the first nonzero time for which $u'(t_0) = 0$. Then

$$F_i^E(x, y) = (u_i(t_0), u_i'''(t_0)).$$

Remark (2.3.4) implies that $M(\mathbf{u}_+^E)$ is conjugate to

$$d(F_1^E \circ F_0^E)(u_+^E(0), (u_+^E)'''(0)).$$

Let us compute $dF_0^E(u_+^E(0),(u_+^E)'''(0))$. To do so we will use the transformation $u(t) = 1 + \epsilon(E)w$ where $\epsilon(E) = \sqrt{\frac{2E}{\lambda_1^4 - 2}}$. One can see that

$$dF_0^E(u_+^E(0),(u_+^E)'''(0)) = d\widetilde{F}^E(w_E(0), w_E'''(0)),$$

where $\widetilde{F}^E$ is defined in the same manner as F_0^E but now u is a solution of Equation (2.5.1) with initial data

$$u(0) = x, \quad u'(0) = 0,$$

$$u''(0) = \sqrt{2\left(\frac{\lambda_1^4}{2} - 1 + x^2 + \epsilon(E)x^3 + \frac{1}{4}\epsilon(E)^2 x^4\right)}, \quad u'''(0) = y.$$

Continuous dependence on E implies that for every $\varepsilon_1 > 0$ there exists an E_1 such that

$$\left\| D\widetilde{F}^E(w_E(0), w_E'''(0)) - D\widetilde{F}^0(w_0(0), w_0'''(0)) \right\| < \varepsilon_1 \text{ for all } 0 < E < E_1, \tag{2.5.6}$$

where $w_0 = -\cos(\lambda_1 t)$. The value of $D\widetilde{F}^E(w_0(0), w_0'''(0))$ in the direction $(\cos\theta, \sin\theta)^T$ for $0 \leq \theta < 2\pi$ can be computed as

$$d\widetilde{F}^0(w_0(0), w_0'''(0)) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} =$$

$$= \frac{d}{d\mu}\widetilde{F}^0(w_0(0) + \mu\cos\theta, w_0'''(0) + \mu\sin\theta)_{\mu=0} =$$

$$= \begin{pmatrix} \partial_\mu y_{\mu,\theta}(P_\theta(0)) \\ -\partial_\mu y_{\mu,\theta}'''(P_\theta(0)) - y_{\mu,\theta}''''(P_\theta(0))\frac{d}{d\mu}P_\theta(0)|_{\mu=0} \end{pmatrix},$$

where $P_\theta(\mu)$ is the first positive time in which $y_{\mu,\theta}'(P_\theta(\mu)) = 0$ has a maximum. The function $y_{\mu,\theta}$ is a solution of Equation (2.5.2) with initial conditions

$$\begin{aligned} y_{\mu,\theta}(0) &= w_0(0) + \mu\cos\theta, \\ y_{\mu,\theta}'(0) &= 0, \\ y_{\mu,\theta}''(0) &= \sqrt{2\left(\frac{\lambda_1^4}{2} - 1 + (w_0(0) + \mu\cos\theta)^2\right)}, \\ y_{\mu,\theta}'''(0) &= w_0'''(0) + \mu\sin\theta. \end{aligned}$$

We can evaluate $\frac{d}{d\mu}P_\theta(0)|_{\mu=0}$ by differentiating the equation

$$y_{\mu,\theta}'(P_\theta(\mu)) = 0,$$

with respect to the parameter μ:

$$\frac{d}{d\mu}P_\theta(\mu)|_{\mu=0} = -\frac{\partial_\mu y'_{\mu,\theta}(P_\theta(0))|_{\mu=0}}{y''_{\mu,\theta}(P_\theta(0))|_{\mu=0}}.$$

Linearity of Equation (2.5.2) enable us to compute all components of $d\widetilde{F}^E(w_0(0), w_0'''(0))(\cos\theta, \sin\theta)^T$ for any θ. By doing so for $\theta = 0$ and $\theta = \frac{\pi}{2}$ we get that

$$d\widetilde{F}^E(w_0(0), w_0'''(0)) = \begin{pmatrix} \cos(\frac{\lambda_2}{\lambda_1}\pi) & -\frac{\lambda_1}{\lambda_2}\sin(\frac{\lambda_2}{\lambda_1}\pi) \\ \frac{\lambda_2}{\lambda_1}\sin(\frac{\lambda_2}{\lambda_1}\pi) & \cos(\frac{\lambda_2}{\lambda_1}\pi) \end{pmatrix},$$

which is conjugate to

$$\begin{pmatrix} \cos(\frac{\lambda_2}{\lambda_1}\pi) & -\sin(\frac{\lambda_2}{\lambda_1}\pi) \\ \sin(\frac{\lambda_2}{\lambda_1}\pi) & \cos(\frac{\lambda_2}{\lambda_1}\pi) \end{pmatrix}.$$

Thus it follows from (2.5.6) that we can choose E_0 in such a way that for all $0 < E < E_0$ the matrix dF_0^E is conjugate to the rotation matrix

$$\begin{pmatrix} \cos(2\tau_E\pi) & -\sin(2\tau_E\pi) \\ \sin(2\tau_E\pi) & \cos(2\tau_E\pi) \end{pmatrix}, \tag{2.5.8}$$

where $|\tau_E - \frac{\lambda_2}{2\lambda_1}| < \frac{\varepsilon}{2}$.

By similar calculation we get the same result for dF_1^E. By composing dF_0^E and dF_1^E one gets that $d(F_1^E \circ F_0^E)$ is also conjugate to the matrix of the form (2.5.8) for some (different) τ_E which satisfies $\left|\tau_E - \frac{\lambda_2}{\lambda_1}\right| < \epsilon$. It follows from (2.3.2) that the rotation number $\tau(\mathbf{u}_+^E) = \tau_E + k$ for some $k \in \mathbb{N}$. Using the fact that $\frac{\lambda_2}{2\lambda_1} < \frac{1}{2}$, for $\alpha > \sqrt{8}$, determines $k = 0$. □

REMARK 2.5.4. From now on, if there is no ambiguity, we will indicate a p-fold of $\mathbf{u}_+^E$ by the same symbol. The rotation number $\tau(\mathbf{u}_+^E)$ of the p-fold $\mathbf{u}_+^E \in \mathcal{E}_{2p}^1$ is p times the rotation number of $\mathbf{u}_+^E$.

Solution u^E with positive energy

We will prove the existence of a solution u of Equation (2.1.1) on the energy level zero as a limit of solutions u^E on positive energy levels, given by the following lemma, for $E \to 0$.

THEOREM 2.5.5. *Let $p, q \in \mathbb{N}$ be relatively prime such that $q < p$ and $\alpha \in (\sqrt{8}, \alpha_{p,q})$. Then for sufficiently small E there exists a solution u^E of (2.1.1) with $\mathbb{E}[u^E] = E$. Its extrema sequence $\mathbf{u}^E$ is $2p$-periodic. Moreover $I(\mathbf{u}^E, \mathbf{u}_+^E) = 2q$ and $I(\mathbf{u}^E, \mathbf{u}_-^E) = 0$, where $\mathbf{u}^E$, $\mathbf{u}_+^E$ and $\mathbf{u}_-^E$ are extrema* *sequences corresponding to the solutions u^E, u^E_+ and u^E_-, seen as points in $\mathcal{E}^1_{2p}$.*

PROOF. To prove this theorem we will employ the relative braid class $[\mathbf{u} \operatorname{rel} \mathbf{v}] \subset \mathcal{E}^1_{2p} \operatorname{rel} \mathbf{v}$. This braid class will turn out to contain a fixed point $\mathbf{u}^E$ which is an extrema sequence of the solution u^E. Let us start by identifying the skeleton

$$\mathbf{v} = \mathbf{v}^1 \cup \mathbf{v}^2 \cup \mathbf{v}^3 \in \mathcal{E}^3_{2p}.$$

We define $\mathbf{v}^1 = \mathbf{u}^E_+$ and $\mathbf{v}^2 = \mathbf{u}^E_-$. To construct the strand $\mathbf{v}^3$ we use the dissipativity of the Lagrangian system generated by Equation (2.1.1). Dissipativity implies the existence of $u^*_1, u^*_2 \in \mathbb{R}$ such that $u^*_1 < v^1_i, v^2_i < u^*_2$ for all i and $\mathcal{R}_{2i}(u_{2i-1}, u^*_1, u_{2i+1}) < 0$ for $u^*_1 < u_{2i\pm1} < u^*_2$ while $\mathcal{R}_{2i+1}(u_{2i}, u^*_2, u_{2i+1}) > 0$, for $u^*_1 < u_{2i}, u_{2i+2} < u^*_2$. For more details see [20]. Let

$$\Omega_i = \begin{cases} \{(u_{i-1}, u_i, u_{i+1}) \in \mathbb{R}^3 : u^*_1 < u_{i\pm1} < u_i < u^*_2\}, i \ \ odd, \\ \{(u_{i-1}, u_i, u_{i+1}) \in \mathbb{R}^3 : u^*_1 < u_i < u_{i\pm1} < u^*_2\}, i \ \ even. \end{cases}$$

Denote by Ω^{2p} the set of $2p$-periodic sequences $\{u_i\}$ for which $(u_{i-1}, u_i, u_{i+1}) \in \Omega_i$. Furthermore define the set

$$C = \{\mathbf{u} \in \Omega^{2p} : I(\mathbf{u}, \mathbf{v}^1) = I(\mathbf{u}, \mathbf{v}^2) = 2p\}.$$

Since $I(\mathbf{v}^1, \mathbf{v}^2) = 0$ the vector field $\mathcal{R}$ is transverse to ∂C. Moreover, the set C is contractible, compact, and $\mathcal{R}$ is pointing outward at the boundary ∂C due to the dissipativity. The set C is therefore negatively invariant for the induced flow Ψ^t. Consequently, there exists a fixed point $\mathbf{v}^3$ of Ψ^t in the interior of C.

We define $[\mathbf{u} \operatorname{rel} \mathbf{v}] \in \mathcal{E}^1_{2p} \operatorname{rel} \mathbf{v}$, by its representant $\mathbf{u}$ satisfying

1) $(-1)^i u_i > (-1)^i v^3_i$,
2) $u_i > v^2_i$,
3) $I(\mathbf{u}, \mathbf{v}^1) = 2q$,

where $0 < 2q < 2p$, see Figure 2.7.

For $p \geq 2$, $[\mathbf{u} \operatorname{rel} \mathbf{v}]$ is a bounded improper and free up-down braid class where $\mathbf{u}$ can collapse only on $\mathbf{v}^1$. It follows from Lemma 2.5.3 and Remark 2.5.4 that for every $\varepsilon_1 > 0$ we can choose $E > 0$ so small that the rotation number of $\mathbf{v}^1 = \mathbf{u}^E_+$ satisfies the inequality

$$\left| \tau(\mathbf{v}^1) - p\frac{\lambda_2}{\lambda_1} \right| < \varepsilon_1,$$

where $\lambda^2_i = \frac{1}{2}[\alpha - (-1)^i\sqrt{\alpha^2 - 8}]$. If $\alpha \in (\sqrt{8}, \alpha_{p,q})$ then $\frac{q}{p} < \frac{\lambda_2}{\lambda_1}$. Therefore for any $\alpha \in (\sqrt{8}, \alpha_{p,q})$ we can choose ε_1 in such a way that

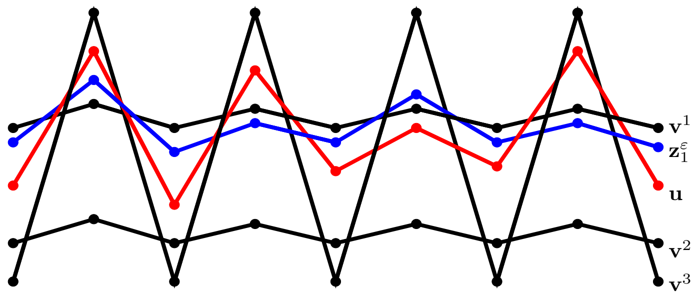

Figure 2.7: A representat of the braid class $[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$ for $p = 4, q = 3, p' = 1$ and $q' = 2$. If we skip the strand $\mathbf{z}_1^\varepsilon$ from the skeleton we get a representant of the braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$.

$\tau(\mathbf{v}^1) > q$. Hence according to Lemma 2.4.13

$$\mathrm{Inv}_\Psi([\mathbf{u} \text{ rel } \mathbf{v}]) = \mathrm{Inv}_{\Phi^\varepsilon}([\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]),$$

for ε small. Hence non-triviality of the index $\mathbf{H}(\mathbf{u} \text{ rel } \overline{\mathbf{v}}^\varepsilon)$, where $\overline{\mathbf{v}}^\varepsilon$ is the augmentation of $\mathbf{v}^\varepsilon$ as explained in Theorem 2.2.11, implies that the braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$ contains a fixed point of the flow Ψ^t. Non-triviality of $\mathbf{H}(\mathbf{u} \text{ rel } \overline{\mathbf{v}}^\varepsilon)$ is given by the following lemma. □

LEMMA 2.5.6. *The homotopy type of the topological invariant* $\mathbf{H}(\mathbf{u} \text{ rel } \overline{\mathbf{v}}^\varepsilon)$ *is given as follows*

$$\mathbf{H}(\mathbf{u} \text{ rel } \overline{\mathbf{v}}^\varepsilon) = S^{q-1} \vee S^q.$$

PROOF. We will use techniques developed in [2, 9] to compute the homotopy invariant $\mathbf{H}(\mathbf{u} \text{ rel } \overline{\mathbf{v}}^\varepsilon)$. We choose a sufficiently simple system (an integrable Hamiltonian system) which exhibits the braids in question and compute the homotopy index by examining the invariant set of this system and its unstable manifold.

Consider the first-order Lagrangian system given by the Lagrangian $L_\lambda(u, u_t) = \frac{1}{2}|u_t|^2 + \lambda F(u)$, where we choose $F(u)$ to be an even four-well potential, with $F''(u) \geq -1$ and $F''(0) = -1$. The Lagrangian system (L_λ, dt) defines an integrable Hamiltonian system on $\mathbb{R}^2$, given by differential equation $u'' = \lambda F'(u)$. The phase portrait of this equation is depicted in Fig. 2.8.

It was shown in [9] that for $0 < \lambda < \pi^2$ the time-1 map defined via the induced Hamiltonian flow Ψ^λ, is an area preserving monotone

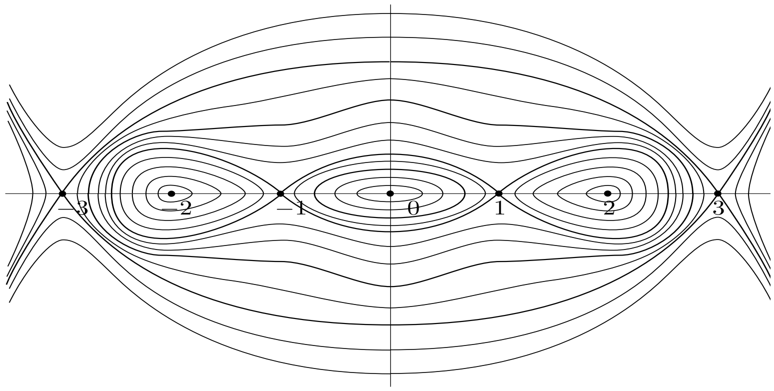

Figure 2.8: The integrable model in the (u, u_t) plane; there are centers at $0, \pm 2$ and saddles in $\pm 1, \pm 3$.

twist map. The generating function of the twist map is given by the minimization problem

$$S_\lambda(u_1, u_2) = \inf_{u \in X(u_1,u_2)} \int_0^1 L(u, u_t) dt,$$

where $X(u_1, u_2) = \{u \in H^1(0,1) : u(0) = u_1, u(1) = u_2\}$. The recurrence function $\mathcal{R}_\lambda(u_{i-1}, u_i, u_{i+1}) = \partial_2 S_\lambda(u_{i-1}, u_i) + \partial_1 S_\lambda(u_i, u_{i+1})$ defines an exact (autonomous) parabolic recurrence relation. We choose the potential F such that the bounded solutions within the heteroclinic loop between -1 and $+1$ have the property that the period T_λ is an increasing function of the amplitude A, and $T_\lambda \to \frac{2\pi}{\sqrt{\lambda}}$, as $A \to 0$. This single integrable system is enough to compute the homotopy index $\mathbf{H}(\mathbf{u} \text{ rel } \overline{\mathbf{v}}^\varepsilon)$.

We begin by identifying the following periodic solutions. Set $\mathbf{v}^1 = \{v_i^1\}, v_i = 0$, and $\mathbf{v}^2 = \{v_i^2\}, v_2 = -1$, and $\mathbf{v}^\pm = \{v_i^\pm\}, v_i^\pm = \pm 3$. Let $\widetilde{u}(t)$ be a solution of $u'' = \lambda F'(u)$ with minimum $\widetilde{u}(0) \in (-3, -2)$ which oscillates around both equilibria ± 2 and $T_1(A(\widetilde{u})) = 2\tau_0 \geq 2\pi, \tau_0 \in \mathbb{N}$. For arbitrary $\lambda \leq 1$ this implies that

$$T_\lambda(A(\widetilde{u})) = \frac{T_1(A(\widetilde{u}))}{\sqrt{\lambda}} = \frac{2\tau_0}{\sqrt{\lambda}},$$

where we choose λ so that $\frac{1}{\sqrt{\lambda}} \in \mathbb{N}$. Set $d = \frac{2\tau_0 p}{\sqrt{\lambda}}$ and define $\mathbf{v}^3 = \{v_i^3\}$ with $v_i^3 = \widetilde{u}(i), i = 0, \ldots, d$. Clearly, $I(\mathbf{v}^3, \mathbf{v}^1) = I(\mathbf{v}^3, \mathbf{v}^2) = 2p$ for λ sufficiently small.

Finally we will identify the strands $\mathbf{z}^\varepsilon$. Let p' be the number of strands in $\mathbf{z}^\varepsilon$ and $2q' = \sum_{\mathbf{z}\in\mathbf{z}^\varepsilon} I(\mathbf{z}, \mathbf{v}^1)$. If we interpret the braid $\mathbf{z}^\varepsilon$ as a single strand in $\mathcal{E}^1_{2pp'}$ then one can see that $2q'$ is the intersection number of this strand with the p-fold of $\mathbf{v}^1$. Hence $\frac{2pp'}{2q'} > 1$ and $\frac{2\tau_0 pp'}{q'} > 2\tau_0 > 2\pi$. Thus we can choose $\widehat{u}(t)$ to be a solution of $u'' = \lambda F(u)$ with minimum $\widehat{u}(0) \in (-1, 0)$ and

$$T_\lambda(A(\widehat{u})) = \frac{2\tau_0 pp'}{q'\sqrt{\lambda}} = d\frac{p'}{q'}.$$

Set $(\mathbf{z}^\varepsilon)^k = \{z_i^k\}$, $z_i^k = \widehat{u}(kd+i)$, for $i = 0, \ldots, d$ and $k = 0, \ldots, p'-1$. On the interval $[0, dp']$ the solutions $\widetilde{u}$ and $\widehat{u}$ have exactly $2q'$ intersections. So $\sum_{\mathbf{z}\in\mathbf{z}^\varepsilon} I(\mathbf{z}, \mathbf{v}^1) = 2q'$, for λ sufficiently small.

We note that in the construction of the proper braid class $[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$ we chose p' and q' in such a way that $q < \frac{q'}{p'}$. Now we look for the solutions which corresponds to the free strand. Period T_λ increases for $A \in (0, 1)$, thus there exists a unique periodic solution $u(t)$, with with minimum $u(0) \in (-1, 0)$ and $T_1(A(u)) = \frac{2\tau_0 p}{q}$. The inequality $q < \frac{q'}{p'}$ implies that the solutions $\widehat{u}$ and u intersects $2q$ times on interval $[0, d]$. This provide us with one parameter family of critical points $M_q = \{\mathbf{u}(s) : u_i(s) = u(i + s\frac{2\tau_0 p}{q\sqrt{\lambda}})\}$, for $s \in \mathbb{R}/\mathbb{Z}$, of the system generated by $\mathcal{R}_\lambda$ and $I(\mathbf{u}(s), \mathbf{v}^1) = I(\mathbf{u}(s), (\mathbf{z}^\varepsilon)^k) = 2q$ while $I(\mathbf{u}(s), \mathbf{v}^3) = 2p$ for λ sufficiently small. Due to the fact that $T_\lambda(A)$ is an increasing function of the amplitude A these are the only stationary points, with the required crossing numbers, of the system generated by $\mathcal{R}_\lambda$. It was shown in [2] that M_q is diffeomorfic to a circle with a $2q$ dimensional unstable manifold, hence

$$\mathbf{H}(\mathbf{u} \text{ rel } \overline{\mathbf{v}}^\varepsilon) = H(M_q) = S^{q-1} \vee S^q.$$

□

Limit process

We proved existence of a solution u^E of (2.1.1) in the parameter range $\alpha \in (\sqrt{8}, \alpha_{p,q})$ on small positive energy levels E. Its sequence of extrema $\mathbf{u}^E$ is $2p$ periodic and $I(\mathbf{u}^E, \mathbf{u}^E_+) = 2q$, while $I(\mathbf{u}^E, \mathbf{u}^E_-) = 0$. We will construct a sequence $\{u_n\}_{n=0}^\infty$ given by $u_n = u^{E_n}$ such that $u = \lim_{n\to\infty} u_n$ is a solution of (2.1.1) in the periodic class $\mathbf{u}_{p,q}$ and $\mathbb{E}[u] = 0$. First we will show that there is a convergent sequence $\{u_n\}_{n=0}^\infty$.

LEMMA 2.5.7. *There exists a convergent sequence $\{u_n\}_{n=0}^{\infty}$ of solutions of (2.1.1) such that $u_n \to u$ for $n \to \infty$ in the C^4 norm. Moreover u is a solution of (2.1.1) on the zero energy level.*

PROOF. Define the sequence $\{u_m\}_{m=0}^{\infty}$ by $u_m = u^{E_m}$ where $E_m \to 0$ as $m \to \infty$. We will show that sequence $\left\{\frac{d^i}{dt^i}u_m(0)\right\}_{m=0}^{\infty}$ is bounded for $i \in \{0,1,2,3\}$. It follows from the construction of solutions u^E that $u_1^* < u_m(t) < u_2^*$ for all t and $u_m'(0) = 0$. Energy equation implies that $u_m''(0) = \sqrt{2E_m + \frac{(u_m^2(0)-1)^2}{2}}$. Therefore $\{u_m''(0)\}_{m=0}^{\infty}$ is bounded. By standard estimates on the third derivative one can get that the sequence $\{u_m'''(0)\}_{m=0}^{\infty}$ is bounded as well.

Then we can choose a subsequence $\{u_n\}_{n=0}^{\infty}$ such that $\frac{d^i}{dt^i}u_n(0) \to u^i$ for $n \to \infty$. The sequence $\{u_n\}_{n=0}^{\infty}$ converges in the C^4 norm to the solution u of (2.1.1) satisfying initial value conditions $\frac{d^i}{dt^i}u(0) = u^i$. Its energy is $\mathbb{E}[u] = \lim_{n\to\infty} \mathbb{E}[u_n] = 0$. □

The following Lemma shows that if the limit solution u is not constant then it is in the periodic class $\mathbf{u}_{p,q}$.

LEMMA 2.5.8. *Let u be the limit of the sequence $\{u_n\}_{n=0}^{\infty}$ given by the previous lemma. If $u \not\equiv 1$ then $u \in \mathbf{u}_{p,q}$.*

PROOF. Let T_n be the period of u_n. Every solution u_n has $2p$ extrema per period and $I(\mathbf{u}_n, \mathbf{u}_+^{E_n}) = 2q$ for $n \in \mathbb{N}$. We denote them $\mathbf{u}_n = (u_0^n, \ldots, u_{2p-1}^n)$. Let $t_n^i \in [0, T_n)$ be the time in which the solution u_n attains the minimum (maximum) u_i^n. The energy $\mathbb{E}[u_n]$ is positive small E, hence any two extremal points u_i^n and u_{i+1}^n are connected by a non-degenerate monotone lap.

According to Lemma 2.5.7 the limit $u = \lim_{n\to\infty} u_n$ lies in the zero energy level, and we want to show that $u \in \mathbf{u}_{p,q}$. First we will show that u is not a constant solution of (2.1.1), i.e., $u \not\equiv \pm 1$. We excluded the case $u \equiv 1$ in the assumption of the lemma. On the other hand it follows from $I(\mathbf{u}_n, \mathbf{u}_+^{E_n}) = 2q$ that for every $n \in \mathbb{N}$ there is t_n such that $u_n(\bar{t}_n) > 1$. Thus u_n can not converge to $u_- \equiv -1$.

We will prove periodicity of u by contradiction. If u is not periodic then $T_n \to \infty$ as $n \to \infty$. Otherwise there would be a constant K such that $T_n < K$ for all $n \in \mathbb{N}$. Then there is a converging subsequence of T_n which converge to some $T \in \mathbb{R}$ and u has to be T periodic. If $T_n \to \infty$ and u is not constant then u consists of a finite number of monotone laps and $u(t)$ has to monotonically converge to $u_\pm = \pm 1$ for $t \to \infty$. This is not possible because the equilibrium points ± 1 are centers.

Next we prove that $u \in \mathbf{u}_{p,q}$. We start with showing that u has $2p$ monotone laps per period. Degenerate monotone laps (inflexion points) can occur on the singular energy level. According to the definition of the solution class we have to count also these degenerate laps. Hence to show that u has $2p$ monotone laps per period it is enough to prove that no sets of more than two extremal points can collapse onto each other and if two extremal points collapse then a degenerated monotone lap is created.

Suppose that there are three different extremal points collapsing onto each other. Then sequences $\{t_n^{i-1}\}, \{t_n^i\}, \{t_n^{i+1}\}$ converge to the same t_0. The equalities $u_n'(t_n^{i-1}) = u_n'(t_n^i) = u_n'(t_n^{i+1}) = 0$ imply that there exist $\widetilde{t}_n \in (t_n^{i-1}, t_n^i)$ and $\widehat{t}_n \in (t_n^i, t_n^{i+1})$ such that $u_n''(\widetilde{t}_n) = u_n''(\widehat{t}_n) = 0$. Finally, there are $\bar{t}_n \in (\widetilde{t}_n, \widehat{t}_n)$ such that $u_n'''(\bar{t}_n) = 0$. By continuity $u'(t_0) = u''(t_0) = u'''(t_0) = 0$. Since $\mathbb{E}[u] = 0$, it holds that $u(t_0) = \pm 1$ and u is a constant solution. However, we already showed that u can not be constant.

If there is a collapsing monotone lap (two extremal points collapse on one) then the same argumentation as above implies that it collapses on an inflexed point and the number of monotone laps is preserved.

Now we will show that the solution u intersects the constant solution $u_+ \equiv 1$ exactly $2q$ times per period. Let $s_n^i \geq 0$ be the time at which the i-th intersection of u_n with u_+ occurs. Between any two crossings of the extrema sequences $\mathbf{u}_n$ and $\mathbf{u}_+^{E_n}$ is at least one anchor point. Thus for every s_n^i there exists t_n^j such that $s_n^i \leq t_n^j \leq s_n^{i+1}$. If two crossing points s_n^i and s_n^j with $i < j$ collapse i.e., $s_n^i - s_n^j \to 0$, then $u_n(t_n^j) \to 1$ as $n \to \infty$ for all j such that $s_n^i \leq t_n^j \leq s_n^j$. We showed that more than two extremal points can not collapse, and thus more than three crossings can not collapse.

Hence suppose that three crossings collapse (the case of just two collapsing crossings is dealt with later). We can assume that crossings are between extremal points $u_i^n, u_{i+1}^n, u_{i+2}^n, u_{i+3}^n$. It is $s_n^i \leq t_n^{j+1} \leq s_n^{i+1} \leq t_n^{j+2} \leq s_n^{i+2}$ and $s_n^i, s_n^{i+2} \to \bar{t}$ as $n \to \infty$. As before one can show that $u(\bar{t}) = 1$ and $u'(\bar{t}) = u''(\bar{t}) = 0$. We assert that $u(t_n^j) \to A \neq 1$ and $u(t_n^{j+4}) \to B \neq 1$ for $n \to \infty$. Otherwise at least three extremal points would collapse. Let $A < B$ (the other case is analogous), then $u'''(\bar{t}) > 0$. By construction $\left| u_+^{E_n} - \left(1 + \sqrt{\frac{E_n}{E_0}} \cos(\lambda_1 t)\right)\right| \to 0$ for $n \to \infty$, where $E_0 = \frac{\lambda_1^4}{2} - 1$. Hence $|(u_+^{E_n})_i - 1| > \frac{1}{2}\sqrt{\frac{E_n}{E_0}}$ for all i and n sufficiently large. It holds for the maximum of the solution u_n at t_n^{i+1}

that $u_n(t_n^{i+1}) > (u_+^{E_n})_{i+1} > 1$ while for the minimum at t_n^{i+2} we get the inequality $u_n(t_n^{i+2}) < (u_+^{E_n})_{i+2} < 1$. We then estimate

$$u_n(t_n^{i+1}) - u_n(t_n^{i+2}) \geq (u_+^{E_n})_{i+1} - (u_+^{E_n})_{i+2} > \sqrt{\frac{E_n}{E_0}}.$$

Let $\delta_n = \sqrt{\frac{E_n}{E_0}}$. It follows form the mean value theorem that for every n there exists $c_n \in (t_{i+1}^n, t_{i+2}^n)$ such that

$$-u_n'(c_n) = \frac{u_n(t_n^{i+1}) - u_n(t_n^{i+2})}{t_n^{i+2} - t_n^{i+1}}.$$

Due to $t_{i+2}^n - t_{i+1}^n \to 0$, we can estimate

$$-u_n'(c_n) > \delta_n > \sqrt{\frac{E_n}{E_0}}, \tag{2.5.9}$$

for n large enough. If we divide the energy equation

$$E_n = -u_n'(c_n)u_n'''(c_n) + \frac{1}{2}(u_n''(c_n))^2 - \frac{\alpha}{2}(u_n'(c_n))^2 - \frac{1}{4}(u_n^2(c_n) - 1)^2.$$

by the positive number $-u_n'(c_n)$ and use Inequality (2.5.9), we get

$$\sqrt{E_0 E_n} \geq u_n'''(c_n) - \frac{\alpha}{2}|u_n'(c_n)| + \frac{(u_n^2(c_n) - 1)^2}{4u_n'(c_n)}. \tag{2.5.10}$$

Let us estimate

$$|u_n(c_n) - 1| < u_n(t_n^{i+1}) - u_n(t_n^{i+2}) = -u_n'(c_n)(t_n^{i+2} - t_n^{i+1}) \leq -u_n'(c_n),$$

and

$$\left|\frac{(u_n^2(c_n) - 1)^2}{u_n'(c_n)}\right| \leq |u_n'(c_n)|(u^2(c_n) + 1).$$

Taking limit for $n \to \infty$ in Inequality (2.5.10) implies that $u'''(\bar{t}) \leq 0$, which is a contradiction with $u'''(\bar{t}) > 0$. Thus three crossings can not collapse.

Now we show by contradiction that two crossings cannot collapse. If two crossings collapse then there exist $s_n^i \leq t_n^j \leq s_n^{i+1}$ such that $s_n^i, s_n^{i+1} \to \bar{t}$. We can assume that $u_n(t_n^{j\pm1}) \not\to 1$ otherwise the proof is analogous to the case of three collapsing intersections. Then for $t = \bar{t}$ the solution u has an extremum and $u(\bar{t}) = 1$. As before, this contradicts that $\mathbb{E}[u] = 0$ and $u \not\equiv 1$.

Finally, $u(t) > -1$ for all t because otherwise there would be an extremum point $\bar{t}$ of u with $u(\bar{t}) = -1$ and again $\mathbb{E}[u] > 0$. $\square$

The final thing we have to show is that $\{u_n\}_{n=0}^{\infty}$ does not converge to the constant solution $u_+ = 1$. Let $\mathbb{E}[u_n] = E_n$ and define the sequences $\{w^n\}_{n=0}^{\infty}$ and $\{w_+^n\}_{n=0}^{\infty}$ as follows

$$u_n = 1 + \varepsilon(n)w^n,$$

$$u_+^{E_n} = 1 + \varepsilon(n)w_+^n,$$

where $\varepsilon(n) = \|u_n - 1\|_{L^\infty}$. Then w^n, w_+^n are solutions of equation

$$w'''' + \alpha w'' + 2w + 3\varepsilon(n)w^2 + \varepsilon^2(n)w^3 = 0.$$

Let $\mathbb{E}_\varepsilon$ be the energy functional related to the previous equation. Then $\mathbb{E}_{\varepsilon(n)}[w^n] = \mathbb{E}_{\varepsilon(n)}[w_+^n] > 0$. If $u_n \to 1$ then $\varepsilon(n) \to 0$, $w^n \to w$ and $w_+^n \to w_+$ where w and w_+ are solutions of the linear equation

$$w'''' + \alpha w'' + 2w = 0, \tag{2.5.11}$$

with $\mathbb{E}_0[w] = \mathbb{E}_0[w_+] = E \geq 0$. By construction $w_+ = \sqrt{\frac{E}{E_0}}\cos(\lambda_1 t)$, where $E_0 = \frac{\lambda_1^4}{2} - 1$.

The following two lemmas summarize the properties of linear equation (2.5.11).

LEMMA 2.5.9. *Let $\alpha > \sqrt{8}$ be such that $\frac{\lambda_2}{\lambda_1}$ is irrational. Then there is no periodic solution of (2.5.11) on the energy level zero. The only periodic solution on a positive energy level is w_+.*

PROOF. Every solution of (2.5.11) can be written as

$$x(t) = A\cos(\lambda_1 t + \varphi_1) + B\cos(\lambda_2 t + \varphi_2), \tag{2.5.12}$$

where $A, B, \varphi_1, \varphi_2 \in \mathbb{R}$. The ratio of the frequencies $\frac{\lambda_2}{\lambda_1}$ is irrational. Thus if x is periodic then either $A = 0$ or $B = 0$. Plugging (2.5.12) into the energy equation proves the lemma. □

LEMMA 2.5.10. *Let $\alpha > \sqrt{8}$ be such that $\frac{\lambda_2}{\lambda_1}$ is rational i.e. there are $p', q' \in \mathbb{N}$ relatively prime and $\frac{\lambda_2}{\lambda_1} = \frac{q'}{p'}$. Assume that $E > 0$ and $w_+ = \sqrt{\frac{E}{E_0}}\cos(\lambda_1 t)$ where $E_0 = \frac{\lambda_1^4}{2} - 1$. Then every solution w of (2.5.11), with $\mathbb{E}[x] = E$, which is not equal to w_+ satisfies that its extrema sequence $\mathbf{v}$ is $2p'$-periodic and intersects $\mathbf{v}_+$ exactly $2q'$ times per period.*

PROOF. Without loss of generality it is enough to prove the statement for the solutions w which attain a minimum for $t = 0$. Since $\frac{\lambda_2}{\lambda_1}$ is rational, it follows from (2.5.12) that all solutions on the positive energy level E are periodic with the period $\frac{2\pi}{\lambda_1}p'$.

First we will show that the number of extremal points per period $\frac{2\pi}{\lambda_1}p'$ is $2p'$ for all solutions of (2.5.11). Let w_1 and w_2 be two different solutions. Then we can interpolate between them. Let $y(s,t)$ be a solution of (2.5.11) for every fixed $s \in [0,1]$ with initial conditions

$$\begin{aligned} y(s,0) &= sw_1(0) + (1-s)w_2(0), \\ y'(s,0) &= 0, \\ y''(s,0) &= \sqrt{2(E + (sw_1(0) + (1-s)w_2(0))^2)}, \\ y'''(s,0) &= sw_1'''(0) + (1-s)w_2'''(0). \end{aligned}$$

For every fixed $s \in [0,1]$ it holds that $\mathbb{E}_0[y(s,t)] = E$. The fact that the energy level $E > 0$ is regular implies that $y(s,t)$ is a concatenation of regular monotone laps (degenerate monotone lap cannot occur) for every fixed s. If two extremal points would collapse or a new one would be created along the path $y(s,t)$ then the degenerate monotone lap occurs which is impossible. Therefore the number of extremal points per period $\frac{2\pi}{\lambda_1}p'$ is constant along the path $y(s,t)$. This implies that w_1 and w_2 have the same number of extremal points per period $\frac{2\pi}{\lambda_1}p'$. By counting the number of extremal points of the solution w_+ on the interval $[0, \frac{2\pi}{\lambda_1}p')$ one gets that this number is $2p'$. Hence the extremal sequence of any solution is $2p'$-periodic.

It follows from the proof of Lemma 2.5.3 that the rotation number $\tau(\mathbf{v}_+) = \frac{\lambda_2}{\lambda_1} = \frac{q'}{p'}$. This combined with the fact that the extrema sequence $\mathbf{v}$ of an arbitrary solution is $2p'$ periodic implies that $I(\mathbf{v}, \mathbf{v}_+) = 2q'$ for all solutions which initial data are sufficiently close to the initial data of the solution w but $w \not\equiv w_+$.

Again by interpolating between the solutions we will prove that $I(\mathbf{v}, \mathbf{v}_+) = 2q'$ for an arbitrary solution not equal to w_+. Let w_1 and w_2 be two solutions such that $w_1, w_2 \not\equiv w_+$ and $y(s,t)$ be the connecting path between them defined as above. It could happen that $y(s,t) = w_+$ for some s_0 but by small perturbation of the path of initial conditions, say varying $y'''(s,0)$ slightly, we can avoid it. Therefore we suppose that $y(s,t)$ is not equal to w_+ for any s. Let $\mathbf{y}(s)$ be an extrema sequence of $y(s,t)$. Now we show that $I(\mathbf{v}, \mathbf{y}(s))$ is constant by contradiction. If it would not be constant then there exists $s_0 \in [0,1]$ for which $\mathbf{v}_+$ and $\mathbf{y}(s)$ have a non-transversal intersection. However according to Lemma 2.2.8 two stationary points $\mathbf{v}_+$ and $\mathbf{y}(s)$ of the flow Ψ^t generated by Equation (2.5.11) can not have a non-transversal intersection. Hence we proved that $I(\mathbf{v}, \mathbf{v}_+) = 2q'$ for an arbitrary solution w not equal to w_+. □

The final lemma completes the proof of Theorem 2.1.2. Let us remind that we are dealing with the parameter range $\alpha \in [\sqrt{8}, \alpha_{p,q})$ where $\frac{q}{p} < \frac{\lambda_2}{\lambda_1}$.

LEMMA 2.5.11. *The sequence $\{u_n\}_{n=0}^{\infty}$ does not converge to the constant solution.*

PROOF. We will prove this by contradiction. Suppose that $u_n \to 1$. Then $w^n \to w$, where w is a solution of the linear equation (2.5.11) and $\mathbb{E}[w] \geq 0$. Moreover, $\|w\|_{\sup} = \lim_{n\to\infty} \|w^n\|_{\sup} = 1$. Let T_n be the period of w^n.

First we assume that $\frac{\lambda_2}{\lambda_1}$ is irrational. Let us start with the case $\mathbb{E}[w] = 0$. It follows from Lemma 2.5.9 that w is not periodic and as we showed in the proof of Lemma 2.5.8 it holds that $T_n \to \infty$. Therefore solution w has at most $2p$ extrema on $\mathbb{R}$ and according to (2.5.12) it holds $w \equiv 0$ which is a contradiction with $\|w\|_{\sup} = 1$.

If $\mathbb{E}[w] > 0$ then w has to be periodic, otherwise we would get the same contradiction as above. It follows from Lemma 2.5.9 that the only periodic solution on this energy level is w_+. Thus $w^n \to w_+$ and $\left\|w^n - w_+^n\right\|_{\sup} \to 0$. The fact that $\tau(\mathbf{v}_+^n) \to \tau(\mathbf{v}_+) = \frac{\lambda_1}{\lambda_2} > \frac{q}{p}$ contradicts the assumption that $\mathbf{v}^n$ is $2p$ periodic and $I(\mathbf{v}^n, \mathbf{v}_+^n) = 2q$.

Now we deal with rational $\frac{\lambda_2}{\lambda_1}$. If $\mathbb{E}[w] = 0$ then $w^+ \equiv 0$ and w^n can not converge to w^+ because $\|w\|_{\sup} = 1$. Hence by repeating the ideas in the proof of Lemma 2.5.8 one gets that $\mathbf{v}$ is $2p$ periodic and intersects zero $2q$ times per period. We will obtain a contradiction by showing that $\mathbf{v}$ is $2p'$ periodic and it intersects zero $2q'$ times per period, where $p', q' \in \mathbb{N}$ such that $\frac{q'}{p'} = \frac{\lambda_2}{\lambda_1} > \frac{q}{p}$. To prove the previous statement about the extrema sequence $\mathbf{v}$ we employ solutions w_L^n of the linear equation (2.5.11) with the same initial conditions as solutions w^n. These solutions converge to w and the energy $\mathbb{E}[w_L^n] > 0$ for all $n \in \mathbb{N}$. It follows from Lemma 2.5.10 that $\mathbf{v}_L^n$ is $2p'$ periodic and $I(\mathbf{v}_L^n, \mathbf{v}_+^n) = 2q'$. Hence as before the limit process for w_L^n implies that $\mathbf{v}$ is $2p'$ periodic and intersects zero $2q'$ times per period. This contradicts the inequality $\frac{q'}{p'} > \frac{q}{p}$.

Finally, if $\mathbb{E}[w] > 0$ then solutions w^n can not converge to w_+, otherwise we would get the same contradiction as in the irrational case. So Lemma 2.5.10 implies that $\mathbf{v}$ is $2p'$ periodic and $I(\mathbf{v}, \mathbf{v}^+) = 2q'$. On the other hand $w^n \to w$ and by repeating the ideas in the proof of Lemma 2.5.8 one can get that $\mathbf{v}$ is $2p$ periodic and $I(\mathbf{v}, \mathbf{v}^+) = 2q$, which is a contradiction. □

CHAPTER 3

Orderings of bifurcation points in fourth order conservative systems

3.1. Introduction

In Section 1.2 we introduced a classification of periodic solutions of the equation

$$u'''' + \alpha u'' - u + u^3 = 0, \quad \alpha \in \mathbb{R}. \tag{3.1.1}$$

In this chapter we study the ordering of these classes imposed by forcing relation.

DEFINITION 3.1.1. The class of solutions $[\sigma^n, p_n]$ precedes $[\sigma^m, p_m]$ if and only if the existence of a solution of the class $[\sigma^n, p_n]$ forces the existence of solution of the class $[\sigma^m, p_m]$, we write $[\sigma^n, p_n] \prec [\sigma^m, p_m]$.

Let us mention the implication for the bifurcation diagram of Equation (3.1.1). If Γ_n and Γ_m are continuous curves corresponding to solutions of the class $[\sigma^n, p_n]$ and $[\sigma^m, p_m]$ then Γ_m has to exist at least as long as Γ_n does. In this chapter we prove the existence of solutions in different classes without showing that they lie on continuous curves, although it seems likely to be the case.

In Section 1.4 we presented basic ideas of forcing. In this chapter, we show that the class $[\sigma_1^2\sigma_2^4, 4]$ precedes a plethora of different classes.

THEOREM 3.1.2. *Let $\alpha \in [0, \sqrt{8})$ and $p \in \mathbb{N}$ such that $p \geq 2$. Then*

$$\left[\sigma_1^2\sigma_2^4, 4\right] \prec \left[\sigma_1^2\sigma_2^{2p}, 2p\right].$$

If $q \in \mathbb{N}$ such that $3 < q < p$ then

$$\left[\sigma_1^2\sigma_2^4, 4\right] \prec \left[\sigma_1^2\sigma_2^{2q}, 2p\right].$$

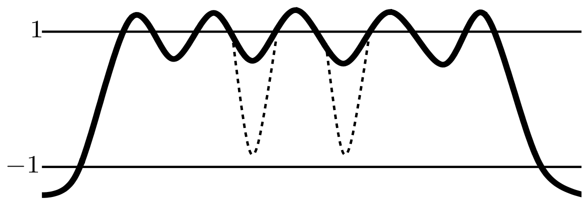

Figure 3.1: Sketch of four different periodic solution $u \in \left[\sigma_1^2\sigma_2^{10}, 10\right]$. The solid black curve represents one solution. The second (third) solution is obtained by replacing the second (third) dip with the dashed curve. The fourth solution is the one for which both dips are replaced by the dashed curves.

Moreover, there are at least $2^{(p-3)}$ geometrically distinct solutions of the class $\left[\sigma_1^2\sigma_2^{2p}, 2p\right]$ and at least $2^{(q-4)}$ of the class $\left[\sigma_1^2\sigma_2^{2q}, 2p\right]$.

Figure 3.1 shows four different solutions of the class $\left[\sigma_1^2\sigma_2^{10}, 10\right]$ and Figure 3.2 shows a solution of the class $\left[\sigma_1^2\sigma_2^{8}, 10\right]$. For Equation (3.1.1) the existence of a solution $\widetilde{u} \in \left[\sigma_1^2\sigma_2^{4}, 4\right]$ is proved in [22] for $\alpha \in [0, 2]$. See Section 1.4 for more details about the solution $\widetilde{u}$. By concatenation of the building blocks and employing the solution $\widetilde{u}$, we obtain the following result.

THEOREM 3.1.3. *Let $\alpha \in [0, 2]$ and $p, q_1, \dots q_n \in \mathbb{N}$ such that $q_i > 1$ and $\sum_{i=1}^n q_i \leq 2p$. Suppose that at least one $q_i > 3$ if $\sum_{i=1}^n q_i < 2p$. Then there exists a solution*

$$u \in \left[\sigma_1^2\sigma_2^{2q_1} \dots \sigma_1^2\sigma_2^{2q_n}, 2p\right].$$

The lower estimate on the number of geometrically different solutions in the class $\left[\sigma_1^2\sigma_2^{2q_1} \dots \sigma_1^2\sigma_2^{2q_n}, 2p\right]$ is a product of the number of solutions in the different blocks given by the previous theorem.

The part of Theorem 3.1.2 which deals with solutions in the class $\left[\sigma_1^2\sigma_2^{2p}, 2p\right]$ is proved in Section 3.2 while the rest of the proof is carried out in Section 3.4.

As we explained in Section 2.2, the problem of finding periodic solutions of Equation (3.1.1) can be reduced to finding fixed points of a vector field generated by a parabolic recurrence relation $\mathcal{R}$, see Theorem 2.2.3. To find fixed points of the vector field generated by $\mathcal{R}$ we employ Conley index theory for braid diagrams which is surveyed in Section 1.5 and Section 2.2

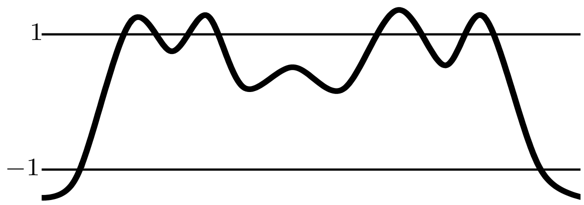

Figure 3.2: Sketch of a periodic solution $u \in \left[\sigma_1^2\sigma_2^8, 10\right]$.

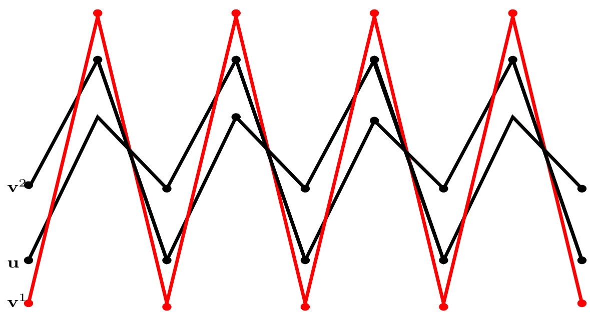

Figure 3.3: Sketch of the strands $\mathbf{v}^1, \mathbf{v}^2$ and $\mathbf{v}^3$.

3.2. Forcing of solutions of the class $[\sigma_1^2\sigma_2^{2p}, 2p]$

In this section we prove the existence of geometrically different solutions of (3.1.1) of the class $[\sigma_1^2\sigma_2^{2p}, 2p]$ under the assumption that there exists a solution $\widetilde{u} \in [\sigma_1^2\sigma_2^4, 4]$. In the rest of this section we consider the flow Ψ^t generated by $\mathcal{R}$ associated to (3.1.1). We start with a simple example, for more examples see Section 1.4.

Example 3.2.1. Let $\widetilde{u} \in [\sigma_1^2\sigma_2^4, 4]$ with the sequence of extrema $\{\widetilde{u}_i\}$. For an arbitrary $p \in \mathbb{N}$ let $\mathbf{v}^1 \cup \mathbf{v}^2 \in \mathcal{E}^2_{2p}$, be defined by $v_i^1 = \widetilde{u}_i$ and $v_2^i = \widetilde{u}_{(i+2) \bmod 2p}$. The system generated by (3.1.1) is dissipative and we can choose $I = [u_0^*, u_1^*]$ where the constants u_0^*, u_1^* are such that $u_0^* < v_i^1, v_i^2 < u_1^*$ and $\mathcal{R}_i(u_{i-1}, u_0^*, u_{i+1}) > 0$ while $\mathcal{R}_i(u_{i-1}, u_1^*, u_{i+1}) < 0$ for $u_{i\pm1} \in I$. Define

$$\Omega_i = \begin{cases} \{(u_{i-1}, u_i, u_{i+1}) \in I^3 | \quad u_0^* \le u_{i\pm1} + \delta \le u_i \le u_1^*\}, & i \text{ odd}, \\ \{(u_{i-1}, u_i, u_{i+1}) \in I^3 | \quad u_0^* \le u_i \le u_{i\pm1} + \delta \le u_1^*\}, & i \text{ even}, \end{cases}$$

where $\delta > 0$ is chosen in such a way that the flow points into the set at the boundary where one of the following relations is satisfied: $u_i =$

$u_{i\pm1} \pm \delta$. Denote by Ω^{2p} the set of $2p$ periodic sequences $\{u_i\}$ for which $(u_{i-1}, u_i, u_{i+1}) \in \Omega^i$. Furthermore define the set

$$C := \{\mathbf{u} \in \Omega^{2p} : \quad \iota(\mathbf{u}, \mathbf{v}^1) = \iota(\mathbf{u}, \mathbf{v}^2) = 2p\},$$

where $\iota(\mathbf{u}, \mathbf{v}^1)$ is the number of intersections of the sequence $\{\mathbf{u}\}_{i=0}^{2p}$ and $\{\mathbf{v}^1\}_{i=0}^{2p}$. Since $\iota(\mathbf{v}^1, \mathbf{v}^2) = p < 2p$ and the system is dissipative, the vector field $\mathcal{R}$ is transverse to ∂C. Moreover, the set C is contractible, compact, and $\mathcal{R}$ is pointing outward at the boundary of ∂C. The set C is therefore negatively invariant for the induced parabolic flow Ψ^t and there exists a global minimum $\mathbf{v}^3$ of W_{2p} which is a fixed point of Ψ^t in the interior of C. Figure 3.3 depicts the strands $\mathbf{v}^1, \mathbf{v}^2$ and $\mathbf{v}^3$.

The Conley index of a relative braid class can be used to detect the existence of a fixed point of Ψ^t within the braid class. However this method works only for a proper bounded braid class $[\mathbf{u} \,\mathrm{rel}\, \mathbf{w}]$ whose skeleton $\mathbf{w}$ is a fixed point of the flow Ψ^t (zero point of $\mathcal{R}$), see Section 2.2 for more details. In our case, there are no strands which are fixed points of the flow Ψ^t and can be used to control the intersection of the free strand $\mathbf{u}$ with $u_\pm = \pm 1$. To overcome this problem we use the sequence $\{u_i^\varepsilon\}_{i=0}^{\infty}$ given by the following lemma to define an isolating neighborhood M as a subset of some proper and bounded braid class $[\mathbf{u} \,\mathrm{rel}\, \mathbf{w}]_\mathcal{E}$. Finally we will show that the homological Conley index of the isolating neighborhood M is the same as $H_*(h(\mathbf{u} \,\mathrm{rel}\, \mathbf{w}))$.

LEMMA 3.2.2 ([22]). *Let* $-\sqrt{8} < \alpha < \sqrt{8}$. *For any* $\varepsilon < 0$ *there exists a sequence* $\{u_i^\varepsilon\}_{i=1}^{\infty}$,

$$0 < (-1)^i (u_i^\varepsilon - 1) < \varepsilon,$$

which satisfies

$$\mathcal{R}_i(u_{i-1}^\varepsilon, u_i^\varepsilon, u_{i+1}^\varepsilon) = 0 \quad \textit{for } i \geq 2.$$

Notice that the previous lemma does not claim that $\mathcal{R}_1(u_0^\varepsilon, u_1^\varepsilon, u_2^\varepsilon) = 0$; u_0^ε is not even defined. The symmetry of Equation (3.1.1) enforces an analogous result near $u_- = -1$. To be explicit, $\overline{u}_i^\varepsilon = -u_i^\varepsilon$. For technical reasons we define

$$\widehat{u}_i^\varepsilon = \overline{u}_{i-2 \bmod 2p}^\varepsilon.$$

In order to find geometrically different solutions we define a braid class $[\mathbf{u}^I \,\mathrm{rel}\, \mathbf{w}]_\mathcal{E} \in \mathcal{E}_{2p}^1 \,\mathrm{rel}\, \mathbf{w}$ for every

$$I = \{j_1, j_2, \ldots, j_n\} \in \mathbb{N}^n, \tag{3.2.1}$$

satisfying

$$1 < j_1 < j_2 < \ldots < j_n < p - 1. \tag{3.2.2}$$

We start with identifying the skeleton strands

$$\mathbf{w} = \mathbf{v}^1 \cup \mathbf{v}^2 \cup \mathbf{v}^3 \cup \mathbf{v}^4 \cup \mathbf{v}^5 \cup \mathbf{v}^6. \tag{3.2.3}$$

The strand $\mathbf{v}^1$ corresponds to a sequence of extrema $\{\widetilde{u}_i\}$ of the solution $\widetilde{u} \in [\sigma_1^2\sigma_2^4, 4]$, i.e. $v_i^1 = \widetilde{u}_i$. The strand $\mathbf{v}^2$ is a shift of the strand $\mathbf{v}^1$ given by $v_i^2 = v^1_{(i \bmod 4)+2}$. We can suppose that $v_1^1 \leq v_2^1$ otherwise we would interchange $\mathbf{v}_1$ with $\mathbf{v}_2$. The strand $\mathbf{v}^3$ corresponds to the sequence of extrema of the solution obtained in Example 3.2.1. It holds that $v_{2i}^3 < \min\{v_{2i}^1, v_{2i}^2\}$ and $v_{2i+1}^3 > \max\{v_{2i+1}^1, v_{2i+1}^2\}$ for all i. The strand $\mathbf{v}^4$ is defined by the sequence of extrema of the solution constructed in Example 1.4.4 and $-1 < v_{2i}^4 < v_{2i+1}^4 < \widetilde{u}_2$. All the strands defined by now are fixed points of Ψ^t. Finally, we define the strands $\mathbf{v}^5$ and $\mathbf{v}^6$ by $v_i^5 = 1 + (-1)^{i+1}\varepsilon_0$ and $v_i^6 = -1 + (-1)^{i+1}\varepsilon_0$, where

$$\varepsilon_0 = \frac{1}{2}\min\{|u_i^\varepsilon - 1|, i = 0, \dots 2p-1\}.$$

and

$$\varepsilon = \frac{1}{2}\min\{-1-\widetilde{u}_0, \widetilde{u}_1 - 1, 1 - \widetilde{u}_2, \widetilde{u}_3 - 1, 1 + v_0^4, \dots, 1 + v_{2p-1}^4\},$$

The set I defines the weaving of the free strand around the skeletal strands $\mathbf{v}^1$ and $\mathbf{v}^2$.

DEFINITION 3.2.3. The braid class $[\mathbf{u}^I \text{ rel } \mathbf{w}]_{\mathcal{E}} \subset \mathcal{E}_{2p}^1$ rel $\mathbf{w}$ is defined by its representant $\mathbf{u}^I$ satisfying:

(1) $u_0^I \in (v_0^1, v_0^6)$,
(2) $u_{2i+1}^I \in (v_1^5, v_1^1)$ for all i,
(3) $u_{2i}^I \in \begin{cases} (v_2^4, v_2^1) & : \text{ if } i \in I, \\ (v_2^1, v_2^5) & : \text{ if } i \neq 0 \text{ and } i \notin I. \end{cases}$

Figure 3.4 shows the braid class $[\mathbf{u}^I \text{ rel } \mathbf{w}]_{\mathcal{E}} \subset \mathcal{E}_{10}^1$ rel $\mathbf{w}$ for $I = \{2, 3\}$. To keep the figure synoptical we do not display the skeleton strand $\mathbf{v}^3$ which crosses all the other skeletal strands in between each two anchor points and makes the braid class bounded. The skeleton $\mathbf{w}$ is not a fixed point of Ψ^t. However any braid class $[\mathbf{u}^I \text{ rel } \mathbf{w}]_{\mathcal{E}}$ is proper and bounded.

Suppose that Φ^t is an arbitrary flow generated by parabolic recurrence relation of up-down type such that $\Phi^t(\mathbf{w}) = \mathbf{w}$. It follows from Proposition 2.2.11 that the set

$$N_{I,\epsilon} := \{\mathbf{u} \in \text{cl}([\mathbf{u}^I \text{ rel } \mathbf{w}]_{\mathcal{E}}) \ : \ (-1)^i(u_{i+1} - u_i) \geq \epsilon \quad \forall i\}$$

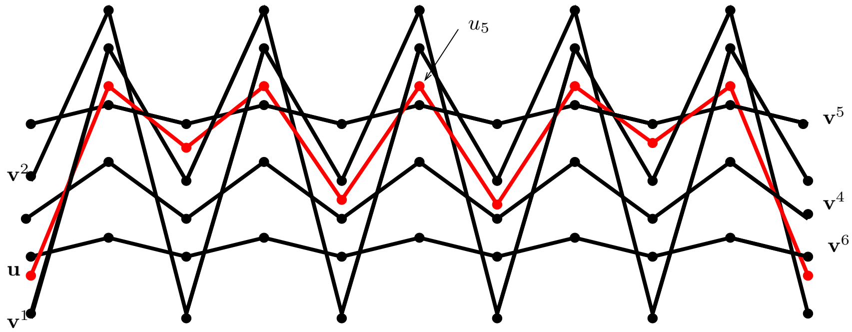

Figure 3.4: A representative of the braid class $[\mathbf{u} \text{ rel } \mathbf{w}]_{\mathcal{E}} \subset \mathcal{E}^1_{10}$ rel $\mathbf{w}$ for $I = \{2, 3\}$. To keep the figure synoptical we do not display the skeletal strand $\mathbf{v}^3$. The strand $\mathbf{v}^3$ makes the the braid class $[\mathbf{u} \text{ rel } \mathbf{w}]_{\mathcal{E}}$ bounded. There would be no upper bound for u_5 without the strand $\mathbf{v}^3$.

is an isolating neighborhood of the flow Φ^t for $\epsilon > 0$ sufficiently small. Moreover

$$h(N_{I,\epsilon}) = h(\mathbf{u}^I \text{ rel } \mathbf{w}).$$

In the rest of this section we denote by $N^-_{I,\epsilon}$ the subset of $\partial N_{I,\epsilon}$ where the flow Φ^t points out of the set $N_{I,\epsilon}$. The set $N^-_{I,\epsilon}$ is the same for every flow Φ^t which fixes the skeleton $\mathbf{w}$. The non-triviality of $H_*(N_{I,\epsilon}, N^-_{I,\epsilon}) = H_*(h(\mathbf{u}^I \text{ rel } \mathbf{w}))$ is given by the following theorem, whose proof we delay until Section 3.3.

THEOREM 3.2.4. *The homology of $h([\mathbf{u}^I \text{ rel } \mathbf{w}])$ is given by:*

$$H_k(h(\mathbf{u}^I \text{ rel } \mathbf{w})) = \begin{cases} \mathbb{Z}, & \text{if } k = 2p - \#I, \\ 0, & \text{otherwise,} \end{cases} \tag{3.2.4}$$

where $\#I$ is the number of elements in I.

Finally we define the set $M_{I,\epsilon}$ which will turn out to be an isolating neighborhood for the flow Ψ^t and $\epsilon > 0$ sufficiently small.

DEFINITION 3.2.5. Let $M_{I,\epsilon}$ be a subset of $N_{I,\epsilon}$ such that $\mathbf{u} \in M_{I,\epsilon}$ if and only if

(1) $u_0 < \widehat{u}^{\varepsilon}_0$,
(2) $(-1)^i u_i < (-1)^i u^{\varepsilon}_i$ for all i.

We denote $M^-_{I,\epsilon}$ the subset of $\partial M_{I,\epsilon}$ where the flow Ψ^t points out of the set $M_{I,\epsilon}$.

LEMMA 3.2.6. *For sufficiently small $\epsilon > 0$ the set $M_{I,\epsilon}$ is an isolating neighborhood for the flow Ψ^t. Moreover,*

$$H_*(M_{I,\epsilon}, M^-_{I,\epsilon}) = H_*(h(\mathbf{u}^I \text{ rel } \mathbf{w})).$$

PROOF. We start by studying the flow Ψ^t at the point $\mathbf{u} \in \partial M_{I,\epsilon}$ for which $u_0 = \widehat{u}_0^\varepsilon$. The monotonicity of $\mathcal{R}$ combined with Lemma 3.2.2 implies that

$$\mathcal{R}_0(u_{2p-1}, \widehat{u}_0^\varepsilon, u_1) > \mathcal{R}_0(\widehat{u}_{2p-1}^\varepsilon, \widehat{u}_0^\varepsilon, \widehat{u}_1^\varepsilon) = 0,$$

and the flow Ψ^t points out of the set M_I. Analogously

$$(-1)^i\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) < (-1)^i\mathcal{R}_i(u_{i-1}^\varepsilon, u_i^\varepsilon, u_{i+1}^\varepsilon) = 0,$$

on the codimension 1 boundaries where only one $u_i = u_i^\varepsilon$. The flow points out of $M_{I,\epsilon}$ at these points. As in the proof of Lemma 40 in [9] the flow is transversal at the rest of the boundary $\partial M_{I,\epsilon}$ for $\epsilon > 0$ sufficiently small.

To relate $H_*(M_{I,\epsilon}, M^-_{I,\epsilon})$ with $H_*(h(\mathbf{u}^I \text{ rel } \mathbf{w}))$ we will show that $M_{I,\epsilon}$ ($M^-_{I,\epsilon}$) is homotopic to $N_{I,\epsilon}$ ($N^-_{I,\epsilon}$). Then

$$H_*(M_{I,\epsilon}, M^-_{I,\epsilon}) = H_*(N_{I,\epsilon}, N^-_{I,\epsilon}) = H_*(h(\mathbf{u}^I \text{ rel } \mathbf{w})).$$

Define $g = (g_0, \ldots, g_{2p-1}) : N_{I,\epsilon} \times [0,1] \to N_{I,\epsilon}$ as follows

$$g_0(\mathbf{u}, t) = \begin{cases} u_0, & \text{if } u_0 < \widehat{u}_0^\varepsilon, \\ (1-t)u_0 + t\widehat{u}_0^\varepsilon, & \text{otherwise} , \end{cases}$$

and

$$g_i(\mathbf{u}, t) = \begin{cases} u_i, & \text{if } (-1)^i u_i < (-1)^i u_i^\varepsilon, \\ (1-t)u_i + tu_i^\varepsilon, & \text{otherwise} , \end{cases}$$

for $i > 0$, where $\mathbf{u} = (u_0, \ldots, u_{2p-1})$. It is straightforward to check that g is a homotopy between $N_{I,\epsilon}$ and $M_{I,\epsilon}$.

Now we concentrate on the sets $N^-_{I,\epsilon}$ and $M^-_{I,\epsilon}$. If $\mathbf{u} \in N_{I,\epsilon}$ and $u_i = v_0^6$ or $u_i = v_i^5$ for some i then $\mathbf{u} \in N^-_{I,\epsilon}$. On the other hand for $\mathbf{u} \in M_{I,\epsilon}$ such that $u_i = \widehat{u}_0^\varepsilon$ or $u_i = u_i^\varepsilon$ for some i it holds that $\mathbf{u} \in N^-_{I,\epsilon}$. Hence $g|_{N^-_{I,\epsilon}}$ is a homotopy between $N^-_{I,\epsilon}$ and $M^-_{I,\epsilon}$. □

The following lemma summarizes the results obtained in this section.

LEMMA 3.2.7. *Let $p \in \mathbb{N}$ and $\alpha \in [0, \sqrt{8})$. Suppose that there exists a solution $\widetilde{u}$ of Equation (3.1.1) of the class $[\sigma_1^2\sigma_2^4, 4]$. Then for every I defined by (3.2.1) and (3.2.2) there exists a solution $u \in [\sigma_1^2\sigma_2^{2p}, 2p]$ whose sequence of extrema $\mathbf{u} = \{u_i\}$ satisfies $\mathbf{u} \in [\mathbf{u}^I \text{ rel } \mathbf{w}]_{\mathcal{E}}$.*

PROOF. It follows from Lemma 3.2.6 that

$$H_*(M_{I,\epsilon}, M_{I,\epsilon}^-) = H_*(h(\mathbf{u}^I \text{ rel } \mathbf{w})).$$

The non-triviality of $H_*(h(\mathbf{u}^I \text{ rel } \mathbf{w}))$ is given by Theorem 3.2.4. Therefore there exists a fixed point $\mathbf{u} \in M_{I,\epsilon} \subset [\mathbf{u}^I \text{ rel } \mathbf{w}]_{\mathcal{E}}$ of the flow Ψ^t. Hence there is a solution u of (3.1.1) with the sequence of extrema $\mathbf{u}$ and $u \in [\sigma_1^2\sigma_2^{2p}, 2p]$. □

3.3. Computation of the homological Conley index

In this section we prove Theorem 3.2.4 which states that

$$H_k(h(\mathbf{u}^I \text{ rel } \mathbf{w})) = \begin{cases} \mathbb{Z}, & \text{if } k = 2p - \#I, \\ 0, & \text{otherwise.} \end{cases}$$

In the rest of this section we omit I from the notation of the braid class $[\mathbf{u}^I \text{ rel } \mathbf{w}]$ and write $[\mathbf{u} \text{ rel } \mathbf{w}]$, although the braid class is always defined for some set I.

We start by simplifying the skeleton $\mathbf{w}$ without changing the index. According to Theorem 2.2.11, the skeleton strands $\mathbf{v}^5$ and $\mathbf{v}^6$ can be deformed to the constant strands $+\mathbf{1}$ and $-\mathbf{1}$ without changing the index $h(\mathbf{u} \text{ rel } \mathbf{w})$. Due to the same theorem we can assume that $v_1^1 = v_1^2 = v_3^1 = v_3^2$. Finally, omitting the skeletal strand $\mathbf{v}^4$ does not change the index either. Compare Figure 3.4, which shows the braid class $[\mathbf{u} \text{ rel } \mathbf{w}]_{\mathcal{E}}$ for $I = \{2, 3\}$, and Figure 3.6 depicting the class $[\mathbf{u} \text{ rel } \mathbf{w}]$ with the simplified skeleton. From now on, $[\mathbf{u} \text{ rel } \mathbf{w}]$ denotes the braid class with the simplified skeleton. First we demonstrate the basic ideas on simple examples.

EXAMPLE 3.3.1. Let $p = 5$ and $I = \{2\}$. The representant of the class $[\mathbf{u} \text{ rel } \mathbf{w}] \subset \mathcal{D}_{10}^1 \text{ rel } \mathbf{w}$ is depicted in Figure 3.5. If $\mathbf{u} \in N_I := \text{cl}[\mathbf{u} \text{ rel } \mathbf{w}]$ then

$$u_i \in \begin{cases} [v_0^1, -1], & \text{if } i = 0, \\ [-1, v_2^1], & \text{if } i = 4, \\ [v_2^1, 1], & \text{if } i \text{ is even and } i \notin \{0, 4\}, \\ [1, v_1^1], & \text{if } i \text{ is odd.} \end{cases} \tag{3.3.1}$$

The set $N_I \cong [0,1]^{2p}$ and the intersection number of the free strand with the skeletal strands increases if $u_4 = -1$ or $u_4 = \widetilde{u}_2$. In the case

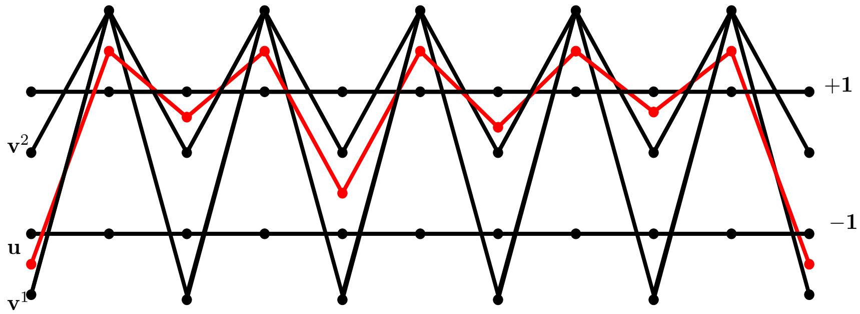

Figure 3.5: The braid class $[\mathbf{u} \text{ rel } \mathbf{w}] \subset \mathcal{D}^1_{10}$ rel $\mathbf{w}$ for $I = \{2\}$. The skeletal strand $\mathbf{v}^3$ is not displayed.

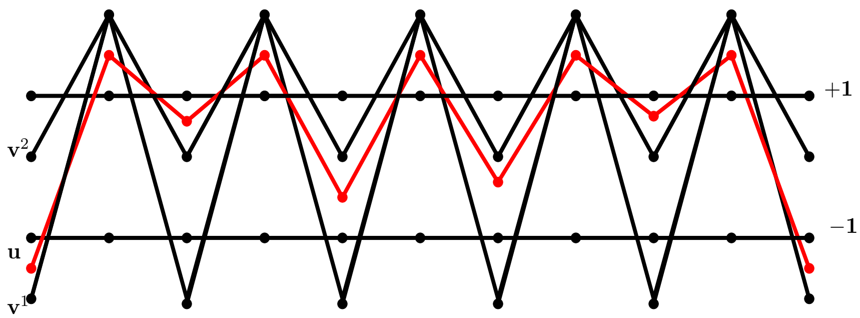

Figure 3.6: The braid class $[\mathbf{u} \text{ rel } \mathbf{w}] \subset \mathcal{D}^1_{10}$ rel $\mathbf{w}$ for $I = \{2, 3\}$ with the simplified skeleton. To keep the figure clear we do not display the strand $\mathbf{v}^3$ which makes the braid class bounded.

that any other anchor point different from u_4 reaches the boundary the intersection number decreases. Hence N_I^- is a boundary of N_I without the faces $\{\mathbf{u} \in N_I : u_4 = -1\}$ and $\{\mathbf{u} \in N_I : u_4 = \widetilde{u}_2\}$. By computing the relative $H_*(N_I, N_I^-)$ we obtain that the index

$$H_k(h(\mathbf{u} \text{ rel } \mathbf{w})) = \begin{cases} \mathbb{Z}, & \text{if } k = 2p - 1, \\ 0, & \text{otherwise.} \end{cases}$$

The same is true for $I = \{3\}$.

EXAMPLE 3.3.2. Let $[\mathbf{u} \text{ rel } \mathbf{w}] \subset \mathcal{D}^1_{10}$ rel $\mathbf{w}$ where $I = \{2, 3\}$, see Figure 3.6. Again the anchor points u_i with $i \notin \{4, 5, 6\}$ are confined to the intervals defined by (3.3.1) and if some of them attain the boundary then the corresponding $\mathbf{u} \in M_I^-$. The configuration space of the anchor points (u_4, u_5, u_6) is given by a union of five 'cubes' shown in

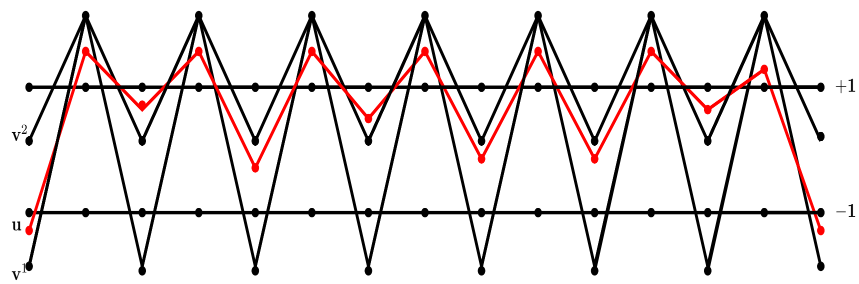

Figure 3.7: The braid class $[\mathbf{u}\,\mathrm{rel}\,\mathbf{w}] \in \mathcal{D}^1_{14}\,\mathrm{rel}\,\mathbf{w}$ for $I = \{2,4,5\}$. The skeletal strand $\mathbf{v}^3$ is not displayed.

Figure 3.8. If (u_4, u_5, u_6) lays in one of the faces depicted in gray then $\mathbf{u} \in N_I^-$. Note that the configuration space of the anchor points u_i with $i \notin \{4,5,6\}$ is independent of the position of the anchor points u_4, u_5, u_6 and vice versa. We will show later on that

$$H_k(h(\mathbf{u}\,\mathrm{rel}\,\mathbf{w})) = \begin{cases} \mathbb{Z}, & \text{if } k = 2p-2, \\ 0, & \text{otherwise.} \end{cases}$$

The following example demonstrates the splitting of anchor points into chunks with the following property. The configuration space of the anchor points in one chunk is the same for any position of the anchor points in the different chunks.

EXAMPLE 3.3.3. Let $[\mathbf{u}\,\mathrm{rel}\,\mathbf{w}] \subset \mathcal{D}^1_{14}\,\mathrm{rel}\,\mathbf{w}$, where $I = \{2,4,5\}$. Define $I_1 = \{4\}$ and $I_2 = \{8,9,10\}$. The configuration space of u_4 is $[-1, \widetilde{u}_2]$ for any position of the other anchor points. As we saw in the previous example the configuration space of u_i with $i \in I_2$ does not depend on the position of any anchor point u_j with j in a complement of I_2. Finally, for $i \in I_0 = \{0, \ldots 2p\} \setminus \overline{I}$, where $\overline{I} = I_1 \cup I_2$, it holds that the configuration space of u_i is always given by (3.3.1).

Decomposition of the sets N_I and N_I^-

First we formalize the splitting of anchor points introduced in Example 3.3.3. This splitting depends on the set $I = \{j_1, j_2, \ldots, j_n\}$. We remind the reader that if we work with the braid class $[\mathbf{u}^I\,\mathrm{rel}\,\mathbf{w}] \subset \mathcal{E}^1_{2p}$ then we assume that

$$1 < j_1 < j_2 < \ldots < j_n < p.$$

DEFINITION 3.3.4. For the set $I = \{j_1, j_2, \ldots, j_n\}$ we define

$$\overline{I} = \{i \in \mathbb{N} : (i = 2j_k) \text{ or } (i = 2j_k + 1 \text{ and } j_{k+1} = j_k + 1) \text{ for some } k\}$$

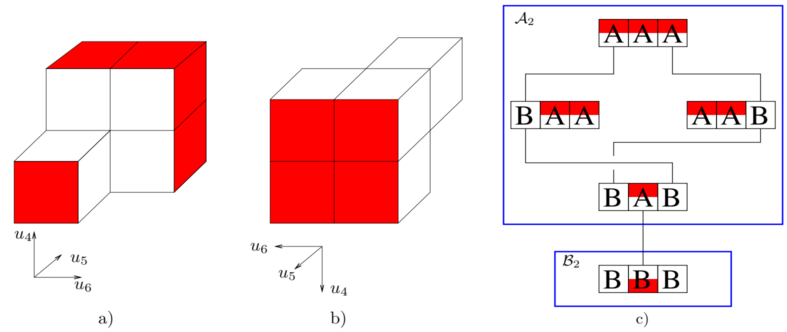

Figure 3.8: a) and b) The configuration space $U(2)$ of the anchor points (u_4, u_5, u_6) for $[\mathbf{u} \text{ rel } \mathbf{w}]$ with $I = \{2, 3\}$. The set $U^-(2)$ consists of the faces displayed in gray. c) schematic representation of $U(2)$. The shaded faces form $U^-(2)$. The cube BBB corresponds to the front cube at a). Division into $\mathcal{A}_1$ and $\mathcal{B}_1$ is displayed.

and

$$I_0 = \{0, 1, \dots, 2p-1\} \setminus \overline{I}.$$

Let $I_1, \dots, I_l$ where $I_k = \{i_1^k, i_2^k, \dots, i_{p_k}^k\}$, be the sets which decompose $\overline{I}$ into consecutive chunks i.e.

$$\overline{I} = I_1 \cup I_2 \cup \ldots \cup I_l,$$

and the elements i_j^k satisfy

$$i_{j+1}^k = i_j^k + 1,$$

$$i_{p_k}^k + 1 < i_1^{k+1}.$$

The following statement generalizes observations from the previous examples for arbitrary set I. If $\mathbf{u} \in N_I$ and $i \in I_0$ then

$$u_i \in \begin{cases} [v_0^1, -1], & \text{if } i = 0, \\ [v_2^1, 1], & \text{if } i \text{ is even and positive}, \\ [1, v_1^1], & \text{if } i \text{ is odd}. \end{cases} \tag{3.3.2}$$

If u_i attains the boundary of the interval given by (3.3.2) then $\mathbf{u} \in N_I^-$. For every set $I_m = \{i_1^m, \dots, i_{p_m}^m\}$, with $m > 0$, it holds that $i_1^m - 1, i_{p_m}^m + 1 \in I_0$ and the anchor points $u_{i_1^m - 1}, u_{i_{p_m}^m + 1}$ lay in the interval given by (3.3.2). The fact that the configuration space of the anchor point depends only on its immediate neighbors implies that the configuration space of the anchor point u_i with $i \in I_m$ depends only on the anchor points with $j \in I_m$. Therefore

$$N_I = U_0 \times U_1 \times \ldots \times U_l, \tag{3.3.3}$$

where

$$U_i = \text{im}(\pi^{I_i}(N_I)),$$

and $\pi^{I_i} : N_I \to \mathbb{R}^{\#I_i}$ is a projection on the coordinates with indices in I_i.

EXAMPLE 3.3.5. Let $[\mathbf{u} \text{ rel } \mathbf{w}]$ be as in Example 3.3.3. Then $N_I = U_0 \times U_1 \times U_2$. The set $U_1 = [-1, \widetilde{u}_2]$ and $U_2 = U(2)$ are depicted in Figure 3.8. It follows from (3.3.2) that U_0 is homotopic to a ten dimensional cube.

By employing the sets U_i we write

$$N_I^- = \bigcup_{j=0}^{l} U_0 \times U_1 \times \ldots \times U_{j-1} \times U_j^- \times U_{j+1} \times \ldots \times U_l, \tag{3.3.4}$$

where

$$U_j^- = \text{im}(\pi^{I_j}(N_{I_j}^-)),$$

and $\mathbf{u} \in N_{I_j}^-$ if and only if $\overline{\mathbf{u}} \in N_I^-$ for every $\overline{\mathbf{u}} \in N_I$ which satisfies $\overline{u}_i = u_i$ for $i \in I_j$. In other words $N_{I_j}^-$ is created by the points which are in N_I^- due to the position of the anchor points u_i with $i \in I_j$.

EXAMPLE 3.3.6. Let $I = \{2, 4, 5\}$. Then $N_I^- = U_0^- \times U_1 \times U_2 \cup U_0 \times U_1^- \times U_2 \cup U_0 \times U_1 \times U_2^-$. The set $U_1^- = \varnothing$, $U_2^- = U^-(2)$ depicted in Figure 3.8 and $U_0^- = \partial U_0$.

First, we compute the homology of the sets U_i and U_i^-. Then we derive the homology N_I and N_I^-. Finally using the exact sequence which relates $H_*(N_I)$ and $H_*(N_I^-)$ to $H_*(N_I, N_I^-)$, we show that

$$H_k(h(\mathbf{u} \text{ rel } \mathbf{w})) = \begin{cases} \mathbb{Z}, & \text{if } k = 2p - \#I, \\ 0, & \text{otherwise.} \end{cases}$$

Homology of U_0 and U_0^-

It follows from (3.3.2) that $U_0 \cong [0,1]^{\#I_0}$ and $U_0^- \cong \partial U_0$. Summarizing

$$H_k(U_0) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{3.3.5}$$

and

$$H_k(U_0^-) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, \#I_0 - 1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.6}$$

Homology of U_i and U_i^- for $i > 0$

The symmetry of the skeleton $\mathbf{w}$ implies that the sets U_i and U_i^- depend only on the number of elements in I_i but not on their values. If $\#I_i = 2m-1$ then $U_i \simeq U(m) \subset \mathbb{R}^{2m-1}$ where $U(m)$ is defined to be U_1 for the set $I = \{2, 3, \ldots, m+1\}$. Therefore it is enough to deal with the sets

$$U(m) = U_1 \subset \mathbb{R}^{2m-1} \quad \text{and} \quad U^-(m) = U_1^- \subset \mathbb{R}^{2m-1},$$

where $I = \{2, 3, \ldots, m+1\}$ and $m \in \mathbb{N}$. The set $U(m)$ is build up of the cubes $\mathbf{C} = C_0 \times C_1 \times \ldots \times C_{2m-2}$ where C_i is either A_i or B_i and

$$A_i = \begin{cases} [v_2^1, 1], & \text{if } i \text{ is even,} \\ [v_1^1, v_1^3], & \text{if } i \text{ is odd,} \end{cases} \tag{3.3.7}$$

$$B_i = \begin{cases} [-1, v_2^1], & \text{if } i \text{ is even,} \\ [1, v_1^1], & \text{if } i \text{ is odd.} \end{cases} \tag{3.3.8}$$

By scaling and shifting we can deform each A_i to the interval $[0, 1]$ and B_i to $[-1, 0]$.

REMARK 3.3.7. We omit the subscript of the sets A_i, B_i and the direct sum $\times$ in the notation. Instead of writing $B_0 \times A_1 \times B_2$ we write BAB. If we want to refer to a face of the cube we will replace the symbol A (B) by the value of the appropriate coordinate i.e. $B[1]B = \{(x_0, x_1, x_2) \in \mathbb{R}^3 : x_1 \in [-1, 0], x_2 = 1, x_3 \in [-1, 0]\}$.

According to Example 3.3.1 the set $U(1) = B$. Figure 3.8 shows the set $U(2)$ consisting of five cubes

$$U(2) = BBB \cup BAB \cup AAB \cup BAA \cup AAA.$$

We denote by $\mathbf{C}^-$ the union of the faces of the cube $\mathbf{C}$ for which the intersection number of the free strand $\mathbf{u}$ with the skeletal strands is smaller than within the braid class $[\mathbf{u} \text{ rel } \mathbf{w}]$. Then

$$U^-(m) = \bigcup \{\mathbf{C}^- : \mathbf{C} \in U(m)\}.$$

It follows from Example 3.3.1 that $U^-(1) = \varnothing$. The case of $U^-(2)$ is already more complicated. Figure 3.8 shows that

$$U^-(2) = \{(x_0, x_1, x_2) \in U(2) : \\ x_1 = -1 \text{ or } x_1 = +1 \text{ or } x_0 = +1 \text{ or } x_2 = +1\}.$$

The set $U(2)$ is contractible and the set $U^{-}(2)$ consists of two disjoint contractible pieces. Hence

$$H_k(U^{-}(2)) = \begin{cases} \mathbb{Z}^2, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.10}$$

For $m > 2$ we can not draw a picture of $U(m)$ any more. Therefore we introduce a schematic representation. The schematic representation of $U(2)$ and $U^{-}(2)$ is depicted in Figure 3.8. Each bar in schematic representation consists of $2m - 1$ boxes and stands for a cube with the coordinates given by its label. If the upper (lower) part of the box, in the bar, is shaded then the upper (lower) face of the cube at the corresponding dimension is in $U^{-}(m)$. For example Figure 3.8 tells us that the face $B[-1]B \subset U^{-}(2)$ because the second box of BBB has its lower part shaded. If there is a connecting line between two cubes then they have a common face which is indicated by the position of the end points of the connecting line.

We already mentioned that $U(2)$ consists of five cubes depicted in Figure 3.8. Now we discuss some important properties of the set $U(m)$ for $m \geq 2$. The representant $\mathbf{u}$ used to define the braid class $[\mathbf{u} \text{ rel } \mathbf{w}]$ for $I = \{2, \ldots, m+1\}$ is chosen in such a way that its anchor points u_i with $i \in I_1$ are in the cube $\mathbf{C} = C_0, \ldots, C_{2m-2}$ where $C_i = B$ for all i. Hence the cube $B \ldots B \subset U(m)$. The representant $\mathbf{u}$ has $2m$ intersections with the strands $\mathbf{v}^1 \cup \mathbf{v}^2$ for $t \in [3, 3 + 2m]$. This number has to be the same for every representant. By inspection of the intersection number we can establish the following rules for cubes in $U(m)$. Let $\mathbf{C} = C_0 \ldots C_{2m-2} \subset U(m)$ then C_{2i} can be both A and B if and only if $C_{2i-1} = B$ while $C_{2i+1} = A$ or $C_{2i-1} = A$ while $C_{2i+1} = B$. The interval C_{2i+1} can be both A and B if and only if $C_{2i} = B$ and $C_{2i+2} = B$. Again by inspection of the intersection number one can show that cubes in $U(m)$ cannot contain the sequences mentioned in the following remark.

REMARK 3.3.8. Let $\mathbf{C} = C_0 \ldots C_{2m-2} \subset U(m)$ then $C_{2i-1}C_{2i}C_{2i+1}$ cannot be of the form BAB or AAA for any i.

The following lemma shows that the set $U(m)$ is contractible.

LEMMA 3.3.9. *The set $U(m)$ is contractible for every $m \in \mathbb{N}$.*

PROOF. Any cube $C_0, \ldots C_{2m-2} \subset U(m)$ contains the point $(0, \ldots, 0)$. Hence for an arbitrary $m \in \mathbb{N}$ the set $U(m)$ is star shaped around the point $(0, \ldots, 0)$ and can be contracted to this point. Therefore the set $U(m)$ is contractible for every $m \in \mathbb{N}$.

□

The complexity of the set $U^-(m)$ increases with m. See Figure 3.9 for a schematic representation of $U(3)$ and $U^-(3)$. We decompose $U^-(m)$ into two sets $\mathcal{A}_m^-, \mathcal{B}_m^-$ i.e.

$$U^-(m) = \mathcal{A}_m^- \cup \mathcal{B}_m^-.$$

The set $\mathcal{A}_m^-$ is a part of the set $U^-(m)$ which is contained in a union of the cubes $\mathbf{C} = C_0C_1 \ldots C_{2m-2}$ with $C_1 = A$ while the set $\mathcal{B}_m^-$ is a restriction of the set $U^-(m)$ to the cubes with $C_1 = B$. The sets $\mathcal{A}_m^-, \mathcal{B}_m^-$ will turn out to be contractible. Moreover, we will prove that

$$H_*(\mathcal{A}_m^- \cap \mathcal{B}_m^-) = H_*(\mathcal{A}_{m-1}^- \cup \mathcal{B}_{m-1}^-).$$

Then the Mayer-Vietoris sequence makes it possible to compute

$$H_*(U^-(m)) = H_*(\mathcal{A}_m^- \cup \mathcal{B}_m^-)$$

if we know $H_*(\mathcal{A}_{m-1}^- \cup \mathcal{B}_{m-1}^-)$ and by induction we can compute the homology of the set $U^-(m)$ for arbitrary $m \in \mathbb{N}$.

DEFINITION 3.3.10. Let us define

$$\mathcal{A}_m = \bigcup\{C_0AC_2 \ldots C_{2m-2} \subset U(m)\},$$

$$\mathcal{B}_m = \bigcup\{C_0BC_2 \ldots C_{2m-2} \subset U(m)\},$$

and

$$\mathcal{A}_m^- = \bigcup\{\mathbf{C}^- : \mathbf{C} \subset \mathcal{A}_m\},$$

$$\mathcal{B}_m^- = \bigcup\{\mathbf{C}^- : \mathbf{C} \subset \mathcal{B}_m\}.$$

REMARK 3.3.11. For $\mathbf{u} \in [\mathbf{u} \text{ rel } \mathbf{w}]$ with $I = \{2, \ldots, m+1\}$ the anchor point $u_3 \in [1, v_1^1]$ which corresponds to B, see (3.3.8). Remark 3.3.8 implies that $C_o = B$ for every cube $\mathbf{C} \in \mathcal{B}_m$.

EXAMPLE 3.3.12. The set $\mathcal{B}_2 = \{BBB\}$ and $\mathcal{B}_2^- = \{B[-1]B\}$ while $\mathcal{A}_2^-$ is the rest of the set $U^-(2)$ and $\mathcal{A}_2$ consists of four remaining cubes, see Figure 3.8. The set $\mathcal{A}_2^- \cap \mathcal{B}_2^- = \mathcal{A}_1^- \cup \mathcal{B}_1^- = \varnothing$.

Before we show that $\mathcal{A}_m^-$ and $\mathcal{B}_m^-$ are contractible, we investigate the diagram of $U^-(3)$ in Figure 3.9. For every cube $C_0AC_2C_3C_4 \subset U(3)$ the face $C_0[1]C_2C_3C_4 \subset U^-(3)$ and $C_0[0]C_2C_3C_4 \not\subset U^-(3)$. By taking into account that the behavior of any anchor point u_i is influenced only by its immediate neighbors u_{i-1} and u_{i+1} one can generalize this for any cube $C_0AC_2C_3 \ldots C_{2m-2} \subset U(m)$ as follows

$$C_0[1]C_2C_3 \ldots C_{2m-2} \subset U^-(m), \tag{3.3.11}$$

$$C_0[0]C_2C_3 \ldots C_{2m-2} \not\subset U^-(m). \tag{3.3.12}$$

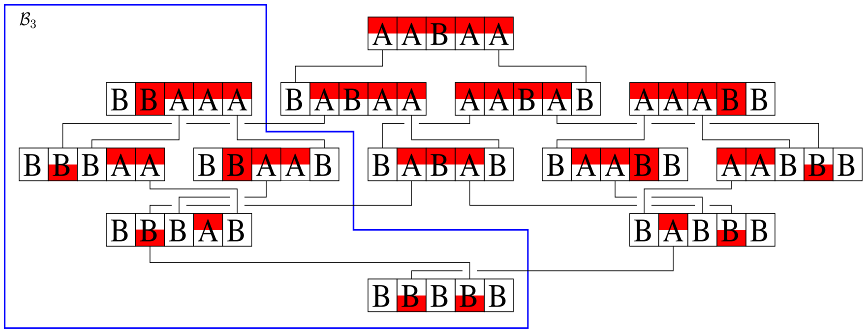

Figure 3.9: Schematic representation of $U(3)$. The shaded faces form $U^-(3)$. The cubes in the polygon form $\mathcal{B}_3$ and the other cubes form $\mathcal{A}_3$.

By inspection of the cubes $BBC_2C_3C_4 \subset U(3)$ we conclude that if $BBC_2C_3 \ldots C_{2m-2} \subset U(m)$ then

$$B[-1]C_2C_3 \ldots C_{2m-2} \subset U^-(m), \tag{3.3.13}$$

$$B[0]C_2C_3 \ldots C_{2m-2} \subset U^-(m) \text{ if and only if } C_2 = A. \tag{3.3.14}$$

LEMMA 3.3.13. *The sets $\mathcal{A}_m^-$ and $\mathcal{B}_m^-$ are contractible for every $m \in \mathbb{N}$.*

PROOF. It follows from (3.3.12) that $C_0[0]C_2 \ldots C_{2m-2} \not\subset \mathcal{A}_m^-$. Hence the map $h : \mathcal{A}_m \times [0,1] \to \mathcal{A}_m$ given by:

$$h(x_0, \ldots, x_{2m-2}, t) = (x_0, x_1 + t(1 - x_1), x_2, \ldots x_{2m-2})$$

is a homotopy between the set $\mathcal{A}_m^-$ and

$$\mathcal{A}_m^-|_{x_1=1} := \{\mathbf{x} \in \mathcal{A}_m^- : x_1 = 1\}.$$

The set $\mathcal{A}_m^-|_{x_1=1}$ is a union of $2m-2$ dimensional cubes

$$C_0[1]C_2 \ldots C_{2m-2}$$

which satisfy that $C_0AC_2 \ldots C_{2m-2} \subset \mathcal{A}_m$. Hence $\mathcal{A}_m^-|_{x_1=1}$ is star shaped around $(0, 1, 0, \ldots, 0)$. This implies that $\mathcal{A}_m^-$ is contractible.

Decompose $\mathcal{B}_m$ into

$$\mathcal{P} = \bigcup \{BBAC_3 \ldots C_{2m-2} \subset \mathcal{B}_m\},$$

$$\mathcal{Q} = \bigcup \{BBBC_3 \ldots C_{2m-2} \subset \mathcal{B}_m\},$$

then

$$\mathcal{B}_m^- = \mathcal{P}^- \cup \mathcal{Q}^-,$$

where $\mathcal{P}^- = \{\mathbf{C}^- : \mathbf{C} \subset \mathcal{P}\}$ and $\mathcal{Q}^- = \{\mathbf{C}^- : \mathbf{C} \subset \mathcal{Q}\}$. For $m \leq 2$ the set $\mathcal{P} = \varnothing$. Note that if $\mathbf{C} = BBAC_3 \ldots C_{2m-2} \subset \mathcal{P}$ then according to

Remark 3.3.8 it holds that $C_3 = A$ and the faces $B[0]AAC_4 \dots C_{2m-2}$, $BB[1]AC_4 \dots C_{2m-2}$ are in $\mathcal{P}^-$, see Figure 3.10. We decompose once more. Let

$$\mathcal{P}_0^- = \bigcup \{B[0]AAC_4 \dots C_{2m-2} \subset \mathcal{P}^-\},$$

and $\mathcal{P}_1^-$ is a union of all the other faces in $\mathcal{P}^-$. Then

$$\mathcal{B}_m^- = \mathcal{P}_0^- \cup (\mathcal{P}_1^- \cup \mathcal{Q}^-).$$

The set $\mathcal{P}_0^-$ is a union of $2m-2$ dimensional cubes which is star shaped around $(0,\dots 0)$ and hence contractible. For any cube $\mathbf{C} = C_0BC_2 \dots C_{2m-2} \subset \mathcal{B}_m$ it holds that the face $C_0[-1]C_2 \dots C_{2m-2}$ is present in $\mathcal{P}_1^- \cup \mathcal{Q}^-$ while $C_0[0]C_2 \dots C_{2m-2}$ is not. The same argument as for $\mathcal{A}_m^-$ furnishes that $\mathcal{P}_1^- \cup \mathcal{Q}^-$ is contractible. To show that $\mathcal{B}_m^- = \mathcal{P}_0^- \cup (\mathcal{P}_1^- \cup \mathcal{Q}^-)$ is contractible it remains to prove that $\mathcal{P}_0^- \cap (\mathcal{P}_1^- \cup \mathcal{Q}^-)$ is contractible.

First, we will prove that

$$\mathcal{P}_0^- \cap (\mathcal{P}_1^- \cup \mathcal{Q}^-) = \mathcal{P}_0^- \cap \mathcal{P}_1^-. \tag{3.3.15}$$

For any cube $\mathbf{C} \in \mathcal{Q}$ we have to show that

$$\mathcal{P}_0^- \cap \mathbf{C}^- \subset \mathcal{P}_0^- \cap \mathcal{P}_1^-.$$

We distinguish three different types of cubes in $\mathcal{Q}$. If

$$\mathbf{C} = BBBAC_4 \dots C_{2m-2} \subset \mathcal{Q}$$

then

$$\mathcal{P}_0^- \cap \mathbf{C}^- \subset \mathcal{P}_0^- \cap (\mathcal{P}_1^- \cap \widetilde{\mathbf{C}}^-) \subset \mathcal{P}_0^- \cap \mathcal{P}_1^-, \tag{3.3.16}$$

where $\widetilde{\mathbf{C}} = BBAAC_4 \dots C_{2m-2} \subset \mathcal{P}$. The second inclusion is trivial. To prove the first inclusion we proceed as follows. The set $\mathcal{P}_0^- \cap \mathbf{C}^- \subset B[0][0]AC_4 \dots C_{2m-2}$ and $\mathbf{C}^- \cap B[0][0]AC_4 \dots C_{2m-2} \subset \mathcal{P}_1^-$. Hence we only have to show that $\mathbf{C}^- \cap B[0][0]AC_4 \dots C_{2m-2} \subset \widetilde{\mathbf{C}}^-$. The fact that the cubes $\mathbf{C}$ and $\widetilde{\mathbf{C}}$ differs only in C_2 implies the previous statement for all faces at coordinates different from $i \in \{1,2,3\}$. By analyzing the intersection number of the free strand with the skeletal strands we get that the faces $BB[0]AC_4 \dots C_{2m-2}$ and $B[0]BAC_4 \dots C_{2m-2}$ are not in $\mathbf{C}^-$. Therefore we only have to check the faces at the third coordinate. Again by analyzing the intersection number we get that $B[0][0][0]C_4 \dots C_{2m-2}$ is not in $\mathbf{C}^-$ ($\widetilde{\mathbf{C}}^-$) and $B[0][0][1]C_4 \dots C_{2m-2}$ belongs both to $\mathbf{C}^-$ and $\widetilde{\mathbf{C}}^-$. This proves the first inclusion in (3.3.16).

For $\mathbf{C} = BBBBBC_5 \dots C_{2m-2} \subset \mathcal{Q}$ relation (3.3.16) holds with $\widetilde{\mathbf{C}} = BBAABC_5 \dots C_{2m-2}$. In this case $\mathcal{P}_0^- \cap \mathbf{C}^- \subset B[0][0][0]BC_5 \dots C_{2m-2}$.

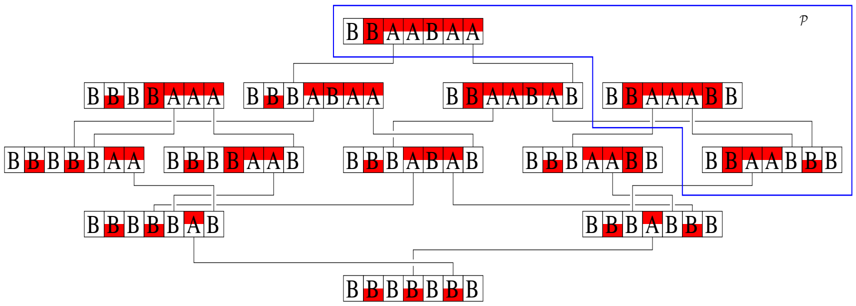

Figure 3.10: Schematic representation of $\mathcal{B}_4$. The shaded faces form $\mathcal{B}_4^-$. The cubes in the polygon form $\mathcal{P}$ and the cubes out of it form $\mathcal{Q}$.

The faces of $\mathbf{C}$ for $i = 0$ and $i = 4$ are not in $\mathbf{C}^-$. As before the faces of $\mathcal{P}_0^- \cap \mathbf{C}^-$ at the coordinates $i > 4$ are contained in $\mathcal{P}_1^- \cap \widetilde{\mathbf{C}}^-$.

Finally, for $m > 3$ there are cubes $\mathbf{C} \subset \mathcal{Q}$ such that $\mathbf{C} = BBBBAAC_6 \dots C_{2m-2}$. From the structure of $\mathcal{P}$ and Remark 3.3.8 it follows that $\mathcal{P} \cap \mathbf{C} \subset \widetilde{\mathbf{C}} \cup \widehat{\mathbf{C}}$, where $\widetilde{\mathbf{C}} = BBAAABC_6 \dots C_{2m-2}$ and $\widehat{\mathbf{C}} = BBAABAC_6 \dots C_{2m-2}$ are two cubes in $\mathcal{P}$. In the same way as in the previous cases one can prove that $\mathcal{P}_0^- \cap \mathbf{C}^- \subset \mathcal{P}_0^- \cap \left\{\mathcal{P}_1^- \cap (\widetilde{\mathbf{C}}^- \cup \widehat{\mathbf{C}}^-)\right\}$. We just point out that intersection of $\mathcal{P}_0^-$ with the face $BBBB[1]AC_6 \dots C_{2m-2}$ is contained in $\widetilde{\mathbf{C}}^-$ while intersection of $\mathcal{P}_0^-$ with all the other faces is in $\widetilde{\mathbf{C}}^-$

According to Remark 3.3.8 every cube $\mathbf{C} \subset \mathcal{P}$ belongs to one of the three types treated above and relation (3.3.15) holds. Hence to finish the proof it is enough to show that $\mathcal{P}_0^- \cap \mathcal{P}_1^-$ is contractible.

If $\mathbf{C} = BBAC_3 \dots C_{2m-2} \subset \mathcal{P}$ then it follows from Remark 3.3.8 that $C_3 = A$ and by analyzing the intersection number we obtain that the face $B[0]AC_3 \dots C_{2m-2} \subset \mathcal{P}_0^-$. All the other faces contained in $\mathbf{C}^-$ belong to $\mathcal{P}_1^-$. Thus $\mathcal{P}_0^- \cap \mathcal{P}_1^- = \mathcal{P}_1^-|_{x_1=0}$. Moreover for every cube in $\mathcal{P}$, the face $BB[1]AC_4 \dots C_{2m-2}$ is in $\mathcal{P}_1^-$ and $BB[0]AC_4 \dots C_{2m-2}$ is not, see Figure 3.10. Hence $\mathcal{P}_0^- \cap \mathcal{P}_1^- = \mathcal{P}_1^-|_{x_1=0}$ is star shaped around $(0, 1, 0, \dots, 0)$. This implies that the set $\mathcal{P}_0^- \cap \mathcal{P}_1^-$ contractible. □

Now we turn our attention to the set $\mathcal{A}_m^- \cap \mathcal{B}_m^-$. To simplify the arguments we define the sets

$$\widetilde{\mathcal{A}}_m := \{\mathbf{C} : \mathbf{C} = BAC_2 \dots C_{2m-2} \subset \mathcal{A}_m\},$$

$$\widetilde{\mathcal{A}}_m^- := \{\mathbf{C}^- : \mathbf{C} = BAC_2 \dots C_{2m-2} \subset \mathcal{A}_m\},$$

which are contractible. The proof is analogous to the one for the sets $\mathcal{A}_m^-$. It holds that $\widetilde{\mathcal{A}}_m^- \cap \mathcal{B}_m^- = \mathcal{A}_m^- \cap \mathcal{B}_m^-$ because if $\mathbf{C} = AAC_2 \dots C_{2m-2} \subset \mathcal{A}_m$ then $\mathbf{C}^- \cap \mathcal{B}_m^- \subset \widehat{\mathbf{C}}^- \cap \mathcal{B}_m^-$ where $\widehat{\mathbf{C}} = BAC_2 \dots C_{2m-2}$. Hence we showed that $H_*(\widetilde{\mathcal{A}}_m^- \cap \mathcal{B}_m^-) = H_*(\mathcal{A}_m^- \cap \mathcal{B}_m^-)$ and $H_*(\widetilde{\mathcal{A}}_m^-) = H_*(\mathcal{A}_m^-)$. By plugging this to the Mayer-Vietoris sequence

$$\dots \xrightarrow{\partial_{k+1}^*} H_k(A \cap B) \xrightarrow{\varphi_k^*} H_k(A) \oplus H_k(B) \xrightarrow{\psi_k^*} H_k(A \cup B) \xrightarrow{\partial_k^*} H_{k-1}(A \cap B) \xrightarrow{\varphi_{k-1}^*} \dots$$

furnishes

$$H_*(\widetilde{\mathcal{A}}_m^- \cup \mathcal{B}_m^-) = H_*(\mathcal{A}_m^- \cup \mathcal{B}_m^-). \tag{3.3.17}$$

Before we start studying the set $\mathcal{A}_m^- \cap \mathcal{B}_m^-$ for an arbitrary m we give a low dimensional example.

EXAMPLE 3.3.14. We claim that

$$H_*(\mathcal{A}_3^- \cap \mathcal{B}_3^-) = H_*(\mathcal{A}_2^- \cup \mathcal{B}_2^-). \tag{3.3.18}$$

According to (3.3.17) it is enough to show that

$$H_*(\widetilde{\mathcal{A}}_3^- \cap \mathcal{B}_3^-) = H_*(\widetilde{\mathcal{A}}_2^- \cup \mathcal{B}_2^-).$$

Let $\mathcal{P}_1 = BABBB \cup BABAB \cup BABAA$, $\mathcal{P}_2 = BAABB$, $\mathcal{Q}_1 = BBBBB \cup BBBAB \cup BBBAA$ and $\mathcal{Q}_2 = BBAAB \cup BBAAA$. The set $\mathcal{P}_i^-$ ($\mathcal{Q}_i^-$) is a union of the faces of the cubes $\mathbf{C} \subset \mathcal{P}_i$ ($\mathcal{Q}_i$) which are in $\widetilde{\mathcal{A}}_3^-$ ($\mathcal{B}_3^-$) and

$$\widetilde{\mathcal{A}}_3^- \cap \mathcal{B}_3^- = \bigcup_{i,j=1}^{2} \mathcal{P}_i^- \cap \mathcal{Q}_j^-.$$

The intersection $\mathcal{P}_1^- \cap \mathcal{Q}_1^- = B[0]B[-1]B \cup B[0]B[1]B \cup B[0]B[1]A \cup B[0]BA[1]$ is homotopic to the set $0 \times 0 \times \{B[-1]B \cup B[1]B \cup B[1]A \cup BA[1]\}$ which can be written as $0 \times 0 \times \{\widetilde{\mathcal{A}}_2^- \cup \mathcal{B}_2^-\}$, see Figure 3.11. Therefore

$$H_*(\mathcal{P}_1^- \cap \mathcal{Q}_1^-) = H_*(\widetilde{\mathcal{A}}_2^- \cup \mathcal{B}_2^-).$$

After a short computation

$$\mathcal{P}_2^- \cap \mathcal{Q}_1^- = B[0][0][-1]B \subset \mathcal{P}_1^- \cap \mathcal{Q}_1^-,$$
$$\mathcal{P}_1^- \cap \mathcal{Q}_2^- = B[0][0]AB \cup B[0][0]AA,$$
$$\mathcal{P}_2^- \cap \mathcal{Q}_2^- = B[0]A[0]B.$$

The set $(\mathcal{P}_2^- \cap \mathcal{Q}_1^-) \cup (\mathcal{P}_2^- \cap \mathcal{Q}_2^-)$ is homotopic to $B[0]B[-1]B \cup B[0]B[1]A \cup B[0]BA[1] \subset \mathcal{P}_1^- \cap \mathcal{Q}_1^-$. Hence $\widetilde{\mathcal{A}}_3^- \cap \mathcal{B}_3^-$ is homotopic to $\mathcal{P}_1^- \cap \mathcal{Q}_1^-$ and

$$H_*(\widetilde{\mathcal{A}}_3^- \cap \mathcal{B}_3^-) = H_*(\mathcal{P}_1^- \cap \mathcal{Q}_1^-) = H_*(\widetilde{\mathcal{A}}_2^- \cup \mathcal{B}_2^-).$$

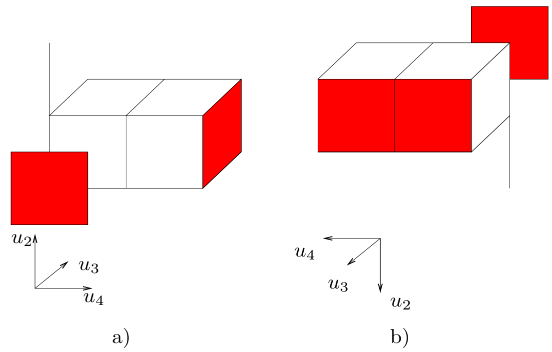

Figure 3.11: a) b) the projection of the set $\mathcal{A}_3^- \cap \mathcal{B}_3^-$ on the coordinates (u_2, u_3, u_4). The set $\mathcal{P}_1^- \cap \mathcal{Q}_1^-$ is shaded.

LEMMA 3.3.15. *Let $m \in \mathbb{N}$ such that $m > 1$. Then*

$$H_*(\mathcal{A}_m^- \cap \mathcal{B}_m^-) = H_*(\mathcal{A}_{m-1}^- \cup \mathcal{B}_{m-1}^-). \tag{3.3.19}$$

PROOF. According to (3.3.17) it holds that

$$H_*(\mathcal{A}_m^- \cap \mathcal{B}_m^-) = H_*(\widetilde{\mathcal{A}}_m^- \cap \mathcal{B}_m^-).$$

Every cube $\mathbf{C} = C_0 C_1 \dots C_{2m-2} \subset \widetilde{\mathcal{A}}_m \cup \mathcal{B}_m$ satisfies that $C_0 = B$ and its faces $[-1]C_1 \dots C_{2m-2}$ and $[0]C_1 \dots C_{2m-2}$ are not present in $\widetilde{\mathcal{A}}_m^- \cup \mathcal{B}_m^-$. This implies that $H_*(\widetilde{\mathcal{A}}_m^- \cap \mathcal{B}_m^-) = H_*(\widetilde{\mathcal{A}}_m^-|_{x_0=0} \cap \mathcal{B}_m^-|_{x_0=0})$ where $\widetilde{\mathcal{A}}_m^-|_{x_0=0} = \{\mathbf{x} \in \widetilde{\mathcal{A}}_m^- : x_0 = 0\}$ and $\mathcal{B}_m^-|_{x_0=0} = \{\mathbf{x} \in \mathcal{B}_m^- : x_0 = 0\}$. In order to evaluate $H_*(\widetilde{\mathcal{A}}_m^-|_{x_0=0} \cap \mathcal{B}_m^-|_{x_0=0})$ we decompose $\widetilde{\mathcal{A}}_m^-|_{x_0=0}$ and $\mathcal{B}_m^-|_{x_0=0}$ as follows

$$\mathcal{P}_1 = \bigcup \{\mathbf{C} \in \widetilde{\mathcal{A}}_m|_{x_0=0} : C_2 = B\}, \quad \mathcal{P}_2 = \bigcup \{\mathbf{C} \in \widetilde{\mathcal{A}}_m|_{x_0=0} : C_2 = A\},$$

$$\mathcal{Q}_1 = \bigcup \{\mathbf{C} \in \mathcal{B}_m|_{x_0=0} : C_2 = B\}, \quad \mathcal{Q}_2 = \bigcup \{\mathbf{C} \in \mathcal{B}_m|_{x_0=0} : C_2 = A\}.$$

The set $\mathcal{P}_i^-$ ($\mathcal{Q}_i^-$) is the union of the faces of the $2m-2$ dimensional cubes $\mathbf{C} = [0]C_1 \dots C_{2m-2} \subset \mathcal{P}_i$ ($\mathcal{Q}_i$), which are in $\widetilde{\mathcal{A}}_m^-|_{x_0=0}$ ($\mathcal{B}_m^-|_{x_0=0}$). Note that if $\mathbf{C} \subset \mathcal{P}_2$ then $C_3 = B$ while if $\mathbf{C} \subset \mathcal{Q}_2$ then $C_3 = A$. The intersection

$$\widetilde{\mathcal{A}}_m^-|_{x_0=0} \cap \mathcal{B}_m^-|_{x_0=0} = \bigcup_{i,j=1}^{2} (\mathcal{P}_i^- \cap \mathcal{Q}_j^-).$$

For every cube $\mathbf{C} = [0]C_1 \dots C_{2m-2} \subset \mathcal{P}_1$ it holds that $C_1 = A$ and $[0][0]C_2 \dots C_{2m-2} \not\subset \mathcal{P}_1^-$. If the cube $\mathbf{C} \subset \mathcal{Q}_1$ then $C_1 = B$ and

$[0][0]C_2 \dots C_{2m-2} \not\subset \mathcal{Q}_1^-$. The fact that $C_2 = B$ for every cube in $\mathcal{P}_1$ and $\mathcal{Q}_1$ implies that $(\mathcal{P}_1^- \cap \mathcal{Q}_1^-) = 0 \times 0 \times (\widetilde{\mathcal{A}}_{m-1}^- \cup \mathcal{B}_{m-1}^-)$ and

$$H_*(\mathcal{P}_1^- \cap \mathcal{Q}_1^-) = H_*(\widetilde{\mathcal{A}}_{m-1}^- \cup \mathcal{B}_{m-1}^-) = H_*(\mathcal{A}_{m-1}^- \cup \mathcal{B}_{m-1}^-).$$

If $\mathbf{C} \subset \mathcal{P}_2 \cap \mathcal{Q}_1$ then

$$\mathbf{C} = [0][0][0]BC_4 \dots C_{2m-2} \subset [0][0]BBC_4 \dots C_{2m-2} \subset \mathcal{P}_1 \cap \mathcal{Q}_1.$$

Non of the first three faces is in $\mathcal{P}_2^- \cap \mathcal{Q}_1^-$. Hence the set $\mathcal{P}_2^- \cap \mathcal{Q}_1^- \subset \mathcal{P}_1^- \cap \mathcal{Q}_1^-$ and $H_*((\mathcal{P}_1^- \cap \mathcal{Q}_1^-) \cup (\mathcal{P}_2^- \cap \mathcal{Q}_1^-)) = H_*(\mathcal{P}_1^- \cap \mathcal{Q}_1^-)$. To prove relation

$$H_*(\widetilde{\mathcal{A}}_m^-|_{x_0=0} \cap \mathcal{B}_m^-|_{x_0=0}) = H_*(\mathcal{P}_1^- \cap \mathcal{Q}_1^-) = H_*(\mathcal{A}_{m-1}^- \cup \mathcal{B}_{m-1}^-), \quad (3.3.20)$$

it is enough to show that

$$V := (\mathcal{P}_1^- \cap \mathcal{Q}_2^-) \cup (\mathcal{P}_2^- \cap \mathcal{Q}_2^-) \quad (3.3.21)$$

is contractible and that its intersection with $\mathcal{P}_1^- \cap \mathcal{Q}_1^-$ is contractible. We prove that V is contractible by showing that both components in (3.3.21) are contractible and so is their intersection.

The set $\mathcal{P}_1$ is build up from cubes of the form $[0]ABC_3 \dots C_{2m-2}$ while cubes in $\mathcal{Q}_2$ have the form $[0]BAAC_4 \dots C_{2m-2}$. The cube $[0]ABAC_4 \dots C_{2m-2} \subset \mathcal{P}_1$ if and only if $[0]BAAC_4 \dots C_{2m-2} \subset \mathcal{Q}_2$. Moreover the face $[0][0][0]AC_4 \dots C_{2m-2} \subset \mathcal{P}_1^- \cap \mathcal{Q}_2^-$. Therefore $\mathcal{P}_1^- \cap \mathcal{Q}_2^-$ is a union of $2n-4$ dimensional cubes $[0][0][0]AC_4 \dots C_{2m-2} \subset \mathcal{Q}_2$ which is star shape around $(0, \dots, 0)$ This concludes that $\mathcal{P}_1^- \cap \mathcal{Q}_2^-$ is contractible.

If $\mathbf{C} \subset \mathcal{P}_2 \cap \mathcal{Q}_2$ then $\mathbf{C} = [0][0]A[0]C_4 \dots C_{2m-2} \subset \mathcal{P}_2$. Actually, $\mathcal{P}_2 \cap \mathcal{Q}_2 = \mathcal{P}_2|_{\{x_2=0, x_3=0\}}$. Moreover $\mathbf{C} \subset \mathcal{P}_2^-$ because of the face at the fourth coordinate and $\mathbf{C} \subset \mathcal{Q}_2^-$ because of the face at the second coordinate. Therefore the set $\mathcal{P}_2^- \cap \mathcal{Q}_2^-$ is a union of the cubes $[0][0]A[0]C_4 \dots C_{2m-2} \subset \mathcal{P}_2$ which is star shaped and contractible .

If $\mathbf{C} = [0][0][0][0]C_4 \dots C_{2m-2} \subset \mathcal{P}_2$ then $\mathbf{C} \subset \mathcal{Q} - 2$. This implies that the set $(\mathcal{P}_1^- \cap \mathcal{Q}_2^-) \cap (\mathcal{P}_2^- \cap \mathcal{Q}_2^-)$ is a union of $2m-5$ dimensional cubes $[0][0][0][0]C_4 \dots C_{2m-2} \subset \mathcal{Q}_2$ which is star shape.We proved that the sets $\mathcal{P}_1^- \cap \mathcal{Q}_2^-$, $\mathcal{P}_2^- \cap \mathcal{Q}_2^-$ and their intersection are contractible. Hence we proved that V is contractible.

The set $(\mathcal{P}_1^- \cap \mathcal{Q}_1^-) \cap V$ is given by faces, of $2m-4$ dimensional cubes $[0][0][0]AC_4 \dots C_{2m-2} \subset \mathcal{Q}_2$, which are present in $(\mathcal{P}_1^- \cap \mathcal{Q}_1^-)$. The face $[0][0][0][1]C_4 \dots C_{2m-2} \subset (\mathcal{P}_1^- \cap \mathcal{Q}_1^-)$ for each of these cubes while $[0][0][0][0]C_4 \dots C_{2m-2}$ is not. As in the proof of Lemma 3.3.13 the set $(\mathcal{P}_1^- \cap \mathcal{Q}_1^-) \cap V$ is contractible. This proves relation (3.3.20). □

The following lemma describes homology of the set $U^-(m)$ for arbitrary $m \in \mathbb{N}$.

LEMMA 3.3.16. *The homology of the set $U^-(m)$, for $m > 0$, is given by*

$$H_*(U^-(m)) = H_*\left(\Gamma^{m-1}\right),$$

where Γ^n is the boundary of the n-dimensional unit cube $[0,1]^n$.

REMARK 3.3.17. We use a convention $H_k\left(\Gamma^0\right) = 0$ for $k \in \mathbb{Z}$.

PROOF. At the beginning of this subsection we proved the lemma for $m = 1, 2$. We use induction to prove the lemma for arbitrary m. The set $U^-(m) = \mathcal{A}_m^- \cup \mathcal{B}_m^-$ and according to Lemma 3.3.13 the sets $\mathcal{A}_m^-$ and $\mathcal{B}_m^-$ are contractible. Lemma 3.3.15 furnishes

$$H_*(\mathcal{A}_m^- \cap \mathcal{B}_m^-) = H_*(\mathcal{A}_{m-1}^- \cup \mathcal{B}_{m-1}^-).$$

The induction argument implies that

$$H_*(\mathcal{A}_m^- \cap \mathcal{B}_m^-) = H_*(\Gamma^{m-2}).$$

By applying the Mayer-Vietoris sequence

$$\ldots \xrightarrow{\partial_{k+1}^*} H_k(A \cap B) \xrightarrow{\varphi_k^*} H_k(A) \oplus H_k(B) \xrightarrow{\psi_k^*} H_k(A \cup B) \xrightarrow{\partial_k^*} H_{k-1}(A \cap B) \xrightarrow{\varphi_{k-1}^*} \ldots$$

we conclude that

$$H_*(\mathcal{A}_m^- \cup \mathcal{B}_m^-) = H_*(\Gamma^{m-1}).$$

□

Computation of $H_*(h(\mathbf{u} \text{ rel } \mathbf{w}))$

First we recall that the sets N_I and N_I^- decompose as follows

$$N_I = U_0 \times U_1 \times \ldots \times U_l,$$

$$N_I^- = \bigcup_{i=0}^{l} U_0 \times U_1 \times \ldots \times U_{i-1} \times U_i^- \times U_{i+1} \times \ldots \times U_l.$$

The homology of the sets U_0 and U_0^- is given by (3.3.5) and (3.3.6). The homology of U_i and U_i^- is computed in Lemma 3.3.9 and Lemma 3.3.16. We use them to compute $H_*(N_I)$ and $H_*(N_I^-)$. Then by using the exact sequence which relates $H_*(N_I)$, $H_*(N_I^-)$ to $H_*(N_I, N_I^-)$ we calculate $H_*(h(\mathbf{u}^I \text{ rel } \mathbf{w})) = H_*(N_I, N_I^-)$.

LEMMA 3.3.18. *The set N_I is contractible and the homology of N_I^- is given by*

$$H_k(N_I^-) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2p - \#I - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.22}$$

PROOF. The set N_I is a direct sum of the contractible sets U_i. Hence it is contractible. It remains to prove that homology of N_I^- is given by (3.3.22). We start by computing homology of the union

$$U_0^- \times U_1 \times U_2 \times \ldots \times U_l \,\cup\, U_0 \times U_1^- \times U_2 \times \ldots \times U_l.$$

The sets U_i are contractible. Therefore

$$H_*(U_0^- \times U_1 \times U_2 \times \ldots \times U_l \cup U_0 \times U_1^- \times U_2 \times \ldots \times U_l) = H_*(U_0^- \times U_1 \cup U_0 \times U_1^-),$$

and

$$H_*(U_0^- \times U_1) = H_*(U_0^-) = H_*(\Gamma^m),$$
$$H_*(U_0 \times U_1^-) = H_*(U_1^-) = H_*(\Gamma^n),$$

where Γ^m is the boundary of an m dimensional cube and $m = \#I_0$ while $n = \#I_1 - 1$. Suppose for now that U_0 is an m dimensional cube and $U_0^- = \partial U_0$, while $U_1^- = \partial U_1$ and U_1 is an n dimensional cube. Then

$$\partial(U_0 \times U_1) \;= \partial U_0 \times U_1 \cup U_0 \times \partial U_1 = U_0^- \times U_1 \cup U_0 \times U_1^-,$$

and

$$H_*(U_0^- \times U_1 \cup U_0 \times U_1^-) = H_*(\partial(U_0 \times U_1)) = H_*(\Gamma^{m+n}).$$

However in our case it only holds that $U_i^- \subset U_i$, where U_i is contractible and $H_*(U_0^-) = H_*(\Gamma^m)$ while $H_*(U_1^-) = H_*(\Gamma^n)$. Therefore we have to prove that

$$H_*(U_0^- \times U_1 \cup U_0 \times U_1^-) = H_*(\Gamma^{m+n}), \tag{3.3.23}$$

in this setting. If $m = 0$ or $n = 0$ then $U_0^- = \varnothing$, or $U_1^- = \varnothing$ and (3.3.23) is trivially satisfied. For $m, n > 0$ we use the long exact sequence

$$\ldots \xrightarrow{\partial^*_{k+1}} H_k(A \cap B) \xrightarrow{\varphi^*_k} H_k(A) \oplus H_k(B) \xrightarrow{\psi^*_k} H_k(A \cup B) \xrightarrow{\partial^*_k} H_{k-1}(A \cap B) \xrightarrow{\varphi^*_{k-1}} \ldots \tag{3.3.24}$$

to prove (3.3.23). Put $A = U_0^- \times U_1$ and $B = U_0 \times U_1^-$. Then

$$A \cap B = U_0^- \times U_1^-.$$

Without the loss of generality we can suppose that $m \geq n$. The homology

$$H_*(A \cap B) = H_*(U_0^- \times U_1^-) = H_*(\Gamma^m \times \Gamma^n)$$

is homology of the cross product of two spheres i.e only the homology groups $0, m, n$ and $m + n$ are nontrivial. If all the inexes are different then these groups are $\mathbb{Z}$. In case that some of the indexes are the same then they refer to the same homology group and the group is $\mathbb{Z}^k$ where k is the number of the indexes which refer to this group. Hence plugging the known homologies $H_*(A)$, $H_*(B)$ and $H_*(A \cap B)$ to the exact sequence (3.3.24) leads to six different cases $m = n = 1$, $m - 1 = n = 1$,

$m-1>n=1$, $m=n>1$ $m-1=n>1$ and $m-1>n>1$. We will deal only with the case $m-1>n>1$. All the other cases can be treated analogously. In this case

$$H_k(A)=\begin{cases}\mathbb{Z}, & \text{if } k=0,m-1,\\ 0, & \text{otherwise,}\end{cases}\qquad H_k(B)=\begin{cases}\mathbb{Z}, & \text{if } k=0,n-1,\\ 0, & \text{otherwise,}\end{cases}$$

$$H_k(A\cap B)=\begin{cases}\mathbb{Z}, & \text{if } k=0,n-1,m-1,m+n-2,\\ 0, & \text{otherwise.}\end{cases}$$

The homology group $H_{m+n-1}(A\cup B)=\mathbb{Z}$ because of the short exact sequence

$$0\overset{\psi^*_{m+n-1}}{\longrightarrow}H_{m+n-1}(A\cup B)\overset{\partial^*_{m+n-1}}{\longrightarrow}\mathbb{Z}\overset{\varphi^*_{m+n-2}}{\longrightarrow}0,$$

which is a part of the long exact sequence. The following part of the exact sequence

$$0\overset{\partial^*_m}{\longrightarrow}\mathbb{Z}\overset{\varphi^*_{m-1}}{\longrightarrow}\mathbb{Z}\overset{\psi^*_{m-1}}{\longrightarrow}H_{m-1}(A\cup B)\overset{\partial^*_{m-1}}{\longrightarrow}0,$$

and relation $\mathrm{im}\varphi^*_{m-1}=\mathbb{Z}$ implies that $H_{m-1}(A\cup B)=0$. Similar calculations provide that $H_{n-1}(A\cup B)=0$. By exploiting the sequence

$$0\overset{\psi^*_1}{\longrightarrow}H_1(A\cup B)\overset{\partial^*_1}{\longrightarrow}\mathbb{Z}\overset{\varphi^*_0}{\longrightarrow}\mathbb{Z}\times\mathbb{Z}\overset{\psi^*_0}{\longrightarrow}H_0(A\cup B)\overset{\partial^*_0}{\longrightarrow}0,$$

where $\mathrm{im}\varphi^*_0=\mathbb{Z}$ and $\ker\varphi^*_0=0$ we obtain that $H_1(A\cup B)=0$ and $H_0(A\cup B)=\mathbb{Z}$. Finally, for all remaining indices the exactness of

$$0\overset{\psi^*_k}{\longrightarrow}H_k(A\cup B)\overset{\partial^*_k}{\longrightarrow}0,$$

implies that $H_k(A\cup B)=0$.

We thus proved that

$$H_*(U_0^-\times U_1\times U_2\times\ldots\times U_l\cup U_0\times U_1^-\times U_2\times\ldots\times U_l)=H_*(\Gamma^{m+n})$$

By repeating the previous computation l times we obtain that

$$H_*(N_I^-)=H_*\left(\Gamma^{\{\#I_0+\sum_{i=1}^{l}(\#I_i-1)\}}\right).$$

According to Definition 3.3.4 it holds that $\#I_0=2p-\sum_{i=1}^{l}(2\#I_i-1)$, hence

$$H_k(N_I^-)=H_k\left(\Gamma^{\{2p-\#I\}}\right)=\begin{cases}\mathbb{Z}, & \text{if } k=0,2p-\#I-1,\\ 0, & \text{otherwise.}\end{cases}$$

□

To conclude the proof of Theorem 3.2.4, we compute the relative homology $H_*(N_I, N_I^-)$. We use the long exact sequence

$$\ldots \xrightarrow{\partial_{k+1}^*} H_k(N_I^-) \xrightarrow{i_k^*} H_k(N_I) \xrightarrow{\pi_k^*} H_k(N_I, N_I^-) \xrightarrow{\partial_k^*} H_{k-1}(N_I^-) \xrightarrow{i_{k-1}^*} \ldots.$$

For $k \notin \{0, 2p - \#I\}$ exactness of

$$0 \xrightarrow{\pi_k^*} H_k(N_I, N_I^-) \xrightarrow{\partial_k^*} 0,$$

implies that $H_k(N_I, N_I^-) = 0$. The exact sequences

$$0 \xrightarrow{\pi_{2p-\#I}^*} H_{2p-\#I}(N_I, N_I^-) \xrightarrow{\partial_{2p-\#I}^*} \mathbb{Z} \xrightarrow{i_{2p-\#I-1}^*} 0,$$

$$\mathbb{Z} \xrightarrow{i_0^*} \mathbb{Z} \xrightarrow{\pi_0^*} H_0(N_I, N_I^-) \xrightarrow{\partial_0^*} 0,$$

furnish

$$H_k(N_I, N_I^-) = \begin{cases} \mathbb{Z}, & \text{if } k = 2p - \#I, \\ 0, & \text{otherwise.} \end{cases}$$

According to Section 2.2 the index $H_*(h(\mathbf{u}^I \text{ rel } \mathbf{w})) = H_*(N_I, N_I^-)$. This concludes the proof of Theorem 3.2.4.

3.4. Forcing of solutions in $[\sigma_1^2\sigma_2^{2q}, 2p]$

In this section we prove that

$$[\sigma_1^2\sigma_2^4, 4] \prec [\sigma_1^2\sigma_2^{2q}, 2p],$$

holds for $3 < q < p$ and estimate the number of solutions of the class $[\sigma_1^2\sigma_2^{2q}, 2p]$. To do so we define a braid class $[\mathbf{u}^I \text{ rel } \mathbf{w}]_{\mathcal{E}}$, for every set $I = \{j_1, \ldots, j_n\} \subset \mathbb{N}^n$ satisfying

$$p - q + 3 < j_1 < \ldots < j_n < 2p,$$

and take a subset M_I of $[\mathbf{u}^I \text{ rel } \mathbf{w}]_{\mathcal{E}}$ which is an isolating neighborhood of the flow Ψ^t generated by the parabolic recurrence relation corresponding to (3.1.1). Non-triviality of the index $H_*(h(\mathbf{u}^I \text{ rel } \mathbf{w}))$ will imply the existence of a fixed point $\mathbf{u}$ of the flow Ψ^t in the set M_I which corresponds to a solution $u \in [\sigma_1^2\sigma_2^{2q}, 2p]$. Solutions obtained for different sets I are geometrically different. By taking different sets I we produce $2^{(q-4)}$ geometrically different solutions.

DEFINITION 3.4.1. Let the skeleton $\mathbf{w}$ be given by (3.2.3). The braid class $[\mathbf{u}^I \text{ rel } \mathbf{w}]_{\mathcal{E}} \in \mathcal{E}_{2p}^1 \text{ rel } \mathbf{w}$ is defined by its representant $\mathbf{u}^I$ satisfying:

(1) $u_0^I \in (v_0^1, v_0^6)$,

(2) $u^I_{2i+1} \in \begin{cases} (v_1^4, v_5^1) & : \quad \text{if } 1 < i < 2+p-q, \\ (v_1^5, v_1^1) & : \quad \text{otherwise,} \end{cases}$

(3) $u^I_{2i} \in \begin{cases} (v_2^4, v_2^1) & : \quad \text{if } i \in I \text{ or } 1 < i < 3+p-q, \\ (v_2^1, v_2^5) & : \quad \text{otherwise.} \end{cases}$

The free strand $\mathbf{u}$ intersects the strand $+\mathbf{1}$ four times then it stays below this strand till the anchor point $u_{4+2(p-q)}$. Then it intersects the strand $+\mathbf{1}$ twice again. After this the free strand $\mathbf{u}$ behaves in the same way as the free strand of the braid class from the previous section. Figure 3.12 shows the braid class $[\mathbf{u}^I \text{ rel } \mathbf{w}]$ with a simplified skeleton for $q = 3$ and $I = \varnothing$. The procedure for simplifying the skeleton is explained in the previous section. For the sake of simplicity we restrict ourselves to the case $I = \varnothing$. However, the same decomposition of the set of anchor points u_i with $i \in \{4+2(p-q), \ldots, 2p\}$ as in Section 3.3 extends the result for non empty sets I. We omit I from the notation.

Now we recall an important property of the parabolic recurrence relation $\mathcal{R}$ generated by (3.1.1) at the zero energy level. In the parameter range $\alpha \in [0, \sqrt{8})$ the two equilibria $u_\pm = \pm 1$ are saddle-foci and there are no solutions which converge monotonically to any of these equilibria. Therefore the twist property, see Section 2.2, implies that for every $(x, y) \in \mathbb{R}^2 \setminus \Delta$ there exists a *finite* $\tau_{x,y}$ and a unique solution $u(t; x, y) : [0, \tau_{x,y}] \to \mathbb{R}$ such that $u(0; x, y) = x$, $u(\tau_{x,y}; x, y) = y$ and $u'|_{(0,\tau_{x,y})} > 0$ if $x < y$ ($u'|_{(0,\tau_{x,y})} < 0$ if $x > y$). The function $\mathcal{R}_i$ is defined as follows

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = u'''(\tau_{u_{i-1},u_i}; u_{i-1}, u_i) - u'''(0; u_i, u_{i+1}).$$

Finally we investigate the function $\mathcal{R}_i$ for u_i close to $u_+ = 1$. We restrict to odd index i and take $u_{i\pm 1} < u_i = 1$. The fact that $\tau_{u_{i-1},1}$ and $\tau_{1,u_{i+1}}$ are always finite combined with Lemma 2.2 in [15] ensures that $u'(\tau_{u_{i-1},1}; u_{i-1}, 1) = u''(\tau_{u_{i-1},1}; u_{i-1}, 1) = 0$ and $u'''(\tau_{u_{i-1},1}; u_{i-1}, 1) \neq 0$. Monotonicity of $u(t; u_{i-1}, 1)$ implies that $u'''(\tau_{u_{i-1},1}; u_{i-1}, 1) > 0$. Analogously $u'''(\tau_{1,u_{i+1}}; 1, u_{i+1}) < 0$ and $\mathcal{R}_i(u_{i-1}, 1, u_{i+1}) > 0$. Due to the uniqueness of the monotone loops (solutions $u(t, x, y)$) the function $\mathcal{R}_i$ is continuous on Ω_i and for sufficiently small $\delta > 0$ it holds that

$$\mathcal{R}_i(u_{i-1}, 1-\delta, u_{i+1}) > 0.$$

In the same way one can show that

$$\mathcal{R}_i(u_{i-1}, -1+\delta, u_{i+1}) > 0.$$

Therefore the same arguments as in the proof of Lemma 3.2.6 show that the set $M_{\epsilon,\delta}$, given by Definition 3.4.2, is an isolating neighborhood for ϵ and δ sufficiently small.

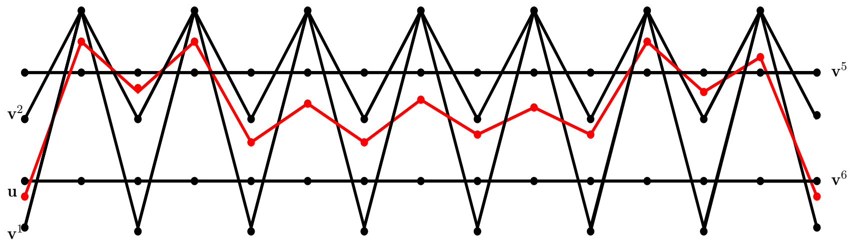

Figure 3.12: A representative of the braid class $[\mathbf{u} \text{ rel } \mathbf{w}]_I \in \mathcal{E}^1_{14}$ rel $\mathbf{w}$ for $q = 4$ and $I = \varnothing$

DEFINITION 3.4.2. Define

$$N_\epsilon = \{\mathbf{u} \in \text{cl}([\mathbf{u} \text{ rel } \mathbf{w}]_{\mathcal{E}}) : (-1)^i(u_{i+1} - u_i) \geq \epsilon\}$$

and let $M_{\epsilon,\delta}$ be a subset of N_ϵ such that $\mathbf{u} \in M_{\epsilon,\delta}$ if and only if

(1) $u_0 < \widehat{u}_0^\varepsilon$,
(2) $-1 + \delta < u_{2i+1} < 1 - \delta$ for $i \in \{2, 3, \ldots, 1 + p - q\}$,
(3) $(-1)^i u_i < (-1)^i u_i^\varepsilon$ for the remaining indices i.

Let $M^-_{\epsilon,\delta}$ denote the subset of $\partial M_{\epsilon,\delta}$ where the flow Ψ^t points out of $M_{\epsilon,\delta}$.

A similar homotopy to the one in the proof of Lemma 3.2.6 furnishes that

$$H_*(M_{\epsilon,\delta}, M^-_{\epsilon,\delta}) = H_*(h(\mathbf{u} \text{ rel } \mathbf{w})).$$

Non-triviality of the index $H_*(h(\mathbf{u} \text{ rel } \mathbf{w}))$ stated in the last lemma concludes the proof of Theorem 3.1.2.

LEMMA 3.4.3. *The index $H_*(h(\mathbf{u} \text{ rel } \mathbf{w}))$ is given by*

$$H_k(h(\mathbf{u} \text{ rel } \mathbf{w})) = \begin{cases} \mathbb{Z}, & \text{if } k = 2q - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4.1}$$

PROOF. The set $N = \text{cl}[\mathbf{u} \text{ rel } \mathbf{w}]$ is a $2p$-dimensional set which is homotopic to the unit cube $[0, 1]^{2p}$ and $\mathbf{u} \in N$ if and only if

(1) $u_0^1 \in [v_0^1, -1]$,
(2) $u_{2i+1} \in \begin{cases} [-1, 1] & : \text{ if } 1 < i < 2 + p - q, \\ [+1, v_1^1] & : \text{ otherwise,} \end{cases}$
(3) $u_{2i} \in \begin{cases} [-1, v_2^1] & : \text{ if } 1 < i < 3 + p - q, \\ [v_2^1, 1] & : \text{ otherwise.} \end{cases}$

If u_{2i+1} reaches the boundary and $i \in \{2, 3, \ldots, 1+p-q\}$ then the crossing number of the free strand $\mathbf{u}$ with the skeleton $\mathbf{w}$ increases. On the other hand if some other anchor point reaches the boundary of its interval then the crossing number decreases. Hence N^- is a set on which some u_j, different from u_{2i+1} with $i \in \{2, 3, \ldots, 1 + p - q\}$ or u_{2i} with $i \in \{2, 3, \ldots, 2+p-q\}$, attains the boundary. The pointed space $[N, N^-]$ is homotopic to the boundary of a $2q - 1$ dimensional cube. Hence $H_*(h(\mathbf{u} \text{ rel } \mathbf{w})) = H_*(N, N^-)$ is given by (3.4.1). $\square$

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Samenvatting

In dit proefschrift bestuderen we dynamische systemen afkomstig van vierde orde conservatieve differentiaalvergelijkingen. De gebruikte technieken combineren variationele en topologische methoden. De dynamica van deze systemen vindt plaats op energieniveaus die de faseruimte foliëren. Aangezien banen in drie-dimensionale energie oppervlakken liggen, kunnen ze beschouwd worden als knopen en vlechten (braids) op het energieniveau. Daarom identificeren we banen van het systeem met vlechtwerken in drie dimensies. De ruimte van vlechtwerken valt uiteen in klassen. We definiëren topologische invarianten voor vlechtwerken die ons in staat stellen forceringsresultaten voor periodieke oplossingen te bewijzen. Om analytische problemen met oneindig dimensionale ruimten te vermijden gebruiken we het begrip van gediscretiseerde vlechtdiagrammen. De variationele structuur van deze systemen wordt gebruikt om deze reductie van een oneindig dimensionale ruimte naar een eindig dimensionale ruimte te bereiken. Het artikel [20] laat zien dat elke begrensde oplossing op een regulier energieniveau van een twist system een aaneenschakeling is van monotone stukken, en door de extrema wordt gekarakteriseerd. De periodieke oplossingen kunnen gerepresenteerd worden in een eindig dimensionale ruimte van continue stuksgewijs lineaire vlechtdiagrammen. Conley index theorie voor niet-gedegenereerde en begrensde vlechtwerken, beschreven in [9], blijkt zeer efficiënt te zijn bij het aantonen van het bestaan van periodieke oplossingen.

Deze techniek wordt gebruikt in hoofdstuk 3 om een partiële ordening, gebaseerd op forceringsrelaties, te definiëren voor oplossingen van de Swift-Hohenberg vergelijking. Ook wordt het bestaan van oneindig veel periodieke oplossingen bewezen. De homologische Conley index van oneindig veel vlechtwerken, waarvan de complexiteit toeneemt met de dimensie, wordt berekend met behulp van volledige inductie naar de dimensie.

In hoofdstuk 2 breiden we de Conley index theorie uit naar gedegenereerde vlechtwerken. In dit geval is het vlechtwerk geen isolerende omgeving, maar is er een evenwichtspunt van de onderliggende gradient stroming op de rand van de gedegenereerde klasse van vlechtwerken. Dit probleem wordt opgelost door een zorgvuldige analyse van de gradient stroming rond dit punt. De eerste ideeën van deze analyse zijn terug te vinden in het artikel [2]. Door dit resultaat toe te passen op de Swift-Hohenberg vergelijking kunnen we het bestaan van een overvloed aan verschillende periodieke oplossingen aantonen en een verklaring geven voor de numeriek waargenomen bifurcatiediagrammen.

Summary

In this thesis we study dynamical systems coming from fourth order conservative equations. The methods used combine variational and topological methods. The dynamics of these systems is restricted to energy manifolds which foliate the phase space. Since the solutions lie on three dimensional energy surfaces the orbits can be regarded as knots and braids in the energy surface. Therefore we identify orbits of the system with braids in three dimensions. The space of braids decomposes into braid classes. We define topological invariants for braids that allow us to prove forcing results for periodic solutions. In order to avoid analytical difficulties of infinite dimensional spaces we use the concept of discretized braid diagrams. To pass from infinite dimensional space to a finite dimensional one, variational techniques are employed. The paper [20] shows that every bounded solution, on a regular energy level, of a twist system is a concatenation of monotone laps and can be encoded by its extrema points. The periodic solutions can be represented in a finite dimensional space of closed piecewise linear braid diagrams. Conley index theory for the proper and bounded braid classes developed in [9] proves very efficient for showing the existence of periodic solutions.

This technique is used in Chapter 3 to impose a partial order on solutions of the Swift-Hohenberg equation based on forcing relations. Also, the existence of infinitely many periodic solutions is proved. The homological Conley index of infinitely many braid classes, of which the complexity increases with their dimension, is computed by using mathematical induction on their dimension.

In Chapter 2 we extend the Conley index theory to non-proper braid classes. In this case the braid class is not an isolating neighborhood. There is a fixed point of the underlying gradient flow on the boundary of the non-proper braid class. We overcome this problem by a careful analysis of the flow near to this point. The ideas of this

analysis go back to [2]. By applying this result to the Swift-Hohenberg equation we show the existence of a plethora of different periodic solutions and give an explanation for numerically observed bifurcation diagrams.