# Optimal Diversity in Investments with Recombinant Innovation

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# Optimal Diversity in Investments with Recombinant Innovation[*]

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## Abstract

We address the notion of dynamic, endogenous diversity and its role in theories of investment and technological innovation. We develop a formal model of an innovation arising from the combination of two existing modules with the objective to optimize the net benefits of diversity. The model takes into account increasing returns to scale and the effect of different dimensions of diversity on the probability of emergence of a third option. We obtain analytical solutions describing the dynamic behaviour of the values of the options. Next we optimize diversity by trading off the benefits of diversity (due to recombinant innovation) and the benefits associated with returns to scale. We derive conditions for optimal diversity under different regimes of returns to scale. When the investment time horizon is beyond a threshold value, the best choice becomes diversity. This threshold will be larger the higher the returns to scale.

**JEL classification:** B52, C61, O31, Q55.

**Key words:** balance, optimal diversity, increasing returns, recombinant innovation, scale effects.

# 1  Introduction

When making decisions on investment and technological innovation, implicitly or explicitly choices are made about diversity of options, strategies or technologies. Nevertheless the notion of diversity has not been systematically addressed in related theoretical analyses. Here we propose a theoretical framework for the description of a generic innovative process resulting from the interaction of two existing but different technologies. The interaction will depend on how these two options match. Matching can occur via spillover or recombination, leading to modular innovation. The model will allow addressing the problem of optimal diversity in the context of modular innovation. The present paper takes the conceptual framework of van den Bergh (2008) as a starting point. The main idea is that in an investment decision problem where available options may recombine and give birth to an innovative option (technology), under particular regimes of returns to scale a certain degree of diversity of parent options can lead to higher benefits than specialization.

Usually in economics and finance, diversity is seen as conflicting with efficiency of specialization. Such efficiency is claimed on the basis of increasing returns to scale arising from fixed costs, learning, network and information externalities, technological complementarities and other self-reinforcement effects. Arthur (1989) studies the dynamics of competing technologies in cases where path dependence and self-reinforcement possibly lead to lock-in. This can be seen as a descriptive approach. Our approach instead is normative in that it studies the efficiency of the system of different options, considering total net benefits of technologies over time, including the innovation-related benefits of diversity.

The positive role of diversity is recognized in option value and real option theories (Arrow and Fisher (1974), Dixit and Pindyck (1994)). But this is concerned merely with the benefits of keeping different options open for later uncertain circumstances. These approaches moreover treat diversity as exogenous. Van den Bergh (2008) proposes an evolutionary model of recombinant innovation where diversity is endogenous, showing how diversity may contribute to the value of the overall system of investment options well beyond the mere opportunity of keeping decisions open. This is because diversity can produce an innovative option (technology) through recombination of already existing modula and through spillovers between technologies. Here we will elaborate this model deriving analytical solutions and general insights.

The economic and policy relevance of our analysis relates to myopia of economic agents. In real world decisions short term interests often prevail, since the advantages of increasing returns are perceived as more clear and certain than the advantages of diversity. The trade-off between short term efficiency and long term benefits from diversity is related to the issue of exploitation versus exploration (March, 1991). Here recombinant innovation can be regarded as a form of exploration.

A model of diversity fits into the approach of evolutionary economics as theorized by Nelson and Winters (1982), Dosi et al. (1988) and Potts (2000), among others. However evolutionary economics tends to avoid the notion or goal of optimality or efficiency in term of maximizing a net present value function. Our approach in fact can be seen as combining diversity-innovation ideas from evolutionary economics with optimality and cost-benefits analysis concepts of neoclassical economics. What economics calls spillover corresponds to recombination or cross-over in genetics and evolutionary computation and to modular innovation in biology and technological innovation studies. Adopting the view of an evolutionary approach we will talk of a population of parent options and an offspring which is the innovative option. Here we will deal with the smallest population possible, namely only two parent options. But the model can be extended to more than two parents as

shown by van den Bergh (2008), in order to analyse the effects of asymmetric diversity.

Within an evolutionary perspective innovation acts against selection in that it creates new individuals while selection reduces diversity according to relative performance or fitness. The first, innovation, is a force leading to disequilibrium. The second, selection, is an equilibrating force instead. Here we will concentrate on the role of innovation. A possible extension of the model could be the addition of selection, creating more complex dynamics.

Following Stirling (2007), we will consider three dimensions of diversity, namely variety, balance and disparity. The first refers to the number of starting options, the elements in the parent population. The second denotes the relative size or distribution of parent options. Finally disparity is the degree of difference between the options, that is, how distinct the different elements are. This represents a sort of distance in technology space when applying the model to technological innovation).

Another motivation of this work is the analysis of the transition from an old to a new basic socio-technological system, possibly given in by considerations of environmental sustainability (Geels, 2002). Diversity may help in preventing early or fast lock-in to an inferior technology. This consideration connects the themes of diversity and lock-in, which is a policy relevant issue, as it may shed light on the problem of how to unlock inferior but dominant technological regimes. In particular unlocking is relevant to fossil-fuel based electricity production, in view of global warming. A diversity analysis can provide useful insights regarding the transition to sustainable energy, since renewable energy technologies require investments in uncertain technological paths. This involves dealing with diversity, innovation, market liberalization, renewable energy and (de)centralized systems (the latter has implications for increasing returns). The general analysis offered here is ultimately meant to generate insights for environmental-energy policy.

The building blocks of our model are endogenous diversity, probabilistic recombinant innovation and returns to scale. The latter play a role in the optimization problem of final benefits from the overall system made of parent and innovative options. There is a trade-off in pushing for more diverse systems: on the one hand the contribution from innovation increases because of a higher probability of recombination, which is assumed to be positively correlated with diversity. On the other hand we lose benefits from lost opportunities of enjoying increasing returns to scale: this can be seen as the cost of diversity. We will see that under some threshold level of returns to scale the benefits of diversity are larger than its costs.

This paper is structured in the following way. Section 2 presents a simple "pilot" model to illustrate the main concepts and their interactions. Section 3 generalizes the model by developing a more general structure of diversity. In section 4 we solve the model obtaining a general solution for the value of the innovative option as a function of time. In section 5 we introduce a size effect into the probability of recombinant innovation. In section 6 we address the optimization problem for the different versions of the model, presenting a complete set of conditions under which either diversity or specialization is the best choice. We also study the effect of the time horizon on the optimal solution. Section 7 concludes and provides suggestions for further research.

# 2 A "pilot" model

Consider a system of two investment options that can be combined to give rise to a third one. Let $I$ denote cumulative investment in the parent options while investment $I_3$ in the new option only occurs if this arises, which happens with probability $P_E$. The growth

rates of parent options are proportional to the capital invested, with shares $\alpha$ and $1 - \alpha$. The growth of the innovative option is non-linear instead and depend on the way and the extent that parent options match. We assumed no depreciation and that the allocation of capital is constant through time. If we indicate the value of the investment options with $O_1$ and $O_2$ for the parents and $O_3$ for the innovative option, the dynamics of the system is described by the following set of equations:

$$
\begin{aligned}
\dot{O}_1 &= I_1 = \alpha I \\
\dot{O}_2 &= I_2 = (1 - \alpha)I \\
\dot{O}_3 &= P_E(O_1, O_2)I_3
\end{aligned}
\tag{1}
$$

The optimization problem that we address is how to set an $\alpha$ that maximizes the final total benefits of parent and innovative options.

The matching factor $P_E$ denotes a probability of recombinant innovation or *Emergence* of the third option. It depends on the interaction of the two preexisting options that form the system under analysis together with the innovative option. The rate of growth of this innovative option can thus be interpreted as the expected value of this recombinant innovation process: $\dot{O}_3 = E[Innovation]$ where $Innovation$ is a random variable representing the value of the innovative option. Such a random variable is subject to a binary event where the new option comes out with probability $P_E$ and nothing happens with probability $1 - P_E$. Then the expected value is simply $P_E$ times the capital invested in the new option $I_3$. We express the probability factor by the product of two main ingredients, namely the balance of the parent options and a scaling factor $\pi$ which can be interpreted as the efficiency of the R&D process underlying the recombinant innovation:

$$
P_E(O_1, O_2) = \pi B(O_1, O_2) = 4\pi \frac{O_1 O_2}{(O_1 + O_2)^2}
\tag{2}
$$

In principle $\pi$ is time dependent since innovation efficiency responds to learning and to progress in general. Here we keep it constant. Diversity is here expressed as the balance of parent options: the more equally present they are, the larger the probability of emergence. A system with two options close to each other is diversified, while the situation with one option close to zero (or relatively very small) corresponds to specialization. The balance function has the following features: $B(O_1, O_2) \in [0, 1]$, $B(O_1 = O_2) = 1$ (maximum diversity or perfect balance), $\lim_{O_i \to 0} B(O_i, O_j)|_{O_j = const} = 0$ with $i, j = 1, 2$ and $i \neq j$. The following figure shows a graph of the diversity function.

Assuming that investment in parent options begins at time $t = 0$, their value at time $t$ is simply $O_1(t) = \alpha I t$ and $O_2(t) = (1 - \alpha)It$. Under this assumption the balance function is independent of time: $B = 4\alpha(1 - \alpha)$. Consequently the probability of emergence is constant and only depends on the initial allocation $\alpha$. The innovative option grows linearly with time then:

$$
O_3(t) = 4\pi I_3 \alpha(1 - \alpha)t
\tag{3}
$$

The optimization problem of this investment decision is addressed considering the joint benefits of parents and innovative options. In order to model the trade-off between diversity and scale advantages of specialization we introduce a returns to scale parameter $s$. This acts on the cumulative investment in each option, in order to capture learning over time. We can then express the overall benefits as follows:

$$
V(\alpha; T) = O_1(T; \alpha)^s + O_2(T; \alpha)^s + O_3(T; \alpha)^s
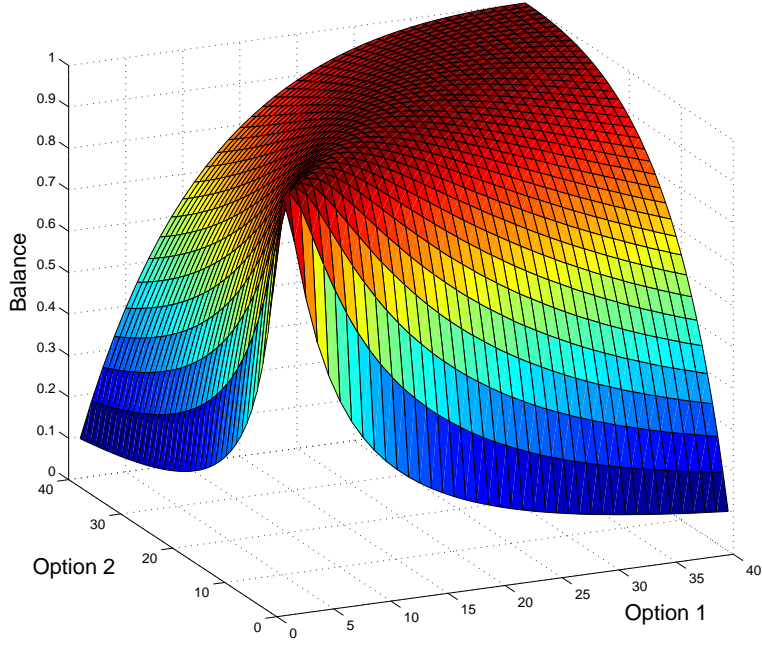\tag{4}
$$

4

Figure 1: Graph of the diversity function with two parent options

Where $t = T$ is the time horizon. According to this expression, once we substitute the expressions of options' values, the maximization problem of the investment decision can be written as

$$\max_{\alpha \in [0,1]} V(\alpha; T) = T^s I^s \big[ \alpha^s + (1 - \alpha)^s + C^s \alpha^s (1 - \alpha)^s \big] \tag{5}$$

where $C = \frac{4\pi I_3}{I}$. This factor weights the contribution of diversity to total benefits. As intuition may tell, such a contribution will be larger for a larger probability of recombinant innovation $\pi$ and a lower total investment ratio $I$. It is useful to normalize the benefits function to its value value in case of specialization. Since the system is symmetric we have $V(\alpha = 0; T) = V(\alpha = 1; T) = I^s T^s$. Then we define

$$\tilde{V}(\alpha) \equiv \frac{V(\alpha; T)}{I^s T^s} = \alpha^s + (1 - \alpha)^s + C^s \alpha^s (1 - \alpha)^s \tag{6}$$

Depending on returns to scale $s$ and the factor $C$, $\tilde{V}$ will be maximum for $\alpha = 1/2$ (maximum diversity) or for either $\alpha = 0$ or $\alpha = 1$ (specialization). It can be instructive to look at some examples of the curve $\tilde{V}(\alpha)$ for different values of returns to scale $s$ and efficiency $\pi$. Setting $I = 4I_3$ we have $C = \pi$. Figure 2 reports the normalized benefits curves for increasing returns to scale with $s = 1.2$ and six different values of the factor $\pi$. As one can see either specialization is to be preferred or diversity, depending on the efficiency of the recombinant innovation process as captured by the probability factor $\pi$. Under this perspective there will be a threshold value $\overline{\pi}$ for this probability such that for $\pi < \overline{\pi}$ the optimal decision is specialization while for $\pi > \overline{\pi}$, i.e. a sufficiently efficient recombinant innovation process, diversity is optimal. Conversely, given an intensity of recombinant innovation $\pi$ one may want to understand what is the turning point $\overline{s}$ of returns to scale at which maximal diversity ($\alpha = 1/2$) becomes optimal[1]. This is given by

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[1]Here and from now onwards with "diversity" we always mean maximum diversity, which is represented
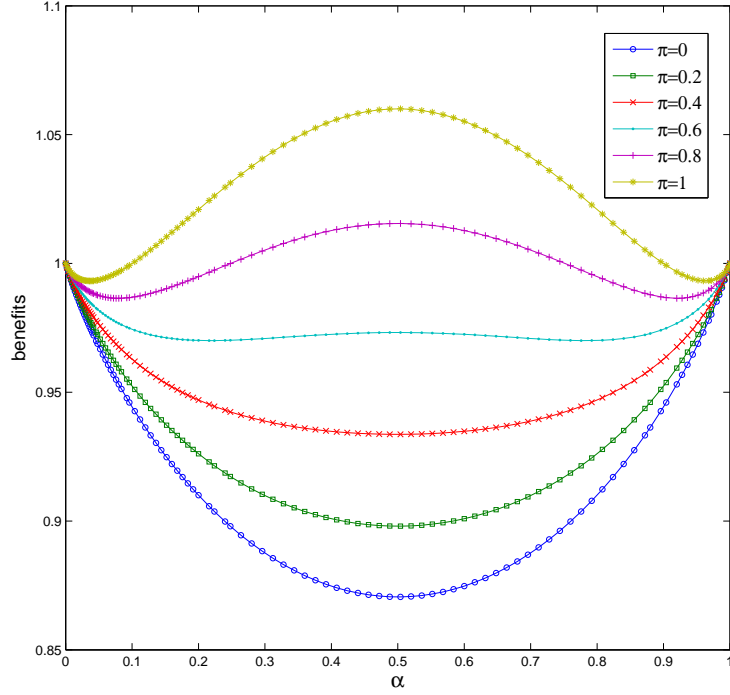
Figure 2: Normalized final benefits $\tilde{V}$ as a function of the investment share $\alpha$ under increasing returns to scale ($s = 1.2$) for different values of the innovation efficiency factor $\pi = 0, 0.2, 0.4, 0.6, 0.8, 1$.

the threshold level $\bar{s}$ that solves the equation

$$\tilde{V}\big(\alpha = 1/2\big) = \frac{1}{2^{\bar{s}}}\left[2 + \left(\frac{C}{2}\right)^{\bar{s}}\right] = 1 \tag{7}$$

If $C = 1$ (for instance with $I = 4I_3$ and $\pi = 1$) the threshold level of returns to scale is given by the equation $2^{s+1} + 1 = 2^{2s}$ and $\bar{s} \simeq 1.2715$. For $s = 1/2$ (decreasing returns) we have $\tilde{V}(1/2)_{s=1/2} = \big(2 + \sqrt{C/2}\big)/\sqrt{2}$ which is always larger than one. For $s = 1$ (constant returns to scale) we have $\tilde{V}(1/2)_{s=1} = 1 + C/4$ which is again always larger than one. Since $C \geq 0$, assuming that a positive capital $I_3$ is assigned to innovation after emergence, a straightforward computation gives the following result:

**Proposition 1.** *It holds $\bar{s} \geq 1$ and $\bar{s} > 1$ iff $\pi > 0$.*

**Corollary 1.** *For all decreasing or constant returns regimes a maximum value of total final benefits is realized for the allocation $\alpha = 1/2$, i.e. for maximum diversity*

This is true no matter what value the factor $C$ assumes[2]. In other words, in all cases of decreasing returns to scale up to constant returns it is better to have maximum diversity and divide equally the investment among the two parent options. One interesting result is that diversity is optimal also in absence of recombinant innovation, when returns to scale are low enough. This situation is summarized in figure 3. Note how for $C = 0$ the threshold

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by $\alpha = 1/2$ since the system is symmetric.

[2]Consider the function $f(s) \equiv \big(2 + (C/2)^s\big)/2^s$. The statement is true if $f(s) \geq 1 \; \forall s \in [0,1]$. $f$ is a decreasing function for fixed $C$, $f'(s) < 0 \; \forall s \geq 0$. For fixed $s$ instead $f$ is an increasing function of $C$. When $C = 0$ $f(1) = 1$ and $f(s) \geq 1 \; \forall s \in [0,1]$. When $C > 0$ $f(1)|_{C>0} > f(1)|_{C=0} = 1$ and $f(s)|_{C>0} > f(s)|_{C=0} = 1 \; \forall s \in [0,1]$. This proves proposition 1.
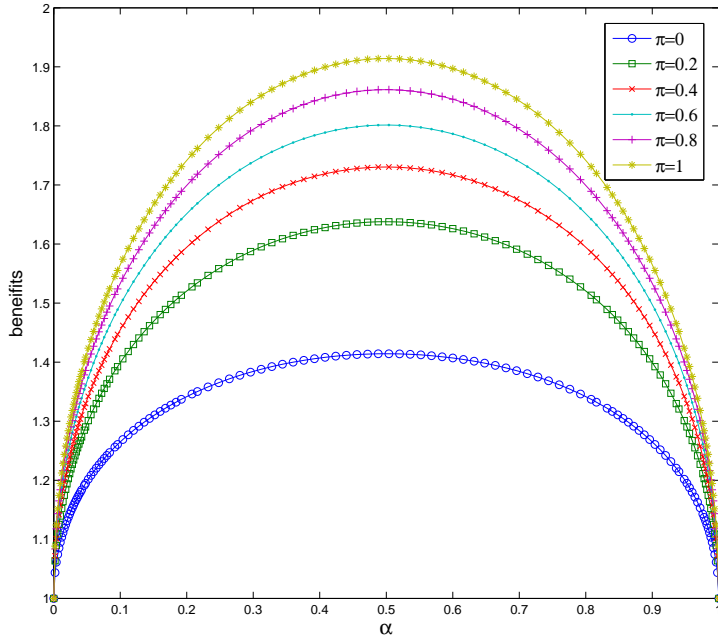
Figure 3: Normalized final benefits $\tilde{V}$ as a function of the investment share $\alpha$ under decreasing returns to scale ($s = 0.5$) for different values of the innovation efficiency factor $\pi = 0, 0.2, 0.4, 0.6, 0.8, 1$.

is $\overline{s} = 1$ and $\tilde{V} = 1$ for all $\alpha$: with constant returns and no recombinant innovation the benefits function is constant.

In order for $\alpha = 1/2$ to give a minimum of benefits the condition $s > \overline{s}$ must hold. When $C > 0$ we have $\overline{s} > 1$, according to equation (7)[3]. For $s = 3/2$ for instance we have $\tilde{V}(1/2)_{s=3/2} = \left[2 + (C/2)^{3/2}\right]/2^{3/2}$. If $C < \overline{C} \simeq 1.764$ we have that $\alpha = 1/2$ is not a maximum of benefits and specialization is optimal, then.

The case of increasing returns to scale is the most interesting and maybe also the one that better represents real cases of technological innovation or investment projects. In this regime we study the tradeoff between scale advantages and benefits from diversity. We have already seen the shape of benefits curve for some values of $C$ when $s = 1.2$ (figure 2). If the probability of recombinant innovation and the ratio $I_3/I$ are insufficiently high, returns to scale may be too high for diversity to be the optimal choice. In figure 2 this holds for the bottom four curves. In general we have the following result, which completes Proposition 1:

**Corollary 2.** *Diversity* $\alpha = 1/2$ *can be optimal also with IRTS ($\overline{s} > 1$) provided that the probability of recombination $\pi$ is large enough.*

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[3]In principle $C$ can have any positive value, but for eq. (7) to hold a higher $C$ requires a higher $\overline{s}$. In the limit, as $C$ approaches 4 such a condition would require an infinite value for the parameter $\overline{s}$.

# 3 A general model

## 3.1 Innovation probability and diversity factors

Now we will build a more general model of recombinant innovation which will relax some of the assumptions of the "pilot" model and at the same time we will enter the structure of the probability of emergence. We will allow for non-zero initial values of parent options and will consider a marginally diminishing effect of options' size on $P_E$. The optimization of diversity is addressed for the more general model then, with successive steps of increasing complexity.

We define the probability of emergence of the innovative option $P_E$ as depending positively on the diversity $\Delta$ of the two parent options' system and positively on their disparity[4] $D$:

$$P_E(O_1, O_2) = k \frac{\Delta(O_1, O_2)}{D^\gamma} \tag{8}$$

The factor $k$ expresses that part of the intensity of recombinant innovation that is not captured by diversity $\Delta$ and disparity $D$. Besides this $k$ can be used to normalize the probability of emergence to one[5]. The parameter $\gamma$ allows for a non-linear effect of disparity $D$. It can be seen to express the concept of "cognitive distance" between two technologies: it may be that two ideas are very different but historical or geographical events make the cognitive distance small, for instance through interdisciplinary research (van den Bergh, 2008).

As observed by Stirling (2007) diversity is a multidimensional concept. In a study of innovation he indicates three dimensions: variety, disparity and balance. Diversity can be expressed as follows:

$$\Delta(O_1, O_2) = \delta NDB(O_1, O_2) \tag{9}$$

Here the three dimensions of diversity are present as multiplicative factors that positively affect diversity. Variety $N$ and disparity $D$ are set exogenously while balance $B$ is a function of the values of the existing options. The factor $\delta$ is a scaling parameter that can be set to normalize maximum diversity to one, given the values of $N$ and $D$. Variety indicates the number of parent options present (technologies, organizations, investment projects, firms, etc.). We fix $N = 2$ in our case. Disparity captures how "different" or how far apart in technology space the two options are. In principle $D$ can assume any positive value since it expresses a degree of differentiation among two alternatives (sort of distance between different species, as in Weitzman, 1992). Balance expresses how (un)equally different options are present in a population, assuming that the more balanced the more diversified a system is. While disparity expresses the substantial diversity of the interacting options, the degree of difference between their nature, balance models their difference in size, a sort of "mechanical" diversity as if we could "weight" options' value on a balance, indeed.

The mathematical expression of $P_E$ shows that disparity has two opposite effects on the probability of recombinant innovation. An example from technological innovation may be instructive. If two very different and apparently unrelated technologies are considered, the

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[4]A basic assumption of the model is that $P_E$ depends deterministically on other quantities and parameters. Assigning a deterministic value to $P_E$ is like to deal with a $\delta$-Dirac probability distribution, centered on $(O_1, O_2)$: all values other than $(O_1, O_2)$ have zero probability of occurring, while the probability distribution value in $(O_1, O_2)$ is infinite. The integral of the probability distribution leaves us with the finite value $P_E$.

[5]in terms of the probability of recombinant innovation $\pi$ that we introduced in the last section, this amounts to require that $\pi \leq 1$.

probability that they meet to produce a new technology through recombinant innovation is very low. Still, if this happens, we may expect the new technology to have a big impact and consequently the rate of growth of its value to be very large. Disparity plays opposite roles within these two effects, the probabilistic one and the innovative one, as the two occurrences of disparity in the expression of the probability of emergence $P_E$ show. The overall effect will depend on the parameter $\gamma$. If $\gamma < 1$ the innovative effect prevails and a larger disparity means a larger $P_E$. If $\gamma > 1$ instead, the probabilistic effect is stronger and disparity negatively affects $P_E$.

The balance function $B(O_1, O_2)$ ranges from the minimum value 0, which is the limiting case where only one option is present, to the maximum value 1 corresponding to perfect symmetry. The idea here is that a higher symmetry results in a larger probability of emergence of the innovative option.

## 3.2 The balance function

The balance function is defined in the positive octant of a $n$-dimensional space. If $n = 2$ as here $B$ is defined on the positive quadrant. A functional specification of the balance of two options $x$ and $y$ is defined by the following properties:

1. it is symmetric in its arguments $B(x, y) = B(y, x)$

2. the maximum value (normalized to one) is attained on the diagonal $B(x, x) \geq B(x, y)$ $\forall x, y \geq 0$

3. the minimum value (lowest balance) is realized when one of the two options has a zero value: $B(x, 0) = B(0, x) < B(x, y)$ $\forall y > 0$

4. it is homogeneous of degree zero: $B(\lambda x, \lambda y) = B(x, y)$

The latter means that the balance of two quantities can be expressed as a function of their ratio $b = O_1/O_2$ (simply put $\lambda = 1/x$) and does not depend on the two values separately. Moreover the balance goes to 0 when one option growths indefinitely while the other stays constant:

$$\lim_{O_i \to \infty} B(O_i, O_j)\Big|_{O_j = \text{const}} = 0 \qquad i, j \in 1, 2, i \neq j$$

The functional specification of the balance that we adopt is the following[6]:

$$B(O_1, O_2) = 1 - \frac{(O_1 - O_2)^2}{(O_1 + O_2)^2} = 4\frac{O_1 O_2}{(O_1 + O_2)^2} \qquad (10)$$

This is the so-called "Gini balance". The main reason for such a choice is the differentiability in $O_1 = O_2$. Expressed as a function of the ratio the above specification reads $B(b) = 4\frac{b}{(1+b)^2}$.

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[6]Other specifications are possible, for instance $B(O_1, O_2) = 1 - \frac{|O_1 - O_2|}{O_1 + O_2}$ and $B(O_1, O_2) = \frac{min\{O_1, O_2\}}{max\{O_1, O_2\}}$ (see also Stirling, 2007). A detailed analysis of the latter specification is available at request. The case $O_1 = O_2 = 0$ is excluded by all these specifications. This is rather a degenerate and irrelevant case however, as we are only interested in systems with at least one option ($\exists\ i = 1, 2\ |\ O_i > 0$). Otherwise we can always define $B(0, 0) = lim_{O_1, O_2 \to 0} B(O_1, O_2) = 1$.

## 3.3 The efficiency factor

Equation (8) contains a scaling factor, $k$. Its value must comply with a normalization of the probability of emergence, that is, it must assure $P_E \leq 1$. The diversity $\Delta$ assumes values in a compact interval $[0, \Delta_{max}]$, depending on the values of variety $N$, disparity $D$ and balance $B$. Diversity can be normalized as well. Variety in our case is set to $N = 2$ since we deal with two parent options. As for disparity, we can restrict ourselves to two discrete values, $D = 1$ (no disparity, identical options) and $D = 2$ (maximum disparity). Balance ranges in the interval $[0, 1]$. Looking at equation (9), the maximum value $\Delta_{max}$ is attained for $N = 2$ and $D = 2$. Setting $\delta = \frac{1}{4}$ we have $\Delta_{max} = 1$. We have the following limit cases:

$$
\begin{aligned}
\Delta &= \Delta_{min} = 0 \quad \text{for } N = 1, B = 0, D = 1 \\
\Delta &= \Delta_{max} = 1 \quad \text{for } N = 2, B = 1, D = 2
\end{aligned}
$$

If we substitute equation (9) into equation (8), the probability of emergence is given by

$$
P_E = \frac{k}{4} N D^{1-\gamma} B
$$

Let us define $\pi = \frac{k}{4} N D^{1-\gamma}$, which is a sort of static probability factor in the expression of $P_E$: the higher the number of available options $N$, the more likely recombinant innovation is, while the contribution of disparity $D$ depends on $\gamma$. Normalization is achieved by requiring that $\pi \leq 1$, which translates in the following condition for $k$

$$
k \leq \frac{4 D^{\gamma-1}}{N} \tag{11}
$$

The factor $k$ captures all other factors of influence on the recombinant innovation process. For instance, two recombinant innovation processes with the same number of parent options $N$, disparity $D$ and balance $B$ may render different values of the innovation likelihood $P_E$ due to different values of recombination efficiency $k$, possibly reflecting different levels of knowledge (education) or experience.

# 4 Solving the dynamic model

Our model of recombinant innovation is made of the system of equations (1) together with the definitions (8) and (9). This section studies the time patterns of the differential equations system (1). These time patterns will then be used to determine optimal diversity maximizing the value function of the options in section 6. We assume a constant allocation over time

$$
\frac{I_1}{I_2} = \frac{\alpha}{1-\alpha} \equiv x(\alpha) \tag{12}
$$

This setting results in a constant linear growth (accumulation) of parent options $O_1$ and $O_2$. The time pattern of the innovative option is non-linear thought. With respect to the pilot model the only difference here is that initial values are different from zero. The value of the investment option at time $t$ is the following:

$$
\begin{aligned}
O_1(t) &= O_{10} + I_1 t \\
O_2(t) &= O_{20} + I_2 t \\
O_3(t) &= I_3 \int_0^t P_E(s) ds
\end{aligned} \tag{13}
$$

When the initial values $O_{10}$ and $O_{20}$ are different from zero this model describes a situation where two existing ideas (technologies, products, institutions, etc.) meet and are matched with the aim of inventing a third one ($O_3(0) = 0$). The first two equations of system (13) are independent and autonomous: a coupling effect only exists between these two and the third equation. Let us focus on the third equation, therefore. The probability of emergence is $P_E(t) = \pi B\big(O_1(t), O_2(t)\big)$, where the scaling factor $\pi \in [0, 1]$ is the probability of recombinant innovation (which in turn depends on variety and on disparity in the way explained in section 3.3). The value of the innovative option at time $t$ is then

$$O_3(t) = I_3 \int_0^t P_E(s)ds = \pi I_3 \int_0^t B(s)ds \tag{14}$$

Before computing the integral (14) we will analyse the dynamic behaviour of the balance function. If the initial value of parent options is zero ($O_{01} = O_{02} = 0$) the balance is constant and defined only for $t > 0$:

$$B(t) = 4\frac{O_1(t)O_2(t)}{\big(O_1(t) + O_2(t)\big)^2} = 4\frac{\alpha It(1 - \alpha)It}{\big(\alpha It + (1 - \alpha)It\big)^2} = 4\alpha(1 - \alpha) = B(\alpha)$$

The function $B(\alpha)$ is a portion of a parabola enclosed in the interval $[0, 1]$ with a maximum value of 1 for $\alpha = 1/2$.

If instead we relax the assumption and allow for generic initial values $O_{10}, O_{20} \neq 0$ we obtain the following function of time

$$B = 4\frac{(O_{10} + \alpha It)(O_{20} + (1 - \alpha)It)}{(O_{10} + O_{20} + It)^2} \longrightarrow 4\alpha(1 - \alpha) \tag{15}$$

where the last limit holds for $t >> O_{i0}/(\alpha I)$, $i = 1, 2$. We see that in the long run the balance converges to a constant value which depends only on the investment shares and is the same function that one has with zero initial values. We can state the following then:

**Proposition 2.** *In the long run the balance converges to the constant value $B(\alpha) = 4\alpha(1 - \alpha)$.*

The dynamics of the balance in the transitory phase ($t \simeq O_{i0}/(\alpha I)$) is different depending on the settings of the systems. There are seven different cases:

**case 1.** $O_{10} < O_{20}$ and $\alpha < 1/2$

Option $O_1$ remains smaller. The balance $B(t)$ never reaches its maximum value and follows a monotonic trend with the following sub-cases:

- if $\frac{O_{10}}{O_{20}} > \frac{\alpha}{1-\alpha}$ then $B(t)$ is decreasing

- if $\frac{O_{10}}{O_{20}} < \frac{\alpha}{1-\alpha}$ then $B(t)$ is increasing

**case 2.** $O_{10} < O_{20}$ and $\alpha > 1/2$

Option $O_1$ starts smaller but grows at a faster rate than option $O_2$. There is a time $t = t^*$ when the two options are equal and the balance is maximal.:

$$t^* = \frac{O_{20} - O_{10}}{(2\alpha - 1)I} \tag{16}$$

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Such a value has to be positive and then it is defined only in this case and in case 3 (see below).

**case 3**. $O_{10} > O_{20}$ and $\alpha < 1/2$

This case is equivalent to case 2 with exchanged roles of options $O_1$ and $O_2$.

**case 4**. $O_{10} > O_{20}$ and $\alpha > 1/2$

This case is the mirror of case 1.

**case 5**. $O_{10} \neq O_{20}$ and $\alpha = 1/2$

The two options start with different values but eventually become equal when $t \gg t^*$. In the long run the balance reaches a maximum, as the following limit shows

$$4\frac{(O_{10} + \frac{It}{2})(O_{20} + \frac{It}{2})}{(O_0 + It)^2} \longrightarrow 1$$

**case 6**. $O_{10} = O_{20}$ and $\alpha \neq 1/2$

The balance is maximal at $t = 0$ and then decreases monotonically, converging to the value given by (15).

**case 7**. $O_{10} = O_{20}$ and $\alpha = 1/2$

This is the setting in which the balance is kept constant at its maximum value. Defining $O_0 = O_{10} + O_{20}$, we have

$$B(t) = 4\frac{(\frac{O_0}{2} + \frac{It}{2})^2}{(O_0 + It)^2} = 1$$

The last case is what one would desire to have if aiming at maximizing the probability of emergence, since it keeps the balance at its maximum forever. Nevertheless this is a particular case of a more general configuration in which the balance stays constant through time:

**Proposition 3.** *The balance is constant through time and equal to $B(\alpha) = 4\alpha(1 - \alpha)$ if and only if*

$$\frac{O_{10}}{O_{20}} = \frac{\alpha}{1 - \alpha} \tag{17}$$

The proof of this proposition is in appendix A. This configuration falls into cases 1, 4 and 7 of the list presented above. As a function of time the balance may or may not have a critical point $t^*$, that is a time when it reaches its maximum value: this depends on the relative value of the ratio of initial values and the ratio of investments shares. In words, if an option starts smaller but has higher investment share, it will overcome the other option eventually. The time $t^*$ when this happens is a maximum point for the balance. The following figure shows two examples of time evolution of the balance function. Here we have set $I = 4$, with initial values $O_{10} = 1$ and $O_{20} = 2$. Example $B$ represents case "2", with $\alpha = 3/4$ (but we would obtain a similar pattern in case 3). In the latter case there is a time $t^* = 1/2$ when the balance is maximum (equal to one). As we will see below this corresponds to the trajectory in $(O_1, O_2)$ space crossing the diagonal. Example $A$ falls into case "1" and the balance happens to be monotonically decreasing because $O_{20}/O_{10} < (1 - \alpha)/\alpha$. If we would have set $\alpha = 2/5$ for instance, $B(t)$ would have been monotonically increasing. In general $B(t)$ is decreasing when $\frac{\alpha}{1-\alpha} < \frac{O_{10}}{O_{20}}$ and increasing when $\frac{\alpha}{1-\alpha} > \frac{O_{10}}{O_{20}}$.
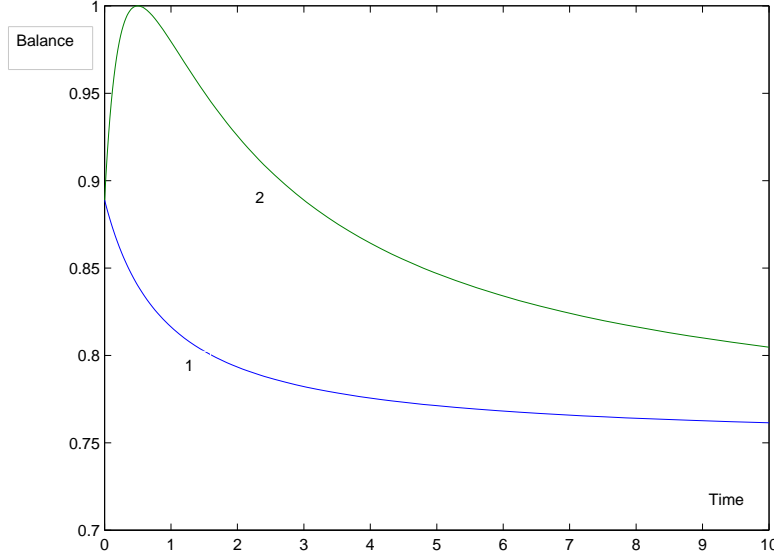
Figure 4: Two cases for the balance as a function of time ($I = 4$, $O_{10} = 1$ and $O_{20} = 2$). Case 1 has $\alpha = 1/4$. Case 2 has $\alpha = 3/4$

The dynamics of this simple two-dimensional system can be understood also looking at options trajectories in $(O_1, O_2)$ space (figure 5). Starting from the expression of the two options' ratio

$$\frac{O_1}{O_2} = \frac{O_{10} + \alpha It}{O_{20} + (1 - \alpha)It}$$

one can eliminate time and express one option in terms of the other:

$$O_2 = O_{20} - \frac{1 - \alpha}{\alpha}O_{10} + \frac{1 - \alpha}{\alpha}O_1$$

The starting point ($t = 0$) of each trajectory is determined by the initial values $(O_{10}, O_{20})$. The slope is the ratio of investment shares. In the following figure we report the trajectories representing the first five cases of the previous list plus a case of constant balance. In case "6" the system starts on the diagonal and moves away. In case "7" the trajectory coincides with the diagonal, representing the perfect maximal balance of two options.

Now we proceed to the integration of balance, giving the value of the innovative option at time $t$. We assume that $k = 4D^{\gamma-1}/N$, so that $\pi = 1$ (maximal efficiency of recombinant innovation). Equation (14) becomes

$$O_3(t) = 4I_3 \int_0^t \frac{(O_{10} + \alpha Is)(O_{20} + (1 - \alpha)Is)}{(O_0 + Is)^2}ds \tag{18}$$

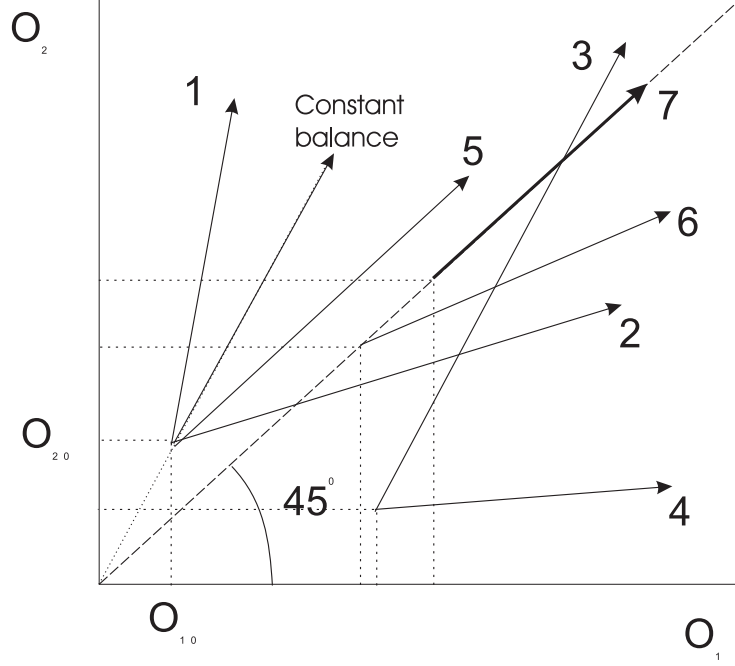The detailed solution of this integral is reported in appendix B. The final result is the following:

Figure 5: Trajectories of the two parent options in the $(O_1, O_2)$ space. Trajectory "1" has $O_{10} < O_{20}$ and $\alpha < 1/2$, trajectory "2" has $O_{10} < O_{20}$ and $\alpha > 1/2$, trajectory "3" has $O_{10} > O_{20}$ and $\alpha < 1/2$, trajectory "4" has $O_{10} > O_{20}$ and $\alpha > 1/2$, trajectory "5" has $O_{10} \neq O_{20}$ and $\alpha = 1/2$, trajectory "6" has $O_{10} = O_{20}$ and $\alpha < 1/2$ and trajectory "7" has $O_{10} = O_{20}$ and $\alpha = 1/2$. The trajectory of constant balance has a slope equal to the ratio of the coordinates of the starting point.

$$O_3(t) = 4\frac{I_3}{I}\left[ \quad (O_{10} - \alpha O_0)^2\left(\frac{1}{O_0 + It} - \frac{1}{O_0}\right) + \right. \tag{19}$$
$$\left. + \quad (O_{10} - \alpha O_0)(1 - 2\alpha)\ln\frac{O_0 + It}{O_0} + \alpha(1 - \alpha)It \quad \right]$$

If condition (17) holds, $O_{10} = \alpha O_0$ and the expression of the innovative option reduces to $O_3(t) = 4I_3\alpha(1-\alpha)t$. This linear behaviour is what we have in the early stages of existence of the innovation, namely when $It << O_0$. In the long run instead the logarithmic term can not be neglected and the value of innovation is approximately given by

$$O_3(t) \simeq 4\frac{I_3}{I}\left[(O_{10} - \alpha O_0)(1 - 2\alpha)\ln\frac{It}{O_0} + \alpha(1 - \alpha)It\right] \tag{20}$$

The first term in this expression is constant and does not affect the growth rate of the innovative options. The coefficient of the logarithmic term will determine whether the time pattern of the innovative option will be concave (positive sign) or convex (when the sign is negative). The first case arises when $\alpha < 1/2$ and $\alpha < O_{10}/O_0$ or $\alpha > 1/2$ and $\alpha > O_{10}/O_0$. These are exactly the conditions of cases "3" ($\alpha < 1/2$ and $O_{10} > O_{20}$) and "2" ($\alpha > 1/2$ and $O_{10} < O_{20}$) respectively, when the balance has a critical point $t^*$. The convex time pattern occurs when balance does not have a critical point instead. For example take $O_0 = 3$, $O_{10} = 1$, $O_{20} = 2$, $\alpha = 2/3$. Since $O_{10}/O_{20} = 1/2 < \alpha/(1 - \alpha) = 2$ we have that option 3 follows a concave time pattern, $O_3(t) = \frac{4}{3}\left[2t + \ln(1 + t) - \frac{t}{1+t}\right]$.

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# 5 A size effect

## 5.1 Specifying the size effect

Up to now the probability of emergence of a third option was related only to the diversity index as a function of the two starting options and the dynamics of the system was driven only by the balance. We now introduce a size effect into the probability of emergence of the third option. This is meant to catch the positive effect that a larger size of one or both options has on the probability of emergence, some kind of scale economy effect in the innovation process. If the size effect is captured by a factor $S(O_1, O_2)$, the probability of emergence of the third option can be expressed as

$$P_E = \pi B(O_1, O_2) S(O_1, O_2) \tag{21}$$

The size effect is defined to have the following properties. First it is increasing in the size of each of the two starting options with diminishing effects (i.e. first derivatives and negative second derivatives). Secondly it should not overlap with the balance factor, which means that only the total sum of the sizes of options matters for the size effect and not their distribution. Third, the size function must be bounded, to guarantee that the probability $P_E$ ranges in the interval $[0, 1]$. One attractive functional specification then is the following:

$$S(O_1, O_2) = 1 - e^{-\sigma(O_1 + O_2)} \tag{22}$$

The parameter $\sigma$ captures the sensitivity of the system to the size: the higher its value, the stronger is the size effect in the sense that the factor $S$ is larger for a given size of the two options[7].

The marginal effect of the options values is diminishing as we can see by taking the first derivative of the factor $S$ with respect to the sizes. Moreover this marginal effect is the same for each of the options values:

$$\frac{\partial S}{\partial O_1} = \frac{\partial S}{\partial O_2} = \frac{\partial S}{\partial O} = \frac{\sigma}{e^{\sigma O}} \tag{23}$$

where $O = O_1 + O_2$. We see how the particular contribution of each option is indistinguishable within the size factor. in other words, the size factor is the same with two equal options sizes as with two very different values that sum up the same total size.

If now we substitute in the size factor the explicit values of the two parent options at time $t$ (which are not affected by the presence of a size factor into the probability of emergence of a third option), we obtain the following expression for the probability of emergence

$$P_E(t) = \pi B(t) \left( 1 - e^{-\sigma(O_0 + It)} \right) \tag{24}$$

where $O_0 = O_{10} + O_{20}$ as before. Note how the effect of size on the probability of emergence is independent of the time $t = 0$ when the investment was done. Only the overall size $O(t) = O_0 + It$ counts. In particular the size factor is not necessarily zero at $t = 0$, as the initial values of the two starting options affect the probability of emergence not only through the balance but also directly via the size effect.

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[7]Alternatively one could allow for heterogeneous effects with the specification $1 - e^{-\sigma_1 O_1 - \sigma_2 O_2}$. For example this one may address two different technologies operating in different sectors with different sensitivities $\sigma_1$ and $\sigma_2$.

The size effect converges to one as the cumulative investment becomes sufficiently large ($It >> O_0$) so that the exponential term is negligible. This is consistent with normalization of the probability $P_E(t)$ as shown by the following limit:

$$lim_{t \to \infty} S(t) = 1 \qquad (25)$$

## 5.2 Time pattern of $P_E$ with constant balance

In order to understand the impact of the size of parent options on the innovation process we look at the behaviour of the probability of emergence through time for few different values of the balance in the particular setting in which the balance is constant (condition (17)). Doing so we select the dynamic size effect. Assume that the efficiency of recombinant innovation is maximal ($\pi = 1$), so that $P_E(t; \alpha) = B(\alpha)S(t)$, with $B(\alpha) = \alpha(1 - \alpha)$. Considering the previous analysis of the balance and the specification of the size effect, in the long run we have

$$lim_{t \to \infty} P_E(t) = B(\alpha) \qquad (26)$$

In the case $\alpha = 1/2$ we have the maximal balance $B(\alpha) = 1$. This means that the third option can arise with certainty only in an infinite time. The size factor $S(t)$ describes a saturation effect of the probability of emergence $P_E$: if we wait for a sufficiently long time this probability is almost one. We might think of the event of birth (emergence) of this third option as occurring suddenly at a time $t_E$. Then we can write $P_E(t) = Prob(t_E < t)$. In cases other than the symmetric one the balance is suboptimal ($B < 1$) and $P_E(t) < 1$ even after an infinite time. This means that the occurrence of a third option is not certain with unbalanced starting options even waiting an infinite amount of time. All this can be summarized in the following proposition:

**Proposition 4.** *When a marginal diminishing size effect is introduced in the probability of emergence, innovation occurs almost surely if and only if the balance is constant and equal to its maximum value ($B = 1$).*

Table 1 helps to get an idea of how the balance and the size factor jointly determine the probability of emergence. The balance is constant and the dynamics is due only to the size effect. We set $\sigma = 1/O_0$ and consider the investment shares $\alpha = 1/2$, $\alpha = 1/3$, $\alpha = 1/4$ and $\alpha = 1/8$:

| $P_E$ | | $It >> O_0$ $(S = 1)$ | $It = 3O_0$ $(S \cong 0.98)$ | $It = 2O_0$ $(S \cong 0.95)$ | $It = O_0$ $(S \cong 0.87)$ | $It = 0$ $(S \cong 0.63)$ |
|---|---|---|---|---|---|---|
| $\alpha = 1/2$ | $B = 1$ | 100% | 98% | 95% | 87% | 63% |
| $\alpha = 1/3$ | $B = 8/9$ | 89% | 87% | 84% | 77% | 56% |
| $\alpha = 1/4$ | $B = 3/4$ | 75% | 74% | 71% | 65% | 47% |
| $\alpha = 1/8$ | $B = 7/16$ | 44% | 43% | 42% | 38% | 28% |

Table 1: Probability of emergence for different values of balance $B$ and size factor $S$.

In the long run the size factor is nearly one and the probability of emergence reflects directly the balance of the two options: according to our assumption, the more symmetric is the system the more likely the third option is to arise.

## 5.3   Solving the dynamic model with the size factor

We now integrate the third equation of the model (1) with a full specification of the probability of emergence, that is taking into account the balance and the size effect. Beforehand it is useful to write down explicitly the probability of emergence as a function of time when both the balance and the size factor are included (again we assume $\pi = 1$):

$$P_E(t) = 4\frac{(O_{10} + \alpha It)(O_{20} + (1 - \alpha)It)}{(O_{10} + O_{20} + It)^2}\left(1 - e^{-\sigma(O_0 + It)}\right) \tag{27}$$

We will proceed by steps in order to better understand the effect of size in the model. First assume that the investment shares are set in a way that their ratio equals the ratio of the initial values of the parent options (condition (17)). In this case we obtain a constant balance $B = 4\alpha(1 - \alpha)$ and the rate of growth $P_E$ of the third option becomes

$$P_E(t) = B\left(1 - e^{-\sigma(O_0 + It)}\right) \tag{28}$$

With this specification of the dynamics we obtain the following time pattern for option the innovative option:

$$O_3(t) = 4\pi I_3 \alpha(1 - \alpha)\left[t + \frac{e^{-\sigma O_0}}{\sigma I}\left(e^{-\sigma It} - 1\right)\right] \tag{29}$$

The first term of this expression is what we have without size factor. The second term comes from the size effect. Here $\dot{O}_3(t) > 0$ and $\ddot{O}_3(t) > 0 \ \forall t \geq 0$.[8] This means the innovative option has a convex time pattern[9]. Such a behaviour accounts for a transitory phase in which the third option "warms up" before getting to work. The time pattern of $O_3(t)$ tends to the asymptote $Bt - B/(\sigma e^{\sigma O_0})$: after a sufficiently long time the innovative option attains linear growth. An indication of the characteristic time interval of transitory phase before linear growth is given by the intercept $\hat{t} = \frac{e^{-\sigma O_0}}{\sigma I}B$. Depending on the sensitivity parameter $\sigma$ and depending on the total initial value of the parent options and their cumulative investment $I$, the transitory phase can last a very long time or may be very brief: the higher the sensitivity $\sigma$ or the initial value $O_0$ or the investment rate $I$, the shorter the transitory phase and the faster the innovative option gets to linear growth. In figure 6 we give an example of function $O_3(t)$: here we have set $B = 1$, $\sigma = 1/4$, $I = 4$ and $O_{10} = O_{20} = 2$. With these values we have $O_3(t) = t + (e^{-t} - 1)/e$ and the asymptote is $t - 1/e$.

In this section we solve the third equation of the model (1) in the general case. This means that we want to compute the time integral of the general expression of the probability

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[8]the first derivative is $\dot{O}_3(t) = P_E(t)$ while the second derivative is $\ddot{O}_3(t) = \sigma I e^{-\sigma(O_0 + It)}$

[9]The profile of $O_3(t)$ suggests to compare the innovative options value to the value of a financial call option. In our case the time plays the role of the underlying asset. The characteristic time $\hat{t}$ resembles the strike price. The first time derivative of the option value, which is equal to the probability of emergence, corresponds to the financial *delta* of the option. This interpretation may not seem to fit well the option generated by recombinant innovation at a first glance and the time as underlying may look a bit weird. But one should think that under our assumption the parent options evolve linearly with time, so that time is just a rescaled value of the parent options. Thinking this we see how the parent option play effectively the role of underlying assets. And the process of recombinant innovation somehow resembles the exercise of a call option. An extension of our model would be to consider stochastic processes for the parent options. Under such assumption the interpretation of the innovative option as derivative would probably be more natural.
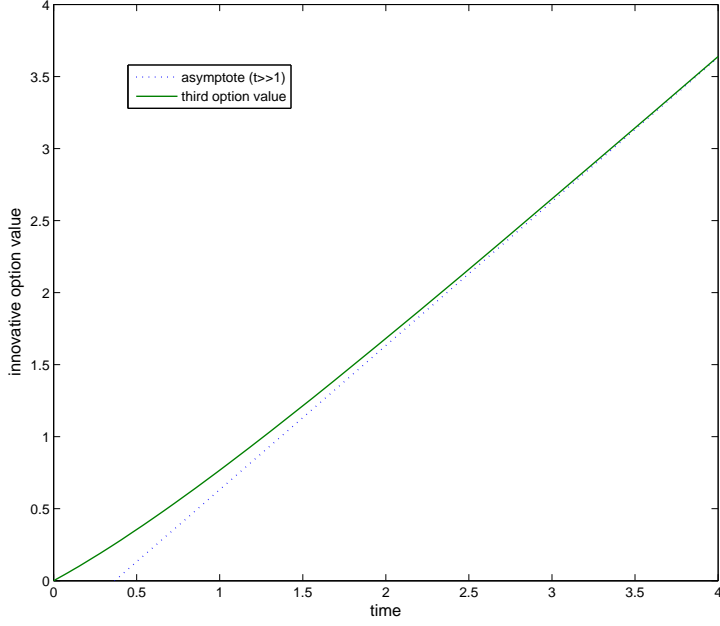
Figure 6: Value of the innovative option at time $t$, case of constant balance ($B = 1$, $\sigma = 1/4$, $I = 4$ and $O_{10} = O_{20} = 2$)

of emergence as given by (27):

$$O_3(t) = 4I_3 \int_0^t \frac{(O_{10} + \alpha Is)(O_{20} + (1 - \alpha)Is)}{(O_0 + Is)^2} \big(1 - e^{-\sigma(O_0 + Is)}\big) ds$$

We will call this solution $O_3^\sigma(t)$ to differentiate it from the solution without size effect. Appendix B contains all the details of the derivation of $O_3(t)$. Here we report the result:

$$
\begin{aligned}
O_3^\sigma(t) &= BI_3 t + B\frac{e^{-\sigma O_0}}{\sigma I}\big(e^{-\sigma It} - 1\big) + \frac{4I_3}{I}\sigma EG \ln \frac{O_0 + It}{O_0} + \\
&+ \frac{4I_3}{I}EG\Big[\frac{1}{O_0}\big(1 - e^{-\sigma O_0}\big) - \frac{1}{O_0 + It}\big(1 - e^{-\sigma(O_0 + It)}\big)\Big] + \\
&+ \frac{4I_3}{I}\Big[\sigma EG - (EH + FG)\Big]\Big[\sum_{k=1}^\infty \frac{\big(-\sigma(O_0 + It)\big)^k}{k \cdot k!} - \sum_{k=1}^\infty \frac{\big(-\sigma O_0\big)^k}{k \cdot k!}\Big]
\end{aligned}
\tag{30}
$$

Here $B = 4\alpha(1 - \alpha)$ is the value of the balance when it does not depend on time. $E = O_{10}(1 - \alpha) - \alpha O_{20}$, $F = \alpha$, $G = O_{20}\alpha - (1 - \alpha)O_{10}$ and $H = (1 - \alpha)$. When the balance is constant we have $O_{10}(1 - \alpha) = O_{20}\alpha$, and the expression of $O_3(t)$ only contains the first two terms since $E = G = 0$. This case was already described in the previous section. When the balance is not constant the time pattern of the third option contains a logarithmic term, a negative exponential divided by a linear function and two infinite sums, one constant and the other dependent on time. As argued in appendix B, the two sums converge to negative exponentials. This means that the infinite sum which depends

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on time goes to zero for $It >> O_0$. In the long run the time pattern of $O_3^\sigma$ is given by the following expression (we make explicit the constants $B$, $E$ and $G$):

$$O_3^\sigma(t) \simeq 4\alpha(1-\alpha)I_3 t - 4\frac{I_3}{I}\sigma[O_{10}(1-\alpha) - O_{20}\alpha]^2 \ln \frac{It}{O_0} \tag{31}$$

Without size effect we have (see equation (20))

$$O_3(t) \simeq 4\alpha(1-\alpha)I_3 t + 4\frac{I_3}{I}[O_{10}(1-\alpha) - O_{20}\alpha](1 - 2\alpha) \ln \frac{It}{O_0}$$

When a size factor is present, the logarithmic term adds negatively to the value of the innovative option in the long run. Without size effect the logarithmic term can be either positive or negative instead. With a size effect the contribution of the logarithmic term depends much on the value of $\sigma$, which relates to the size effect of the parent options on the probability of emergence and which should be estimated empirically.

# 6 Optimization of diversity

Now we address the problem of optimal diversity. The objective function depends on the benefits from parent and innovative options at some final date $t$:

$$V(\alpha; t) = O_1(t; \alpha)^s + O_2(t; \alpha)^s + O_3(t; \alpha)^s \tag{32}$$

This function contains a parameter $s > 0$ which indicates the returns to scale from investment in each option. This definition clarifies the relevance of the trade-off between returns to scale from specialization and benefits from recombinant innovation. Our control variable is the time-constant investment share $\alpha$. The maximization problem is then defined as follows:

$$\max_{\alpha \in [0,1]} O_1(t; \alpha)^s + O_2(t; \alpha)^s + O_3(t; \alpha)^s \tag{33}$$

We will solve this optimization problem for different versions of the model, namely with and without initial values of parent options and with and without the size factor. The simple case of zero initial values and no size effect was the pilot model that we addressed initially. Before passing on to more complex specifications we study in some detail the first order conditions for that simple case because they give a general indication of the qualitative properties of the benefits curve.

## 6.1 The shape of the benefits curve

Substituting the solutions $O_i(t)$ of the pilot model (see section 2), the maximization problem becomes

$$\max_{\alpha \in [0,1]} V(\alpha; T) = T^s I^s \big[\alpha^s + (1-\alpha)^s + C^s \alpha^s (1-\alpha)^s\big] \tag{34}$$

Here $C = 4\pi I_3/I$ as usual and $\pi$ is the efficiency of recombinant innovation defined in section 3.3. We prefer to examine the normalized version of the benefits function, which is obtained dividing $V$ by the benefits from specialization $V(\alpha = 0; t) = V(\alpha = 1; t) = I^s t^s$. Such a normalization does not affect the solution of the optimization problem since $I^s t^s$ is a factor that does not depend on $\alpha$. The first order necessary condition for maximization of final benefits is

$$\frac{\partial \tilde{V}}{\partial \alpha} = s\alpha^{s-1} - s(1-\alpha)^{s-1} + C^s s\big[\alpha(1-\alpha)\big]^{s-1}(1 - 2\alpha) = 0 \tag{35}$$

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There may be one or three interior solutions to this equation. The symmetric solution $\alpha = 1/2$ always exists. Depending on the returns to scale parameter $s$ two other solutions are present, $\alpha_1(s)$ and $\alpha_2(s)$. They are symmetric with respect to $\alpha = 1/2$ (in accordance with the symmetry of the system under study) so that $\alpha_1 + \alpha_2 = 1$ and if they exist they always give a minimum value of benefits, while $\alpha = 1/2$ may be either a minimum or a maximum. The transition from $\alpha = 1/2$ as a minimum to $\alpha = 1/2$ as a maximum depends on the appearance of these two roots. In general for a given value of the factor $C$ there is a threshold level of returns to scale $\hat{s}$ at which $\alpha = 1/2$ is neither a maximum or a minimum. This threshold value is given by a tangency requirement

$$\left.\frac{\partial^2 \tilde{V}}{\partial \alpha^2}\right|_{\alpha=1/2} = 0$$

Computing the second derivative in $\alpha = 1/2$ and setting it to zero one works out the condition

$$\hat{s} = \left(\frac{C}{2}\right)^{\hat{s}} + 1 \tag{36}$$

This means that for a given probability of recombinant innovation ($C$ given) the threshold value of returns to scale $\hat{s}$ is a fixed point of the function $f(s) = \left(\frac{C}{2}\right)^s + 1$. Note that $\hat{s} > 1$ since $C \geq 0$. Then we have the following proposition:

**Proposition 5.** *Necessary condition to have only 1 stationary point ($\alpha = 1/2$, local and global minimum) is increasing returns to scale. With decreasing returns there are always 3 stationary points.*

Conversely if the returns to scale are set to some value $s$ then one can obtain an explicit solution for the threshold value of the probability of recombinant innovation, $\hat{C} = 2(s-1)^{1/s}$.

A graphical analysis is illustrative. With $C = 1$ (for instance with $I = 4I_3$ and $\pi = 1$) we have $\hat{s} \simeq 1.3833$. In the following figures we report the graphs of $\tilde{V}(\alpha)$ and its derivative for two different cases:

$$\tilde{V}'(\alpha)/s = \alpha^{s-1} - (1-\alpha)^{s-1} + \left[\alpha(1-\alpha)\right]^{s-1}(1-2\alpha)$$

In the first case ($s = 1.5$, figure 7) the only stationary point is $\alpha = 1/2$, a local and global minimum of final benefits. Global maxima are the corner solutions $\alpha = 0$ and $\alpha = 1$. In the second case ($s = 1.5$, figure 8) there are three stationary points: $\alpha = 1/2$ is now a local maximum (possibly also global), while the two symmetric stationary points appear, $\alpha_1$ and $\alpha_2$, which are local and global minima.

If we now go back to the analysis of section 2, we can compare the transition value $\hat{s}$ with the threshold value $\overline{s}$ between diversity and specialization as optimal solution for maximum final benefits.

**Lemma** *Without recombinant innovation ($\pi = 0$) we have $\hat{s} = \overline{s} = 1$. With recombinant innovation ($\pi > 0$) it holds $\hat{s} > \overline{s} > 1$.*

This means that three different regions can be identified in the returns to scale domain, as figure 9 shows.
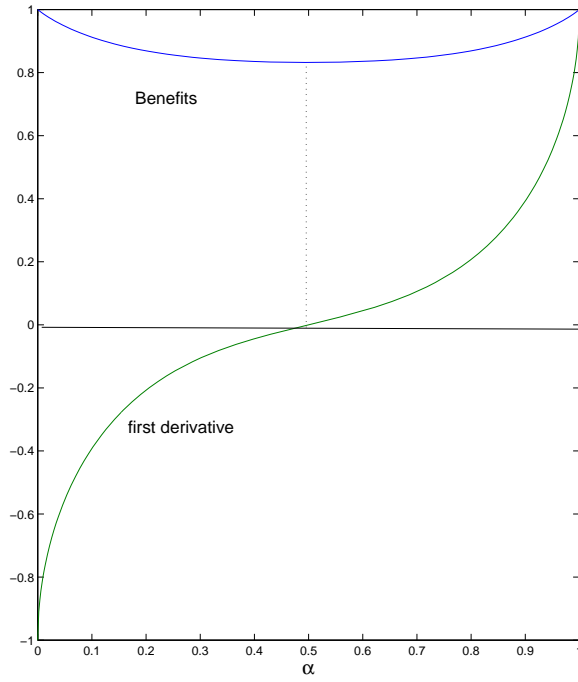
Figure 7: Normalized final benefits $\tilde{V}(\alpha)$ and its derivative. Case $s = 1.5$
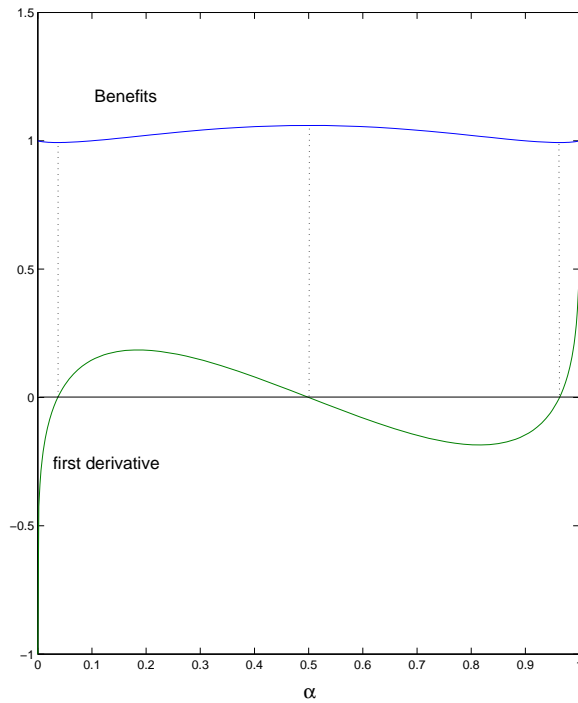


Figure 8: Normalized final benefits $\tilde{V}$ and its derivative. Case $s = 1.2$
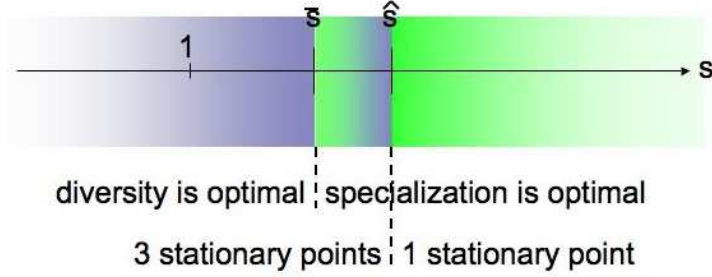
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Figure 9: With a positive probability of recombinant innovation $\pi > 0$ we have $\hat{s} > \overline{s} > 1$.

## 6.2 Optimization with size effect and zero initial values

In this subsection we consider zero initial values for the parent options and a probability of emergence $P_E$ containing both the balance and the size factors. As we know from section 2 the balance is constant when initial values are zero. But $P_E$ depends on time because of the size effect. The objective function for the problem of maximization of final benefits is defined as in the pilot model. The expression of the innovative option is given by (29). Substituting this, the final benefits read

$$V(\alpha, t) = (\alpha I t)^s + \big((1 - \alpha) I t\big)^s + \big[4\pi I_3 \alpha(1 - \alpha)\big]^s \big[t + g(t)\big]^s \tag{37}$$

where $g(t) = (e^{-\sigma I t} - 1)/\sigma I$. If we normalize the objective function dividing it by $(It)^s$ (benefits from specialization) we have

$$\tilde{V}(\alpha, t) = \alpha^s + (1 - \alpha)^s + C^s m(t)^s \alpha^s (1 - \alpha)^s \tag{38}$$

where the constant factor C is the same as we defined for the pilot model ($C = 4\pi I_3/I$). Now a time dependent factor shows up, $m(t) = 1 + \frac{e^{-\sigma I t} - 1}{\sigma I t}$, with $m'(t) > 0$, $\lim_{t \to 0} m(t) = 0$ and $\lim_{t \to \infty} m(t) = 1$. The factor $m(t)$ monotonically modulates the contribution of innovative recombination to final benefits, being very small at early stages and converging to one at the end of the investment. This feature adds considerably to the realism of the model.

In the long run ($It >> O_0$) the model converges to the pilot model, where only $C$ appears in the expression of final benefits. This consideration can be pushed even further incorporating $m(t)$ into $C$ and defining a function $C(t) = Cm(t)$. As one can see looking at expression (38), final benefits with size effect are formally the same as in the pilot model (6). Only difference is that constant $C$ now depends on time. This consideration is maximally important for the optimization of diversity. Even if the size effect makes the investment system dynamic, still the optimal solution will be either $\alpha = 0, 1$ or $\alpha = 1/2$. The optimal diversity now is time dependent but it can be just one of these values. This is better understood by looking at figures 2 and 3. As time flows, the factor $C(t)$ increases and the benefits curve goes from the lower curve $\pi = 0$ (representing $C = 0$) to the upper curve $\pi = 1$ (which stands for $C = 1$).

The first order necessary condition for optimization of diversity in this dynamic setting is the following:

$$s\alpha^{s-1} - s(1 - \alpha)^{s-1} + C(t)^s \big[\alpha(1 - \alpha)\big]^{s-1}(1 - 2\alpha) = 0 \tag{39}$$

The analysis of section 6.1 can be repeated here by substituting the constant factor $C$ with the function $C(t)$. In particular the threshold value $\hat{s}$ of returns to scale where the two

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symmetric roots appear and $\alpha = 1/2$ becomes a (local) maximum of benefits is given by the counterpart of equation (36):

$$\hat{s}(t) = \left(\frac{C(t)}{2}\right)^{\hat{s}(t)} + 1 \tag{40}$$

Now the transition value is a function of time. It may also be interesting to think in terms of transition time $\hat{t}$: for a given value of returns to scale $s$ one computes the factor C that satisfies the equation above:

$$C(\hat{t}) = 2(s-1)^{1/s} \tag{41}$$

Similarly to the transition from one to three stationary points, also the threshold analysis for optimal diversity is formally the same as in the pilot model. We define the threshold value $\overline{s}(t)$ as the returns to scale level at which, for a given time horizon $t$, the benefits with $\alpha = 1/2$ are the same as the benefits from specialization (either $\alpha = 0$ or $\alpha = 1$). Using expression (38) this requirement turns into the following equation

$$\tilde{V}\left(\alpha = 1/2\right) = \frac{1}{2^{\overline{s}}}\left[2 + \left(\frac{C(t)}{2}\right)^{\overline{s}}\right] = 1 \tag{42}$$

We can express this result with the following proposition[10]:

**Proposition 6.** *For a given time horizon $t$ diversity ($\alpha = 1/2$) is optimal if and only if $s < \overline{s}(t)$.*

How $\overline{s}(t)$ behaves? The larger $t$, the larger $\overline{s}(t)$. The intuition is the following: $C(t)$ is increasing and this means that time works for recombinant innovation. As time goes, the threshold value of returns to scale above which specialization is optimal becomes larger and larger. The region where diversity is optimal enlarges. This fact is represented in the following figure.
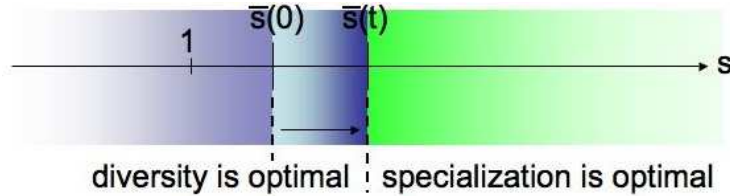


Figure 10: As time goes, the region of returns to scale where diversity is optimal becomes larger

Alternatively one can define a threshold time horizon $\overline{t}$:

$$C(\overline{t}) = 2^{s+1}\left(2^{s-1} - 1\right) \tag{43}$$

The threshold value $\overline{t}$ that satisfies (43) for a given level of returns to scale $s$ must be computed numerically. Its interpretation is the following: for $t < \overline{t}$ specialization (in one of parent options) is optimal. For $t \geq \overline{t}$ diversity ($\alpha = 1/2$) is optimal. We may want to understand how such a threshold time depends on returns to scale. The function $C(t)$ is monotonically increasing: the inverse $C^{-1}(\cdot)$ can be defined (and it is increasing as well) and a solution $\overline{t}$ exists. The right hand side of (43) is increasing[11] in $s$. We have the following result then:

---

[10]This generalizes proposition 1 of the pilot model

[11]We have $\frac{d}{ds}2^{s+1}\left(2^{s-1} - 1\right) = 2^{s+1}\ln 2(2^s - 1) > 0$ since $s > 0$

**Proposition 7.** *For higher returns to scale s the threshold time horizon $\bar{t}$ is larger and it takes a longer time for diversity ($\alpha = 1/2$) to become the optimal decision.*

Concluding, the size effect introduces a dynamical scale effect into the system. The optimal solution may change through time, but it can only switch from $\alpha = 0, 1$ to $\alpha = 1/2$ if it ever would. This happens if and only if the probability of recombination $\pi$ is large enough (see corollary 2 in section 2).

A last important consideration. in the limit of infinite time ($It >> O_0$) the size effect saturates ($\lim_{t \to \infty} S(t) = 1$). This means that if one faces a time horizon long enough the size factor can be discarded in the probability of emergence of recombinant innovation. Not paying attention to the transitory phase, the solution at time $t$ for optimal diversity in such a dynamic model will converge to the static pilot model solution.

## 6.3 The effect of non-zero initial values on optimization

Now we want to see what happens if we consider the initial value of parent options in the optimization of final benefits. Equation (19) shows the value of the innovative option in this case:

$$O_3(t) = C\left[f(\alpha, t) + \alpha(1 - \alpha)It\right] \tag{44}$$

where $C = 4\pi I_3 / I$. Comparing this with the expression that we used in the model of section 2 we have one more term:

$$f(\alpha, t) = \left(O_{10} - \alpha O_0\right)^2 \left(\frac{1}{O_0 + It} - \frac{1}{O_0}\right) + \left(O_{10} - \alpha O_0\right)(1 - 2\alpha) \ln \frac{O_0 + It}{O_0}$$

This is the sum of two terms: one is hyperbolic and converges to a negative value as time goes to infinity. The other is logarithmic and monotonically increasing or decreasing depending on the factor $(O_{10} - \alpha O_0)(1 - 2\alpha)$. The objective function for maximization of final benefits is

$$V(\alpha, t) = \left(O_{10} + \alpha It\right)^s + \left(O_{20} + (1 - \alpha)It\right)^s + C^s\left[f(\alpha, t) + \alpha(1 - \alpha)It\right]^s \tag{45}$$

Normalizing this function to $(It)^s$ as done before is less meaningful since with non-zero initial values $(It)^s$ does no longer represent the value of benefits with specialization. Nevertheless this normalization leaves us with an adimensional function and allows to compare the results with other versions of the model. The normalized benefits are

$$\tilde{V}(\alpha, t) = \left(\frac{O_{10}}{It} + \alpha\right)^s + \left(\frac{O_{20}}{It} + 1 - \alpha\right)^s + C^s\left[\frac{f(\alpha, t)}{It} + \alpha(1 - \alpha)\right]^s \tag{46}$$

The first order necessary condition for a maximum is

$$\left(\frac{O_{10}}{It} + \alpha\right)^{s-1} - \left(\frac{O_{20}}{It} + 1 - \alpha\right)^{s-1} + \tag{47}$$

$$+ \ C^s\left[\frac{f(\alpha, t)}{It} + \alpha(1 - \alpha)\right]^{s-1}\left(\frac{1}{It}\frac{\partial f(\alpha, t)}{\partial \alpha} + 1 - 2\alpha\right) = 0$$

The solution to this equation is rather complicated. But the important result is a breaking of symmetry in the system because $O_{10} \neq O_{20}$ in general. A first consequence is that

$\alpha = 1/2$ is not a solution to the above equation in general[12]. Initial values of parent options make the investment system truly dynamic. Now diversity as optimal solution is represented by a function of time $\alpha^*(t)$. Given the time horizon $t$, equation (47) must be solved numerically in order to find $\alpha^*(t)$. But there is more to it. If the time horizon is long enough ($It >> O_0$) the initial values do not affect the system anymore and we are back in the situation of the pilot model. In the long run symmetry is restored. In the long run one can disregard the complicate dynamics of the system produced by initial values and simply find the optimal diversity of the investment system as if facing the conditions of the pilot model.

## 6.4   Optimization in the general case

In this last section we address the optimization of the more general model, with a size factor and initial values different from zero. The value of the innovative option at time $t$ is given by (30). We rewrite it here as

$$
\begin{aligned}
O_3(t) \;=\; & C\Bigg\{ B(\alpha)It + B(\alpha)\frac{e^{-\sigma O_0}}{\sigma}\big(e^{-\sigma It} - 1\big) + \sigma E(\alpha)G(\alpha)\ln\frac{O_0 + It}{O_0} + \\
& + \; E(\alpha)G(\alpha)\Big[\frac{1}{O_0}\big(1 - e^{-\sigma O_0}\big) - \frac{1}{O_0 + It}\big(1 - e^{-\sigma(O_0+It)}\big)\Big] + \\
& + \; \Big[\sigma E(\alpha)G(\alpha) - \big(E(\alpha)H(\alpha) + F(\alpha)G(\alpha)\big)\Big]\Big[\sum_{k=1}^{\infty}\frac{\big(-\sigma(O_0+It)\big)^k}{k\cdot k!} + \\
& - \; \sum_{k=1}^{\infty}\frac{\big(-\sigma O_0\big)^k}{k\cdot k!}\Big]\Bigg\}
\end{aligned}
\tag{48}
$$

Here we explicitly indicated which factors depending on the control variable $\alpha$: $B(\alpha) = \alpha(1 - \alpha)$ and for the other functions of $\alpha$ we refer to equation (30). Note that it is not possible to separate this expression into two factors dependent separately on $t$ and $\alpha$ as we managed to do in section 6.2. The value of final benefits from the overall investment is similar to the one given by equation (45)

$$
\begin{aligned}
V(\alpha, t) \;=\; & \big(O_{10} + \alpha It\big)^s + \big(O_{20} + (1-\alpha)It\big)^s + \\
& + \; C^s\Big[B(\alpha)It + B(\alpha)\frac{e^{-\sigma O_0}}{\sigma}\big(e^{-\sigma It} - 1\big) + h(\alpha, t)\Big]^s
\end{aligned}
\tag{49}
$$

where $h(\alpha, t)$ collects all terms in the expression of $O_3$ but the first two. The contribution of innovation (the term multiplied by $C^s$) consists of three terms. The first is the basic and linear one, which already appeared in the pilot model. The second is a direct effect of the size factor. The third one is due to the presence of non-zero initial values of parent options. This expression combines the effects that we have been analysing separately so far. If we normalize this expression dividing it by $I^s t^s$ as we have done before we obtain

$$
\tilde{V}(\alpha, t) = \Big(\frac{O_{10}}{It} + \alpha\Big)^s + \Big(\frac{O_{20}}{It} + 1 - \alpha\Big)^s + C^s\Big[B(\alpha)n(t) + \frac{h(\alpha, t)}{It}\Big]^s
\tag{50}
$$

where $n(t) = 1 + e^{-\sigma O_0}/(\sigma It)(e^{-\sigma It} - 1)$. This time factor can be expressed in terms of factor $m(t)$ that we have introduced in section 6.2: $n(t) = e^{-\sigma O_0}m(t) + 1 - e^{-\sigma O_0}$, $n(0) \simeq 1 - e^{-\sigma O_0}$,

---

[12]$\alpha = 1/2$ is still a solution to the particular case of equal initial values $O_{10} = O_{20}$, where the system is symmetric.

$n'(t) = e^{-\sigma O_0} m'(t) > 0$ and $\lim_{t \to \infty} n(t) = 1$. The smaller the sum of initial values $(O_0)$ the closer $n(t)$ is to $m(t)$. In the limit, $\lim_{O_0 \to 0} n(t) = m(t)$. The qualitative effect of $n(t)$ on final benefits is the same as the effect of $m(t)$ anyway: as time goes by, this term drives the benefits curve from lower values where the contribution of innovation is negligible to higher values where diversity may be the optimal choice eventually. The presence of non-zero initial values brakes the symmetry of the system and $\alpha = 1/2$ is not a solution to the first order condition in general. The first order necessary condition for optimization is

$$
\left( \frac{O_{10}}{It} + \alpha \right)^{s-1} - \left( \frac{O_{20}}{It} + 1 - \alpha \right)^{s-1} + \tag{51}
$$
$$
+ \quad C^s \left[ B(\alpha) n(t) + \frac{h(\alpha, t)}{It} \right]^{s-1} \left( \frac{1}{It} \frac{\partial h(\alpha, t)}{\partial \alpha} + n(t)(1 - 2\alpha) \right) = 0
$$

In the long run $(It >> O_0)$ the initial values become negligible, as already noticed, and the size factor converges to one. This means that also in the more general case, where both initial values and size effect are important, if the time horizon is long enough, one can address the much simpler pilot model, doing as if initial values and size effects were absent. The solution for optimal diversity in the corresponding version of the pilot model will be the long run solution of the general model.

# 7 Conclusions and further research

This study has proposed a model of recombinant innovation which was applied to an investment decision problem where the decision maker faces a tradeoff between scale advantages and the benefits of diversity. We considered three different versions of the model with increasing levels of complexity. First a pilot model was developed to express the core elements of recombinant innovations. A more general model devoted attention to the detailed structure of diversity and allowed initial values of parent options to be different from zero. Finally a third version introduced a diminishing marginal size effect in the probability of emergence of a recombinant innovation.

The initial part of the analysis consisted of deriving a solution for the three model versions. We have shown the condition for constant diversity of the system of parent options: in order to have constant diversity the ratio of investment shares must be equal to the ratio of initial values of parent options. When this is not the case, diversity changes over time and its qualitative behaviour depends on which ratio is larger. Nevertheless in all cases diversity converges to the same constant value in the long run. The investment shares and the initial values of parent options also determine the shape of the time pattern of the innovative option. In the long run we have a linear plus a logarithmic term and the shape may be either convex (faster than linear growth) or concave (slower than linear).

In order to account for the diminishing marginal contribution of total parent options' size, a size effect is added into the probability of emergence. The time pattern of the innovative option then becomes quite complicated. In the long run it reduces again to a linear plus a logarithmic term. What is interesting is that now such a logarithmic term always adds negatively and causes a convex shape of the innovative option through time.

We then moved on to optimize diversity given a final benefits function, which comes down to finding an optimal balance or an optimal trade-off between the benefits of diversity due to recombinant innovation and the benefits associated with returns to scale. We derived conditions for optimal diversity under different regimes of returns to scale. Maximum

diversity, expressed by a perfectly symmetric system with $\alpha = 1/2$, may be either a local maximum or a local minimum of final benefits, depending on the level of returns to scale. When diversity is a local maximum two other stationary points of final benefits are present. We have defined two threshold values of returns to scale: the first one is the value where the system makes a transition from one to three stationary points of final benefits. The second threshold is the returns to scale level below which diversity is a global maximum of final benefits.

The presence of a size factor in the probability of emergence makes the returns to scale threshold time dependent. This suggests a second dimension of a threshold analysis, which is in the time domain: for a given level of returns to scale, when the investment time horizon is beyond a threshold value, the best choice becomes diversity. This threshold time horizon will be larger the higher are the returns to scale. Introducing the initial values of parent options breaks the symmetry of the system. In particular an investment share $\alpha = 1/2$ is not a general solution to the maximization problem. In the long run symmetry is restored, that is, approximated through convergence and diversity ($\alpha = 1/2$) may become optimal eventually.

A number of directions for future research can be identified. Basic extensions are the use of a NPV (CBA) objective and the inclusion of a discount factor. Somewhat related (mathematically equivalent to discounting in fact) is adding a depreciation term for investment dynamics. Currently the assumption is that somehow it represents knowledge that does not depreciate over time. Another line of research is to endogenize investment in the innovative option (which now is fixed and exogenous) and make it part of the (optimal) allocation decision. Next, one can study recombinant innovation in the context of three or more parent options. Then the marginal effect of adding options can be assessed (e.g., diminishing returns under certain conditions), and possibly something can be said about the optimal number of options in general. Moreover, three or more options allow for an examination of the role and dynamics of disparity, one of the dimensions of diversity, between parent options. Finally, one could consider the value of parent options as stochastic processes. This suggests an analogy between the innovative option and a financial derivative: parent options can then be regarded to play the role of underlying assets.

# Appendix A  Condition for constant balance

Here we give a proof of the necessary and sufficient conditions of constant balance for the "Gini" specification.

In order to prove necessity we differentiate the expression $B\big(O_1(t), O_2(t)\big)$ with respect to time and see under which conditions the derivative is equal to zero. Using the chain rule we have

$$\frac{dB}{dt} = \frac{\partial B}{\partial O_1}\frac{dO_1}{dt} + \frac{\partial B}{\partial O_2}\frac{dO_2}{dt} \tag{52}$$

where

$$\frac{\partial B}{\partial O_i} = \frac{O_j(O_j - O_i)}{(O_i + O_j)^3} \qquad i,j = 1,2 \qquad i \neq j$$

Time derivatives are given by the specifications of the model (1). If now one substitutes the time flow of each option value, $O_1(t) = O_{10} + \alpha It$ and $O_2(t) = O_{20} + (1 - \alpha)It$, the time derivative of balance becomes

$$\frac{dB}{dt} = \frac{O_{10} - O_{20} + (2\alpha - 1)It}{(O_{10} + O_{20} + It)^3}\Big[(O_{10} + \alpha It)(1 - \alpha)I - (O_{20} + (1 - \alpha)It)\alpha I\Big] \tag{53}$$

Setting this derivative to zero we obtain

$$(O_{10} + \alpha It)(1 - \alpha) = (O_{20} + (1 - \alpha)It)(\alpha I)$$

This equation must hold true for any value of $t$. For instance taking $t = 1/I$ we have

$$\frac{O_{10}}{O_{20}} = \frac{\alpha}{1 - \alpha}$$

which is condition (17).

This is also a sufficient condition for constant balance as one can see by direct substitution:

$$\begin{aligned}
B(t) &= 4\frac{(O_{10} + \alpha It)(O_{20} + (1 - \alpha)It)}{(O_{10} + O_{20} + It)^2} = 4\frac{(O_{10} + \alpha It)(O_{10}\frac{1-\alpha}{\alpha} + (1 - \alpha)It)}{(O_{10} + O_{10}\frac{1-\alpha}{\alpha} + It)^2} \\
&= 4\frac{(1 + \frac{\alpha}{O_{10}}It)(\frac{1-\alpha}{\alpha} + \frac{1-\alpha}{O_{10}}It)}{(1 + \frac{1-\alpha}{\alpha} + \frac{It}{O_{10}})^2} = 4\frac{1-\alpha}{\alpha}\frac{(1 + \frac{\alpha}{O_{10}}It)^2}{(\frac{1}{\alpha} + \frac{It}{O_{10}})^2} = 4\frac{1-\alpha}{\alpha}\alpha^2 = 4\alpha(1 - \alpha)
\end{aligned}$$

# Appendix B   General solution to the dynamics of the innovative option

Here we report the steps of the integration of the probability of emergence as defined in (27), that is the integration of the third equation of the model (1) leading to the time value of the third option $O_3$. For less general definitions of the probability of emergence (for instance without the size factor) the solution can be easily obtained by considering only the required terms out of those in which we split the general integral.

$$O_3(t) = \int_0^t 4\frac{(O_{10} + \alpha Is)(O_{20} + (1-\alpha)Is)}{(O_0 + Is)^2}\left(1 - e^{-\sigma(O_0 + Is)}\right)ds \tag{54}$$

We substitute $s = (x - O_0)/I$ and obtain

$$O_3 = \frac{4}{I}\int_{O_0}^{O_0+It}\frac{(E + Fx)(G + Hx)}{x^2}\left(1 - e^{-\sigma x}\right)dx \tag{55}$$

where $E = O_{10}(1-\alpha) - \alpha O_{20}$, $F = \alpha$, $G = O_{20}\alpha - (1-\alpha)O_{10}$ and $H = (1-\alpha)$. Observe that $E = -G$. The expression above contains the difference of two integrals (for ease of notation we avoid reporting the extremes of integration and consider indefinite integrals for the moment). The first one is

$$
\begin{aligned}
\int \frac{(E + Fx)(G + Hx)}{x^2}dx &= EG\int\frac{dx}{x^2} + (EH + FG)\int\frac{dx}{x} + FH\int dx\\
&= -\frac{EG}{x} + (EH + FG)lnx + FHx
\end{aligned}
$$

As for the second integral we have

$$
\begin{aligned}
\int \frac{(E + Fx)(G + Hx)}{x^2}e^{-\sigma x}dx &= EG\int\frac{e^{-\sigma x}}{x^2}dx + (EH + FG)\int\frac{e^{-\sigma x}}{x}dx +\\
&+ FH\int e^{-\sigma x}dx =\\
&= -\frac{FH}{\sigma}e^{-\sigma x} - EG\frac{e^{-\sigma x}}{x} +\\
&+ [EH + FG - \sigma EG]\left[lnx + \sum_{k=1}^{\infty}\frac{(-\sigma x)^k}{k \cdot k!}\right]
\end{aligned}
$$

When substituting the latter two results into equation (55) we obtain

$$
\begin{aligned}
\int \frac{(E + Fx)(G + Hx)}{x^2}\left(1 - e^{-\sigma x}\right)dx &= -\frac{EG}{x} + FHx + FH\frac{e^{-\sigma x}}{\sigma} +\\
&+ EG\frac{e^{-\sigma x}}{x} + \sigma EGlnx +\\
&+ [\sigma EG - (EH + FG)]\sum_{k=1}^{\infty}\frac{(-\sigma x)^k}{k \cdot k!}
\end{aligned}
$$

It is instructive to look first at the case of constant balance. The necessary and sufficient condition can be written as $O_{10}(1 - \alpha) = O_{20}\alpha$. Then $EG = 0$, $EH + FG = 0$ and $FH = \alpha(1 - \alpha)$ and the integral above simplifies to

$$\int \frac{(E + Fx)(G + Hx)}{x^2}\left(1 - e^{-\sigma x}\right)dx\bigg|_{B=const} = \alpha(1 - \alpha)\left(x + \frac{e^{-\sigma x}}{\sigma}\right) \qquad (56)$$

The solution for the value of the third option as a function of time is then

$$O_3(t) = \frac{4}{I}\alpha(1 - \alpha)\left(x + \frac{e^{-\sigma x}}{\sigma}\right)\bigg|_{x=O_0}^{x=O_0+It} = Bt + B\frac{e^{-\sigma O_0}}{\sigma I}\left(e^{-\sigma It} - 1\right) \qquad (57)$$

where $B = 4\alpha(1 - \alpha)$. It is useful to check the "physical" dimensions of the solution just obtained. The first term $Bt$ is a *time* (balance is dimensionless). The second term is *time* too since in the denominator we have the product $\sigma I$ where $\sigma$ is the inverse of an option value (money for instance), while $I$ is an investment rate, that is money per unit of time. Not surprisingly $O_3$ has a *time* dimension, since at the very beginning of our analysis we have normalized the investment in the third option: $I_3 = 1$, so that $[O_3] = time$.

Relaxing the condition of constant balance we have the following general result for the value of the innovative option at time $t$:

$$\begin{aligned}
O_3(t) &= \frac{4}{I}\int_{x=O_0}^{x=O_0+It}\frac{(E + Fx)(G + Hx)}{x^2}\left(1 - e^{-\sigma x}\right)dx = \qquad (58) \\
&= Bt + B\frac{e^{-\sigma O_0}}{\sigma I}\left(e^{-\sigma It} - 1\right) + \frac{4}{I}\sigma EG\log\frac{O_0 + It}{O_0} + \\
&+ \frac{4}{I}EG\left[\frac{1}{O_0}\left(1 - e^{-\sigma O_0}\right) - \frac{1}{O_0 + It}\left(1 - e^{-\sigma(O_0+It)}\right)\right] + \\
&+ \frac{4}{I}\left[\sigma EG - (EH + FG)\right]\left[\sum_{k=1}^{\infty}\frac{\left(-\sigma(O_0 + It)\right)^k}{k \cdot k!} - \sum_{k=1}^{\infty}\frac{\left(-\sigma O_0\right)^k}{k \cdot k!}\right]
\end{aligned}$$

The first two terms are what we have with constant balance (see section 5.3). In the short run ($It << O_0$) we have $O_3(t) \simeq Bt$. A bit more complex is the analysis of the long run behaviour ($t >> O_0/I$). The part referring to constant balance will tend to a linear growth, as we have seen already in the main text. In the logarithmic term the value of the new investment $It$ overcomes the initial option value $O_0$. The second part of the third term vanishes even faster than the exponential term of the part relative to constant balance, because of the presence of $t$ in the denominator. Finally the infinite sum containing $t$ goes to zero at least exponentially: this can be seen by noting that for even values of $k$ we have ($O_0 + It = y$)

$$\frac{(-y)^k}{2^k \cdot k!} < \frac{(-y)^k}{k \cdot k!} < \frac{(-y)^k}{k!}$$

For odd values of $k$ the inequalities are reversed. This means that our series is bounded between the functions $-1 + e^{-(O_0+It)}$ and $-1 + e^{-(O_0+It)/2}$, implying that it goes to zero at least exponentially:

$$\sum_{k=1}^{\infty} \frac{\left(-\sigma(O_0 + It)\right)^k}{k \cdot k!} = -\sigma(O_0 + It) + \frac{\sigma^2(O_0 + It)^2}{2 \cdot 2} - \frac{\sigma^3(O_0 + It)^3}{3 \cdot 3!} + \frac{\sigma^4(O_0 + It)^4}{4 \cdot 4!} - \cdots$$

$$< -\sigma(O_0 + It) + \frac{\sigma^2(O_0 + It)^2}{2} - \frac{\sigma^3(O_0 + It)^3}{3!} + \frac{\sigma^4(O_0 + It)^4}{4!} - \cdots$$

$$= -1 + e^{-\sigma(O_0 + It)} \le 0$$

$$\sum_{k=1}^{\infty} \frac{\left(-\sigma(O_0 + It)\right)^k}{k \cdot k!} = -\sigma(O_0 + It) + \frac{\sigma^2(O_0 + It)^2}{2 \cdot 2} - \frac{\sigma^3(O_0 + It)^3}{3 \cdot 3!} + \frac{\sigma^4(O_0 + It)^4}{4 \cdot 4!} - \cdots$$

$$> -\frac{\sigma(O_0 + It)}{2} - \frac{\sigma(O_0 + It)}{2} + \frac{\sigma^2(O_0 + It)^2}{2^2 \cdot 2!} - \frac{\sigma^3(O_0 + It)^3}{2^3 \cdot 3!} + \cdots$$

$$= -1 - \frac{\sigma(O_0 + It)}{2} + e^{-\frac{\sigma(O_0 + It)}{2}}$$

Alternatively one can think that for $k >> 1$ we have $k \cdot k! \simeq k e^{k \log k - k} \simeq k!$. This means that the infinite sums in the expression of $O_3(t)$ do not differ too much from negative exponential functions. In particular the one depending on $t$ goes to zero as time is long enough ($It >> O_0$). Consequently we are left with the following long run functional behaviour:

$$O_3(t) \simeq B\left(t - \frac{e^{-\sigma O_0}}{\sigma I}\right) + \frac{4}{I}\sigma EG \log \frac{It}{O_0} + \tag{59}$$

$$+ \ \frac{4}{I}EG\left[\frac{1}{O_0}\left(1 - e^{-\sigma O_0}\right)\right] - \frac{4}{I}\left[\sigma EG - (EH + FG)\right]D(\sigma, O_0)$$

The factor $D(\sigma, O_0) = \sum_{k=1}^{\infty} \frac{(-\sigma O_0)^k}{k \cdot k!}$ only depends on parameters $\sigma$ and $O_0$; similarly to what we have noticed for the series dependent on $t$ we can say that such a quantity is bounded between $e^{-O_0}$ and $e^{-O_0/2}$. In particular one can easily see that $C(\sigma, O_0)$ is finite:

$$\sum_{k=1}^{\infty} \frac{\left(-\sigma O_0\right)^k}{k \cdot k!} = -\sigma O_0 + \frac{\sigma^2 O_0^2}{2 \cdot 2} - \frac{\sigma^3 O_0^3}{3 \cdot 3!} + \frac{\sigma^4 O_0^4}{4 \cdot 4!} - \cdots$$

$$< -\sigma O_0 + \frac{\sigma^2 O_0^2}{2} - \frac{\sigma^3 O_0^3}{3!} + \frac{\sigma^4 O_0^4}{4!} - \cdots$$

$$= -1 + e^{-\sigma O_0} \le 0$$

$$\sum_{k=1}^{\infty} \frac{\left(-\sigma O_0\right)^k}{k \cdot k!} = -\sigma O_0 + \frac{\sigma^2 O_0^2}{2 \cdot 2} - \frac{\sigma^3 O_0^3}{3 \cdot 3!} + \frac{\sigma^4 O_0^4}{4 \cdot 4!} - \cdots$$

$$> -\frac{\sigma O_0}{2} - \frac{\sigma O_0}{2} + \frac{\sigma^2 O_0^2}{2^2 \cdot 2!} - \frac{\sigma^3 O_0^3}{2^3 \cdot 3!} + \frac{\sigma^4 O_0^4}{2^4 \cdot 4!} - \cdots$$

$$= -1 - \frac{\sigma O_0}{2} + e^{\frac{-\sigma O_0}{2}}$$

Obviously the expression in (59) must be positive. The third and the fourth terms are constant and since we are in the long run regime it does not really matter whether they are positive or negative (actually the third term is negative, while the fourth can be either negative or positive depending on $\sigma$, the investment share $\alpha$ and the initial values $O_{10}$ and $O_{20}$). The second term is negative, since $G = -E$. But in the long run the linear function overcomes the logarithmic one. Then we can be sure that what we obtain for $O_3(t)$ in the long run is a positive quantity.

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