L.S. Penrose's limit theorem: proof of some special cases

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Abstract

L.S. Penrose was the first to propose a measure of voting power (which later came to be known as 'the [absolute] Banzhaf (Bz) index'). His limit theorem—which is implicit in his booklet (1952) and for which he gave no rigorous proof—says that in simple weighted voting games (WVGs), if the number of voters increases indefinitely while the quota is pegged at half the total weight, then—under certain conditions—the ratio between the voting powers (as measured by him) of any two voters converges to the ratio between their weights. We conjecture that the theorem holds, under rather general conditions, for large classes of variously defined WVGs, other values of the quota, and other measures of voting power. We provide proofs for some special cases.

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Article Outline

1. Introduction

In his [Penrose, 1946] paper, Lionel Penrose gave the first definition of a priori voting power. According to this definition, as slightly amended in his [Penrose, 1952] booklet the voting power of voter *a* equals the probability ψ_a of *a* ‘being able to influence a decision either way’. Here it is assumed a priori that all voters other than *a* vote independently of one another, each voting ‘yes’ and ‘no’ with equal probability; so that all divisions of those voters into ‘yes’ and ‘no’ camps are equiprobable. Then ψ_a is the probability of the event that those voters are so divided that *a*'s vote will tip the balance: if *a* votes ‘yes’ the act in question will be adopted, and if she/he votes ‘no’ the act will be blocked.[1]

Penrose always assumes that decisions are subject to the simple majority rule, whereby each voter must vote either ‘yes’ or ‘no’ (so that no abstentions are admitted) and a proposed bill is adopted iff it receives over half of all votes. However, he allows the formation of blocs, so that a bloc-voter can have any positive integral number of votes. Thus the decision rules he considers are a special case of what is known in the voting-power literature as a ‘weighted voting game’ (WVG).

Let us recall briefly the definition of a WVG. A WVG $\mathcal{W}$ consists of a finite set *N* of voters together with an assignment of a non-negative real weight w_x to each voter $x \in N$, and a real $q \in (0, 1)$. A bill is passed under $\mathcal{W}$ iff the coalition (set of voters) *A* voting for it satisfies the condition:

$$\sum_{x \in A} w_x \geq q \sum_{x \in N} w_x. \tag{1}$$

We refer to *N* as the *assembly* of $\mathcal{W}$ and to *q* as the latter's *relative quota*. The whole right-hand side of (1), namely *q* multiplied by the total weight of *N*, is the *absolute quota*, or simply the *quota*.

Penrose confines his attention to the special case in which q equals or slightly exceeds (1/2).[2] For such WVGs, he derives in (1952) the following approximation for the voting power ψ_a of voter a:

$$\psi_a \approx w_a \sqrt{\frac{2}{\pi \sum_{x \in N} w_x^2}}. \tag{2}$$

In deriving (2) he assumes that the number of voters is large, and w_a is small compared with the sum S of all weights.[3] Note that as w_a/S becomes vanishingly small, so do both sides of (2). Thus ≈ must be taken to mean that the *relative* error of the approximation tends to 0; in other words, the ratio between the two sides tends to 1.

Implicit in this approximation formula is a limit theorem about the behaviour of the ratio between the voting powers of any two voters, a and b: if the number of voters increases indefinitely, while existing voters always keep their old weights and the relative quota is pegged at 1/2, then (under suitable conditions):

$$\frac{\psi_a}{\psi_b} \to \frac{w_a}{w_b}. \tag{3}$$

Penrose does not present a rigorous proof of (2) and (3), but merely outlines an argument, which is presumably based on some version of the central limit theorem of probability theory.

Unfortunately, (2) or (3) do not always hold under the conditions assumed by Penrose. For example, let $0<w'<w$, and for any positive integer n put:

$$\mathcal{W}^{(n)} := \left[\frac{(w' + nw)}{2};\quad w', \underbrace{w, w, \ldots, w}_{n \text{ times}} \right]. \tag{4}$$

Thus, voters 2, …, n+1 have the same weight, which is greater than that of voter 1; and a bill is adopted iff it receives at least (and hence in fact more than) half the total weight.[4] Clearly, for any fixed n the voting powers $\psi_i[\mathcal{W}^{(n)}]$, for i=2, …, n+1, are positive and equal to one another. But:

(5)

$$\psi_1[\mathcal{W}^{(n)}] = \begin{cases} 0 & \text{if } n \text{ is odd.} \\ \psi_2[\mathcal{W}^{(n)}] & \text{if } n \text{ is even.} \end{cases}$$

Hence (3) does not hold in this case for a=1 and b>1.

Nevertheless, experience suggests that such counter-examples are atypical, contrived exceptions. Both real-life and randomly generated WVGs with many voters provide much empirical evidence that (3) holds in most cases, as a general rule: if the distribution of weights is not too skewed (in other words, the ratio of the largest weight to the smallest is not very high), then the relative powers of the voters tend to approximate closely to their respective relative weights. Moreover, this is the case not only for multi-voter WVGs with q=1/2, but also for those with any $q \in (0, 1)$.

By the *relative power* of voter a in a WVG $\mathcal{W}$ we mean here a's Banzhaf (briefly, Bz) index β, obtained by normalising (or relativising) the Penrose measure:

(6)

$$\beta_a[\mathcal{W}] := \frac{\psi_a[\mathcal{W}]}{\sum_{x \in N} \psi_x[\mathcal{W}]}.$$

Similarly, a's *relative weight* $\bar{w}_a$ in $\mathcal{W}$ is obtained by dividing a's weight by the total weight of all voters:

(7)

$$\bar{w}_a[\mathcal{W}] := \frac{w_a}{\sum_{x \in N} w_x}.$$

The typical tendency of the values of β to approximate to the respective relative weights in multi-voter WVGs is illustrated in Table 1 and Table 2. The WVGs shown in these tables are taken from [Felsenthal and Machover, 2001]. Both are decision rules designed for the so-called qualified majority voting (QMV) in the EU's Council of Ministers following its prospective enlargement to 27 member states. $\mathcal{N}_{27}$ (Table 1) is prescribed in the [Treaty of Nice, 2001]; [5] Rule B (Table 2) is a ‘benchmark’ rule proposed in [Felsenthal and Machover, 2001].

Table 1. QMV under $\mathcal{N}_{27}$

Country	(1)	(2)	(3)	(4)	(5)	(6)
	w	w (%)	β (%)	(3):(2)	ϕ (%)	(5):(2)
Germany	118	8.5199	7.7145	0.905	8.6799	1.019
UK	117	8.4477	7.7145	0.913	8.6670	1.026
France	117	8.4477	7.7145	0.913	8.6670	1.026
Italy	117	8.4477	7.7145	0.913	8.6670	1.026
Spain	108	7.7978	7.3732	0.946	7.9888	1.024
Poland	108	7.7978	7.3732	0.946	7.9888	1.024
Romania	56	4.0433	4.2771	1.058	3.9925	0.987
Netherlands	52	3.7545	3.9900	1.063	3.6866	0.982
Greece	48	3.4657	3.7092	1.070	3.3977	0.980
Czech Republic	48	3.4657	3.7092	1.070	3.3977	0.980
Belgium	48	3.4657	3.7092	1.070	3.3977	0.980
Hungary	48	3.4657	3.7092	1.070	3.3977	0.980
Portugal	48	3.4657	3.7092	1.070	3.3977	0.980
Sweden	40	2.8881	3.1126	1.078	2.8137	0.974
Bulgaria	40	2.8881	3.1126	1.078	2.8137	0.974
Austria	40	2.8881	3.1126	1.078	2.8137	0.974
Slovakia	28	2.0217	2.1984	1.087	1.9594	0.969
Denmark	28	2.0217	2.1984	1.087	1.9594	0.969
Finland	28	2.0217	2.1984	1.087	1.9594	0.969
Ireland	28	2.0217	2.1984	1.087	1.9594	0.969
Lithuania	28	2.0217	2.1984	1.087	1.9594	0.969
Latvia	16	1.1552	1.2603	1.091	1.1209	0.970
Slovenia	16	1.1552	1.2603	1.091	1.1209	0.970
Estonia	16	1.1552	1.2603	1.091	1.1209	0.970
Cyprus	16	1.1552	1.2603	1.091	1.1209	0.970
Luxembourg	16	1.1552	1.2603	1.091	1.1209	0.970
Malta	12	0.8664	0.9514	1.098	0.8310	0.959
Total	1385	100.0001	100.0002		99.9997	

Full-size table (<1K)

Quota: 1034=74.66% of 1385. Note: For explanations see main text.

Table 2. Rule B (benchmark QMV rule for enlarged CM)

Country	(1)	(2)	(3)	(4)	(5)	(6)
	w	$\bar{w}$ (%)	β (%)	(3):(2)	ϕ (%)	(5):(2)
Germany	954	9.5381	9.6184	1.008	9.9894	1.047
UK	810	8.0984	8.1441	1.006	8.3359	1.029
France	809	8.0884	8.1338	1.006	8.3246	1.029
Italy	799	7.9884	8.0312	1.005	8.2122	1.028
Spain	661	6.6087	6.6219	1.002	6.6886	1.012
Poland	655	6.5487	6.5606	1.002	6.6232	1.011
Romania	499	4.9890	4.9813	0.998	4.9627	0.994
Netherlands	418	4.1792	4.1665	0.997	4.1221	0.982
Greece	342	3.4193	3.4080	0.996	3.3470	0.979
Czech Republic	338	3.3793	3.3649	0.996	3.3066	0.978
Belgium	337	3.3693	3.3549	0.996	3.2965	0.978
Hungary	335	3.3493	3.3349	0.996	3.2761	0.978
Portugal	333	3.3293	3.3149	0.996	3.2559	0.978
Sweden	313	3.1294	3.1151	0.995	3.0553	0.976
Bulgaria	302	3.0194	3.0061	0.995	2.9453	0.975
Austria	299	2.9894	2.9751	0.994	2.9153	0.975
Slovakia	245	2.4495	2.4365	0.995	2.3755	0.970
Denmark	243	2.4295	2.4166	0.995	2.3556	0.970
Finland	239	2.3895	2.3766	0.995	2.3157	0.969
Ireland	204	2.0396	2.0277	0.994	1.9706	0.966
Lithuania	203	2.0296	2.0176	0.994	1.9604	0.965
Latvia	164	1.6397	1.6299	0.994	1.5783	0.963
Slovenia	148	1.4797	1.4706	0.994	1.4223	0.961
Estonia	127	1.2697	1.2615	0.994	1.2175	0.959
Cyprus	91	0.9098	0.9042	0.994	0.8693	0.955
Luxembourg	69	0.6899	0.6854	0.993	0.6580	0.954
Malta	65	0.6499	0.6457	0.994	0.6200	0.954
Total	10 002	100.0000	100.0000		99.9999	

Full-size table (<1K)

Quota: 6000=59.99% of 10 002. Note: For explanations see main text.

In each of these tables, column (1) gives the weights of the voters. The absolute and relative quota are stated at the bottom of the table. Column (2) gives the respective relative weights $\bar{w}$ as percentages. Column (3) gives the relative voting powers as measured by the Bz index β, also in percentage terms. Column (4) gives the ratio of the Bz index to the respective relative weight. Note that all the figures in this column are quite close to 1. In Table 1 they are well within the range 1±0.1. In Table 2—where the quota is nearer half the total weight—the approximation is even better: the ratios are all well within the range 1±0.01.

The same tendency is also apparent in Table 3, which is based on a WVG model of the Electoral College that elects the President of the US. The figures for β are quite close to those for $\bar{w}$.

Table 3. US Presidential Electoral College (1970 Census)

Number	w	$\bar{w}$ (%)	Φ (%)	β (%)	ψ	ψ approximate
1	45	8.3643	8.8309	8.8816	0.379366	0.403527
1	41	7.6208	7.9727	7.9513	0.339629	0.359921
1	27	5.0186	5.0963	5.0457	0.215522	0.224441
2	26	4.8327	4.8977	4.8499	0.207159	0.215510
1	25	4.6468	4.6999	4.6533	0.198844	0.206653
1	21	3.9033	3.9169	3.8865	0.166007	0.171900
2	17	3.1599	3.1465	3.1308	0.133730	0.138057
1	14	2.6022	2.5767	2.5708	0.109809	0.113130
2	13	2.4164	2.3882	2.3852	0.101879	0.104923
3	12	2.2305	2.2004	2.2000	0.093970	0.096728
1	11	2.0446	2.0133	2.0152	0.086078	0.088564
4	10	1.8587	1.8270	1.8308	0.078202	0.080426
4	9	1.6729	1.6413	1.6468	0.070341	0.072314
2	8	1.4870	1.4563	1.4631	0.062493	0.064224
4	7	1.3011	1.2719	1.2796	0.054656	0.056153
4	6	1.1152	1.0883	1.0964	0.046830	0.048100
1	5	0.9294	0.9053	0.9133	0.039012	0.040061
9	4	0.7435	0.7230	0.7305	0.031201	0.032034
7	3	0.5576	0.5413	0.5477	0.023396	0.024017
Total 51	538	99.9998	100.0009	100.0005		

Full-size table (<1K)

Quota: 270=50.19% of 538. Note: For the purpose of this table, the Electoral College is regarded as a WVG, in which each ‘voter’ is a bloc of Electors for a State, or for the District of Columbia. The number of Electors in each bloc is taken as the weight w of this bloc-voter. The first column, headed ‘Number’ shows the number of blocs with a given weight w. This way of modelling the Electoral College involves some over-simplification, because there may be more than two candidates, and since 1969 the Electors of Maine did not have to vote as a single bloc. (Since 1993, the same applies to Nebraska). We use this model here for the sake of computational illustration, and for comparison with Table XII.4.1 of [Owen, 1995], (p. 297), which is based on the same model. For further explanations, see Remark 3.5.

Moreover, a similar phenomenon is observable not only for the Bz index, but for also for some other indices of voting power, notably the Shapley–Shubik (briefly, S–S) index φ.[6] This typical behaviour of φ is also illustrated in Table 1 and Table 2. In these tables, column (5) gives the values of the S–S index φ in percentage terms and column (6) gives the ratio of these values to the respective relative weights. Note that all these ratios are well within the range 1±0.05. The same tendency is evident also in Table 3: compare the figures for φ with those for w.

This suggests a general problem: under what conditions does the ratio of the voting powers of any two voters, as measured by a given index, converge to the ratio of their weights?

In order to make this problem more precise, let us introduce the following framework.

Definition 1.1. Let

$$N^{(0)} \subsetneq N^{(1)} \subsetneq N^{(2)} \subsetneq \ldots \tag{8}$$

be an infinite increasing chain of finite non-empty sets, and let

$$N = \bigcup_{n=0}^{\infty} N^{(n)}. \tag{9}$$

Let w be a function that assigns to each $a \in N$ a positive real number w_a as *weight*; and let q be a real $\in (0, 1)$. For each $n \in \mathbb{N}$ let $\mathcal{W}^{(n)}$ be the WVG whose assembly is $N^{(n)}$—each voter $a \in N^{(n)}$ being endowed with the pre-assigned weight w_a—and whose relative quota is q. We shall then say that $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ is a *q-chain* of WVGs. Further, let ξ be an index of voting power. We shall say that *Penrose's Limit Theorem* (*PLT*) *holds for the q-chain* $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ *with respect to the index* ξ if for any $a, b \in N$

$$\lim_{n\to\infty} \frac{\xi_a[\mathcal{W}^{(n)}]}{\xi_b[\mathcal{W}^{(n)}]} = \frac{w_a}{w_b}. \tag{10}$$

Remark 1.2.

(i)In what follows, whenever we shall refer to a q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$, we shall assume that the $N^{(n)}$, N and w are as specified in Definition 1.1: $N^{(n)}$ is the assembly of $\mathcal{W}^{(n)}$, N is given by (9), and w is the weight function.

(ii)Note that $\xi_a[\mathcal{W}^{(n)}]/\xi_b[\mathcal{W}^{(n)}]$ in (10) is undefined if $a \notin N^{(n)}$ or $b \notin N^{(n)}$, but this does not matter because $a, b \in N^{(n)}$ for all sufficiently large n.

(iii)Definition 1.1 may be extended to weighted ternary voting games, in which voters have the option of abstaining—cf. [Felsenthal and Machover, 1997 and Felsenthal and Machover, 1998]. The only change that needs to be made to the definition is that each $\mathcal{W}^{(n)}$, instead of being a (binary) WVG, is the ternary decision rule whereby a bill is passed iff the total weight of those voting for it is at least q times the total weight of those voting against it. Of course, ξ must then be an index defined for such games.

In preparation for what follows, we introduce two items of notation.

First, note that if $a \in N^{(n)}$ the relative weight of a in $\mathcal{W}^{(n)}$—unlike a's absolute weight w_a—depends on n. We denote this relative weight by $\bar{w}_a^{(n)}$; thus:

(11)

$$\bar{w}_a^{(n)} := \frac{w_a}{\sum_{x \in N^{(n)}} w_x}.$$

Second, for each $a \in N$ we put:

(12)

$$N_a^{(n)} := \{ x \in N^{(n)} : w_x = w_a \}.$$

The members of $N_a^{(n)}$ have the same weight as a, and we shall therefore refer to them as *replicas* of a.

2. PLT for replicative q-chains and the S–S index

In this section, we shall prove that PLT holds with respect to the S–S index for a special class of chains. The main special property of these chains is that each $a \in N$ is eventually (that is, for sufficiently large n) accompanied by sufficiently many replicas in N_n. Let us make this more precise.

Definition 2.1. We shall say that the q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ is *replicative* if it satisfies the following two conditions. First,

(13)

$$\lim_{n\to\infty} \max\{ \bar{w}_a^{(n)} : a \in N^{(n)} \} = 0.$$

Second, for each $a \in N$ there is a positive constant C_a such that for all sufficiently large n

(14)

$$\sum_{x\in X_a^{(n)}} \bar{w}_x^{(n)} > C_a.$$

Remark 2.2. Condition ([13](#)) is essentially the one assumed by Penrose: the relative weight of each *individual* voter becomes negligibly small. This condition is automatically satisfied if the values of *w* are bounded from above and bounded away from 0. The second condition ([14](#)) ensures that nevertheless, the total relative weight of the voter's replicas does not become negligibly small.

Our main result in this section is

Theorem 2.3. *If* $\{W^{(n)}\}_{n=0}^{\infty}$ *is a replicative q-chain then PLT holds for it with respect to the S–S index φ.*

Proof. We shall show that for each $a \in N$

(15)

$$\lim_{n\to\infty} \frac{\Phi_a[W^{(n)}]}{\bar{w}_a^{(n)}} = 1.$$

from which our theorem clearly follows. To this end, we invoke a result of [[Neyman, 1982](#)], Theorem 9.8), according to which ([13](#)) implies that:

(16)

$$\lim_{n\to\infty} \sum_{x\in I^{(n)}} |\Phi_x[W^{(n)}] - \bar{w}_x^{(n)}| = 0.$$

Now let $a \in N$. Then we have, a fortiori,

(17)

$$\lim_{n\to\infty} \sum_{x\in X_a^{(n)}} |\Phi_x[W^{(n)}] - \bar{w}_x^{(n)}| = 0.$$

which can be written as:

(18)

$$\lim_{n\to\infty} \sum_{x\in X_a^{(n)}} \bar{w}_x^{(n)} \left| \frac{\Phi_x[W^{(n)}]}{\bar{w}_x^{(n)}} - 1 \right| = 0.$$

However, all the $x \in N_a^{(n)}$ are replicas of a, so they all have the same value of φ and the same weight as a. Hence (18) can be written as follows:

$$\lim_{n\to\infty}\left|\frac{\Phi_a[\mathcal{W}^{(n)}]}{w_a^{(n)}}-1\right|\sum_{x\in N_a^{(n)}}\pi_x^{(n)}=0. \tag{19}$$

It now follows from (14) that (15) holds—as claimed. □

Remark 2.4. In the definition of WVG, the blunt inequality ≥ in (1) can be replaced by a sharp inequality>. The two definitions are equivalent: they determine the same class of structures. However, the relative quota q of a WVG in the blunt sense may not work for the sharp sense, but may need to be slightly adjusted (and vice versa). Consequently, the corresponding definitions of q-chain and replicative q-chain in the sharp sense do not yield the same classes as our present Definition 1.1 and Definition 2.1Definition 1.1 and Definition 2.1. Nevertheless, Theorem 2.3 applies to replicative q-chains in the sharp sense as well, because Neyman's result, on which our proof depends, also covers this case—see [Neyman, 1981], Lemma 3.2).

3. PLT for some 1/2-chains and the Banzhaf index

Given a q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ of WVGs (see Definition 1.1), we associate with it the family $\{ \Upsilon_x : x \in N \}$ of independent random variables indexed by N, such that for every $a \in N$,

$$\mathrm{Prob}\{\Upsilon_a = w_a\} = \mathrm{Prob}\{\Upsilon_a = 0\} = \frac{1}{2}. \tag{20}$$

We consider the chain

$$\Upsilon := \{\{\Upsilon_x : x \in N^{(n)}\} : n \in \mathbb{N}\} \tag{21}$$

of (finite) sets of these random variables.

For any $a \in N$ let us put:

(22)

$$S_{\neg a}^{(n)} := \left(\sum_{x\in N^{(n)}} Y_x\right) - Y_a, \qquad \mu_{\neg a}^{(n)} := \mathbf{E}S_{\neg a}^{(n)}, \qquad \sigma_{\neg a}^{(n)} := (\mathrm{Var}\, S_{\neg a}^{(n)})^{1/2}.$$

And let $\tilde{S}_{\neg a}^{(n)}$ be the 'standardised' form of $S_{\neg a}^{(n)}$, i.e.

(23)

$$\tilde{S}_{\neg a}^{(n)} := \frac{S_{\neg a}^{(n)} - \mu_{\neg a}^{(n)}}{\sigma_{\neg a}^{(n)}}.$$

Using the definition of the Y_a it is easy to obtain the following explicit expressions for $\mu_{\neg a}^{(n)}$ and $\sigma_{\neg a}^{(n)}$.

(24)

$$\mu_{\neg a}^{(n)} = \frac{\left(\sum_{x\in N^{(n)}} w_x\right) - w_a}{2}.$$

(25)

$$\sigma_{\neg a}^{(n)} = \frac{\left[\left(\sum_{x\in N^{(n)}} w_x^2\right) - w_a^2\right]}{2}.$$

Definition 3.1. We shall say that the chain $\mathcal{Y}$ *satisfies the special local central limit* (*SLCL*) *condition* if, for every $a \in N$,

(26)

$$\lim_{n\to\infty} \mathrm{Prob}\left\{\tilde{S}_{\neg a}^{(n)} \in \left[-\frac{w_a}{2\sigma_{\neg a}^{(n)}}, \frac{w_a}{2\sigma_{\neg a}^{(n)}}\right)\right\} \frac{\sigma_{\neg a}^{(n)}}{w_a} = \frac{1}{\sqrt{2\pi}};$$

and for all $a, b \in N$,

(27)

$$\lim_{n\to\infty} \frac{\sigma_{\neg a}^{(n)}}{\sigma_{\neg b}^{(n)}} = 1.$$

Remark 3.2. The $\tilde{S}_{\neg a}^{(n)}$ are evidently discrete random variables with mean 0. We shall be interested in cases where their standard deviations, $\sigma_{\neg a}^{(n)}$, tend to ∞ with n. Then Eq. (26) says that the average density of $\tilde{S}_{\neg a}^{(n)}$ in a half-open interval around 0, whose length becomes vanishingly small, approaches the value of the standard normal density function \at 0, namely $1/\sqrt{2\pi}$. This means that $\mathcal{Y}$ obeys a special case (namely, at 0) of the local central limit theorem of probability theory.

The main result in this section is

Proposition 3.3. *Let* $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ *be a* 1/2*-chain of WVGs. If its associated chain* $\mathcal{Y}$ *satisfies the SLCL condition, then PLT holds for* $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ *with respect to the Bz index.*

Proof. Let $a \in N$ and take n large enough so that $a \in N^{(n)}$. Then, by definition, the Penrose power of a in $\mathcal{W}^{(n)}$ is given by:

$$\psi_a[\mathcal{W}^{(n)}] = \text{Prob}\left\{S^{(n)}_{-a} \in \left[\frac{1}{2}\left(\sum_{x \in N^{(n)}} w_x\right) - w_a, \frac{1}{2}\sum_{x \in N^{(n)}} w_x\right)\right\}. \tag{28}$$

Using (23) and (24), this can be re-written as:

$$\psi_a[\mathcal{W}^{(n)}] = \text{Prob}\left\{\bar{S}^{(n)}_{-a} \in \left[-\frac{w_a}{2\sigma^{(n)}_{-a}}, \frac{w_a}{2\sigma^{(n)}_{-a}}\right)\right\}. \tag{29}$$

Invoking (26) we obtain

$$\lim_{n\to\infty} \psi_a[\mathcal{W}^{(n)}]\frac{\sigma^{(n)}_{-a}}{w_a} = \frac{1}{\sqrt{2\pi}}. \tag{30}$$

Hence by (27)

$$\lim_{n\to\infty} \frac{\psi_a[\mathcal{W}^{(n)}]}{\psi_b[\mathcal{W}^{(n)}]} = \frac{w_a}{w_b}. \tag{31}$$

Finally, using (6) we get:

$$\lim_{n\to\infty} \frac{\beta_a[\mathcal{W}^{(n)}]}{\beta_b[\mathcal{W}^{(n)}]} = \frac{w_a}{w_b}. \tag{32}$$

as claimed. □

Combining (30) and (25) we get:

Corollary 3.4. *If (26) holds, then*

$$\psi_a[W^{(n)}] \approx w_a \sqrt{\frac{2}{\pi\left\{\left(\sum_{x\in V^{(n)}} w_x^2\right) - w_a^2\right\}}}. \tag{33}$$

This is our slightly improved version of Penrose's approximation formula (2). Of course, if—as Penrose assumes—each individual weight w_a becomes relatively negligible, then the difference between the two approximations is likewise negligible. □

Remark 3.5. [Owen, 1995], pp. 272, 297) gives approximation formulas for ψ as well as for φ in multi-voter WVGs. His approximations are based on an interval version of the central limit theorem (as opposed to the local form used by us), and are stated without proof and without specifying the precise conditions under which they hold.[7] Nevertheless, the numerical approximations he obtains for the Penrose powers φ of the bloc-voters in the US Presidential Electoral College—shown in the last column of Table XII.4.1 of [Owen, 1995], p. 297)—are closer than ours, which are based on (33) above and shown in the last column of our Table 3. (The exact values of ψ, correct to six decimal figures, are shown in the penultimate column of Table 3).

As an example of an application of Proposition 3.3, we prove the following:

Theorem 3.6. *Let* $\{W^{(n)}\}_{n=0}^{\infty}$ *be a* 1/2*-chain such that its weight function w assumes only finitely many values, all of them positive integers; and such that the greatest common divisor of those values* w_a *that occur infinitely often is* 1. *Then the associated chain* $\mathcal{V}$ *satisfies the SLCL condition. Hence PLT holds for* $\{W^{(n)}\}_{n=0}^{\infty}$ *with respect to the Bz index. Also, (33) holds.*

Proof. To prove that (26) holds for any $a\in N$, observe that since all possible values of $S_{\neg a}^{(n)}$ are integers, all possible values of $S_{\neg a}^{*(n)}$ belong to a lattice whose span is $1/\sigma_{\neg a}^{(n)}$. In the half-open interval:

(34)

$$\left[-\frac{w_a}{2\sigma_{\neg a}^{(n)}}, \frac{w_a}{2\sigma_{\neg a}^{(n)}}\right)$$

there are exactly w_a points of this lattice: say $x_i^{(n)}$, i=1, 2, …, w_a. We now invoke a well-known version of the local central limit theorem—see [Petrov, 1975], p. 189, Theorem 2). ,[8] From this theorem it follows that if n is sufficiently large then for each i=1, 2, …, w_a the product:

$$\text{Prob}\{S_{\neg a}^{(n)} = x_i^{(n)}\}\sigma_{\neg a}^{(n)} \tag{35}$$

is arbitrarily close to $\phi(x_i^{(n)})$. Also, from (25) it is clear that $\lim_{n\to\infty} \sigma_{\neg a}^{(n)}=\infty$; thus for sufficiently large n each of the $x_i^{(n)}$ is arbitrarily close to 0, hence the product (35) is arbitrarily close to $\phi(0)=(2\pi)^{-1/2}$. But the left-hand side of (26) is simply the arithmetic mean of the w_a products (35); so it also gets arbitrarily close to $(2\pi)^{-1/2}$, as required. As for (27): we have just seen that as n increases, $\sigma_{\neg a}^{(n)}$ grows without bound. Clearly, the term $w_a{}^2$ in (25) becomes relatively negligible. Therefore, (27) holds. □

Remark 3.7. Note that the chain defined in our counter-example (4) fails to satisfy the condition of Theorem 3.6, even if the weights w and w' in (4) are integers. In this case, only the greater weight, w, occurs infinitely often, and so it is trivially also the gcd of the weights that occur infinitely often; but w>1.

4. Discussion

PLT may best be regarded not as a single theorem but—like the central limit theorem of probability theory, with which it has some affinity—as an open-ended research programme covering many related results. Our present results are merely a modest contribution to this programme.

On the basis of empirical-computational evidence, we conjecture that similar results hold for other classes of q-chains (including those of suitably defined weighted games that admit abstentions) with respect to the S–S , Bz as well as other indices of voting power.

In fact, it seems to us likely that PLT holds almost always, in a sense that can be made precise, along the following lines.

Let $\mathbb{N}^+$ be the set of positive integers and consider the Cartesian product space:

(36)

$$\mathcal{W} = (0, 1) \times \mathbb{N}^{+\mathbb{N}}.$$

Each member of $\mathcal{W}$ is then an infinite sequence of the form (q; w_0, w_1, …) where $q \in$ (0, 1) and the w_n are positive integers. Such a sequence gives rise to a q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$, where $N^{(n)}=\{0, 1, \ldots, n\}$ for each $n \in \mathbb{N}$.

Further, we can regard $\mathcal{W}$ as a product *probability* space by taking (0, 1) with the Lebesgue probability measure, and each copy of $\mathbb{N}^+$ with a reasonable probability distribution: say a geometric distribution (Prob$\{k\}=2^{-k}$), or a Poisson distribution (Prob$\{k\}=e^{-1}/(k-1)!$).

Or, instead of confining ourselves to integer weights, we can allow arbitrary positive real weights. To this end we can replace $\mathbb{N}^+$ by the set $\mathbb{R}^+$ of positive reals, with some reasonable probability measure on each copy—using, say, a Gaussian density f on the positive half-line:

(37)

$$f(x) = \sqrt{\frac{2e^{-x^2}}{\pi}}.$$

It now makes precise sense to talk about the probability that PLT holds, with respect to a given index, for the chain corresponding to a randomly chosen member of $\mathcal{W}$.

We conjecture that PLT holds with probability 1 with respect to both the $\mathbb{S}$–$\mathbb{S}$ and the Bz index.

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References

Treaty of Nice, 2001. Treaty of Nice, 2001. Conference of the Representatives of the Governments of the Member States, Brussels, 28 February 2001. Treaty of Nice amending the Treaty on European Union, the Treaties establishing the European Communities and certain related Acts. EU document CONFER 4820/00.

Felsenthal and Machover, 1997. D.S. Felsenthal and M. Machover, Ternary voting games. *International Journal of Game Theory* **26** (1997), pp. 335–351. MathSciNet | **Full Text** via CrossRef | View Record in Scopus | Cited By in Scopus (31)

Felsenthal and Machover, 1998. D.S. Felsenthal and M. Machover. *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes*, Edward Elgar, Cheltenham (1998).

Felsenthal and Machover, 2001. D.S. Felsenthal and M. Machover, The treaty of nice and qualified majority voting. *Social Choice and Welfare* **18** (2001), pp. 431–464. **Full Text** via CrossRef | View Record in Scopus | Cited By in Scopus (24)

Neyman, 1981. A. Neyman, Singular games have asymptotic values. *Mathematics of Operations Research* **6** (1981), pp. 205–212. MathSciNet

Neyman, 1982. A. Neyman, Renewal theory for sampling without replacement. *Annals of Probability* **10** (1982), pp. 464–481. MathSciNet

Owen, 1995. G. Owen. *Game Theory* (3rd Edition ed.),, Academic Press, San Diego (1995).

Penrose, 1946. L.S. Penrose, The elementary statistics of majority voting. *Journal of the Royal Statistical Society* **109** (1946), pp. 53–57. **Full Text** via CrossRef

Penrose, 1952. L.S. Penrose. *On the Objective Study of Crowd Behaviour*, H.K. Lewis and Co, London (1952).

Petrov, 1975. V.V. Petrov. *Sums of Independent Random Variables*, Springer, Berlin, Heidelberg, New York (1975).

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[1] We have stated the a priori assumption more fully than Penrose, who merely says that the other voters are assumed to act ‘at random’. The definition he had given in (1946) took $\psi_a/2$ rather than ψ_a itself as *a*'s voting power; the difference is of course inessential. Penrose's measure ψ is often referred to in the literature as ‘the

[absolute] Banzhaf index' and denoted by 'β''. In using 'ψ' we are following [Owen, 1995].

[2] In fact, he seems to be thinking of (1) with> instead of ≥, and q=1/2. We shall return to this minor point below; see Remark 2.4 (ii).

[3] In stating (2) and the assumptions under which it is derived, we are paraphrasing Penrose. For his own formulations see his (1952, p. 715).

[4] For the square bracket notation see, for example, [Felsenthal and Machover, 1998], Definition 2.3.14).

[5] $\mathcal{N}_{27}$ is not stated in the treaty in this simple form, as a WVG; but it can be reduced to the form shown in Table 1. For details, see [Felsenthal and Machover, 2001], Section 3).

[6] Thus, in multi-voter WVGs in which the distribution of weights is not extremely skewed, the respective values of β and φ tend, as a general rule, to be quite close to each other. This phenomenon has helped to foster the widespread fallacy that these two indices always behave alike, and so must have more or less the same meaning. This fallacy is criticised in [Felsenthal and Machover, 1998].

[7] Rigorous validation of these approximations is not straightforward. In cases where the approximation is expected to hold, both the relative voting power of each voter and the term approximating it tend to 0 as the number of voters increases. In order to validate the approximation, it must be proved not only that the error term—the difference between the true value and the approximating term—also tends to 0, but that it does so faster than the approximating term.

[8] This theorem deals with a sequence of independent integer-valued random variables each having finite variance, such that the set of distinct distributions of these variables is finite. The key condition is that the greatest common divisor of the maximal spans of those distributions that occur infinitely often in the sequence is 1. For details see [Petrov, 1975], ibid).