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A SPECIAL CASE OF PENROSE'S LIMIT THEOREM WHEN ABSTENTION IS ALLOWED

ABSTRACT. In general, analyses of voting power are performed through the notion of a simple voting game (SVG) in which every voter can choose between two options: 'yes' or 'no'. Felsenthal and Machover [Felsenthal, D.S. and Machover, M. (1997), *International Journal of Game Theory* 26, 335–351.] introduced the concept of ternary voting games (TVGs) which recognizes abstention alongside. They derive appropriate generalizations of the Shapley–Shubik and Banzhaf indices in TVGs. Braham and Steffen [Braham, M. and Steffen, F. (2002), in Holler, et al. (eds.), *Power and Fairness*, Jahrbuch für Neue Politische Ökonomie 20, Mohr Siebeck, pp. 333–348.] argued that the decision-making structure of a TVG may not be justified. They propose a sequential structure in which voters first decide between participation and abstention and then between 'yes' or 'no'. The purpose of this paper is two-fold. First, we compare the two approaches and show how the probabilistic interpretation of power provides a unifying characterization of analogues of the Banzhaf (Bz) measure. Second, using the probabilistic approach we shall prove a special case of *Penrose's Limit Theorem* (PLT). This theorem deals with an asymptotic property in weighted voting games with an increasing number of voters. It says that under certain conditions the ratio between the voting power of any two voters (according to various measures of voting power) approaches the ratio between their weights. We show that PLT holds in TVGs for analogues of Bz measures, irrespective of the particular nature of abstention.

KEY WORDS: limit theorems, ternary voting games, voting power, weighted voting games

JEL CLASSIFICATION: C 71, D71

1. INTRODUCTION

In real-life decisions, the option to abstain is one that can undoubtedly influence the outcome of a vote. This is clearly evident in the most commonly used rule in decision-making bodies: the simple majority, whereby a resolution passes if and only if more voters vote for it than against it. Unless specified otherwise, this rule does not treat abstentions as tantamount to either 'yes' or 'no'. Certainly, in some real-life decision rule abstention is not a distinct third option. For example, in the Council of Ministers of the European Union, abstention usually counts as a 'no', except when an issue to be vote upon is basic constitutional. In this case abstention counts as a 'yes'. However, these are exceptions since in most real-life situations abstention is a *tertium quid*. In the United Nations Security Council (UNSC) abstention plays a key role: an abstaining permanent member is usually not interpreted as a voter. Since Article 27 of the UN Charter requires a minimum of nine affirmative members abstention it is not treated tantamount to 'yes' either.[1] In each of the two houses of the US Congress the rule is that for a proposal to pass a certain percentage[2] of the members *present* has to be achieved (provided that a quorum of half the membership is present).

Somewhat surprisingly, the literature has only recently started to take any notice of it. The widely used instrument to analyze voting power is that of a simple voting game (SVG) which is *binary* in that it assumes that each voter has just two options: 'yes' and 'no'. This is even more surprising as social choice theory does not in general impose strict preference orderings, i.e. it allows for indifference over alternatives. In their 1998 paper, Felsenthal and Machover criticize the 'misreporting' of some authors to squeeze rules into the SVG corset when abstention is a distinct third option. Felsenthal and Machover (1997, 1998) overcome this shortcoming by proposing a setup called a *ternary voting game* (TVG). By adding abstention as a third option alongside 'yes' and 'no' they define an appropriate generalization of a SVG.[3]

In their TVG setup they offer analogous definitions of the Banzhaf measure (1965) and Shapley–Shubik index (1954), the classical measures of voting power in SVGs. Whereas an analogous definition of the Banzhaf (Bz) measure follows more or less naturally, the translation of the Shapley–Shubik

(S-S) index is less obvious. The authors construct it by means of an alternative representation of this index (Felsenthal and Machover, 1996).

Any reasonable extension of a power concept in the more general setting of games with abstentions should of course provide the prevalent classical power measures designed for SVGs games when the option to abstain is equivalent to either 'yes' or 'no'. However, several generalizations may have the same projection to the binary case, such that it is a mistake to latch on to a particular formulation of the original idea. More recently, Braham and Steffen (2002) remarked that the simple majority rule is often specified as *counting only the votes of those voting* ('yes' or 'no') so that abstention can be seen as tantamount to 'non-participation'. From this they argue that in contrast to Felsenthal and Machover who treat 'abstain' as symmetric to 'yes' and 'no', abstentions are to be treated separately. They point out that the TVG structure assumes that voters can choose simultaneously between 'yes', 'no' and 'abstain', when in fact the 'counting the votes of those voting' implies a sequential choice structure: a voter first decides whether to vote at all, and then to vote 'yes' or 'no'. In particular this approach suggests other generalizations of the Bz and S-S index than the ones proposed by Felsenthal and Machover.

This paper provides a probabilistic characterization of Bz power in games with abstentions which unifies both approaches of taking abstention into account. It will be achieved by recourse to a probabilistic interpretation of voting power, such that it is expressed as a voter's *expected contribution* to the outcome of the vote (i.e. the practical difference that a voter makes). Furthermore, the fact that the Bz power in TVG behaves mathematically as expectations allows to apply the powerful tools of stochastics, primarily important for approximation purposes and analyzing asymptotic properties. Using the probabilistic approach we shall prove a special case of *Penrose's Limit Theorem* (PLT).

L S Penrose (1946) was the first to propose a measure of voting power (which later came to be known as 'the (absolute)

Bz measure'). His limit theorem – which is implicit in Penrose (1952) – says that, in simple weighted voting games, if the number of voters increases indefinitely while the quota is pegged at half the total weight, then – under certain conditions – the ratio between the voting powers (as measured by him) of any two voters converges to the ratio between their weights. Penrose gave no rigorous proof of this limit theorem and there are in fact counter-examples to his claim (see Lindner and Machover, 2004). Nevertheless, experience suggests that such counter-examples are atypical, contrived exceptions. Both real-life and randomly generated weighted voting games (WVGs) with many voters provide much empirical evidence that the following holds in most cases as a general rule: if the distribution of weights is not too skewed (in other words, the ratio of the largest weight to the smallest is not very high), then the relative powers of the voters tend to approximate closely to their respective relative weights.

This typical tendency is illustrated in Table I which is based on a WVG model of the Electoral College that elects the President of the US. Here, each 'voter' is a bloc of Electors for a State, or for the District of Columbia.[4] California, one of the most populous states, can cast 55 electoral votes while Alaska may cast only 3 votes. The column headed 'No.' shows the number of states with a given weight w. Column (1) gives the weights of the voters. The absolute and relative quota are stated at the bottom of the table. Column (2) gives the respective relative weights $\bar{w}$ as percentages. Column (3) to (5) show various measures of voting power in percentage terms. Column (3) gives the voting powers as measured by the S-S index ϕ. Column (4) gives the relative voting powers as measured by the Bz index β. Column (5) gives the *generalized Bz index* as proposed by Felsenthal and Machover (1997, 1998) for voting games with abstention. Note that we reserve the term 'index' for measures whose values for all voters always add up to 1. Hence, β and $\tilde{\beta}$ are obtained by normalizing the (absolute) Bz measures.

TABLE I
US Presidential Electoral College (2000 Census)

No.	(1) w	(2) $\bar{w}$ (%)	(3) ϕ (%)	(4) β (%)	(5) $\tilde{\beta}$ (%)
1	55	10.2230	11.0358	11.4021	11.0402
1	34	6.3197	6.4988	6.3927	6.3875
1	31	5.7621	5.8884	5.7948	5.7959
1	27	5.0186	5.0869	5.0119	5.0199
2	21	3.9033	3.9100	3.8654	3.8784
1	20	3.7175	3.7166	3.6771	3.6903
1	17	3.1599	3.1411	3.1157	3.1289
3	15	2.7881	2.7612	2.7442	2.7569
1	13	2.4164	2.3841	2.3746	2.3863
1	12	2.2305	2.1967	2.1904	2.2016
4	11	2.0446	2.0099	2.0066	2.0171
4	10	1.8587	1.8239	1.8231	1.8329
3	9	1.6729	1.6385	1.6400	1.6489
2	8	1.4870	1.4538	1.4571	1.4652
4	7	1.3011	1.2698	1.2744	1.2816
3	6	1.1152	1.0865	1.0920	1.0982
5	5	0.9294	0.9038	0.9097	0.9150
5	4	0.7435	0.7218	.7276	0.7318
8	3	0.5576	0.5404	0.5456	0.5488
Total 51	538	99.9977	100.0002	100.0006	100.0006
$D(\xi, w) * 10^3$			23.9940	25.6880	18.3940
$d(\xi, w) * 10^3$			73.6512	103.4108	74.0204
Quota: 270 = 50.19% of 538					

Note For the purpose of this table, the Electoral College is regarded as a WVG, in which each 'voter' is a bloc of Electors for a State, or for the District of Columbia. The number of Electors in each bloc is taken as the weight w of this bloc-voter.

Let ξ be an index of voting power. The overall discrepancy between ξ_k and the relative voting weight $\bar{w}_k$ is given by

$$D(\xi, w) := \sum_{k=1}^{n} |\xi_k - \bar{w}_k| . \tag{1}$$

The *local* distortion is given by

$$d(\xi, w) := max_{1 \leq k \leq n} \left| 1 - \frac{\xi_k}{\bar{w}_k} \right| . \tag{2}$$

Table I illustrates the typical tendency of various indices ξ to approximate closely to their respective relative weigths $\bar{w}$.

This suggests a general problem: under what conditions does the ratio of the voting powers of any two voters, as measured by a given index, converge to the ratio of their weights? Penrose's claim implies that this asymptotic property holds with respect to the Bz index β if the relative quota is 1/2 and the relative weight of each voter tends to 0. However, Lindner and Machover (2004) show by means of a simple counterexample that these conditions are insufficient. They prove the asymptotic property for β for the relative quota $q = 1/2$ satisfying more stringent conditions. They further provide sufficient conditions with respect to the S-S index ϕ for a large class of WVGs with arbitrary $q \in (0, 1)$.

However, on the basis of empirical-computational evidence these sufficient conditions are most likely to be too strict. In addition, similar results seem to apply to other measures of voting power besides the classical indices ϕ and β (as illustrated above for $\tilde{\beta}$). Hence, PLT may best be regarded not as a collection of a single theorem but – like the central limit theorem of probability theory, with which it has some affinity – as an open-ended research programme covering many related results. The present paper is a contribution to this programme. PLT suggests *relative irrelevance*: with increasing number of voters the power ratio of any two voters converges to the ratio of the voting weights, irrespective of the specific measure chosen, irrespective of the nature of the abstention decision in particular.[5]

The paper is organized as follows. Section 2 gives a general probabilistic characterization for power in voting games with abstentions. Section 3 introduces the different concepts of the nature of abstention as prevalent in the literature. Section 4 discusses PLT in weighted voting games with abstentions.

2. PROBABILISTIC INTERPRETATION

Let N be a non-empty finite set to which we shall refer as *assembly*. The elements of N are called *voters* and we shall identify them with the integers $1, 2, \ldots, n$, where $n = |N|$. We shall briefly recall the definition of the classical measures of voting power in SVGs (in which abstention is not a *tertium quid*).

A SVG consists of N together with a characteristic function v on the power set of N such that $v(S) = 1$ iff $S \subseteq N$ is a winning coalition and 0 otherwise. In SVGs the S-S index and Bz measure of a voter $a \in N$ is represented as the (weighted) sum of *contributions* $C_a(S) := v(S) - v(S - \{a\})$ voter a brings to each possible coalition $S \subset N$ in which s/he is a member. The contribution $C_a(S)$ is 1 if a joining S makes a practical difference to the outcome; it is 0 otherwise. The contributions are weighted with coalition-specific factors $f_a(S)$. Let ξ_a stand for either the Bz measure or S-S index of voter a. Then ξ_a can be written as

$$\xi_a = \sum_{S \subset N|\ a \in S} f_a(S) C_a(S), \tag{3}$$

where $f_a(S) = (|S| - 1)!(n - |S|)!/n!$ when ξ_a stands for the S-S index, $f_a(S) = 1/2^{n-1}$ when ξ_a stands for the Bz measure, respectively.

The following definition is taken from Felsenthal and Machover (1997, 1998).[6]

DEFINITION 2.1. A *tripartition* of a set N is a map T from N to, $\{-1, 0, 1\}$. We denote by T^-, T^0 and T^+ the inverse images of $\{-1\}, \{0\}$ and $\{1\}$, respectively under T:

$$\begin{aligned} T^- &= \{k \in N \mid T(k) = -1\}, \\ T^0 &= \{k \in N \mid T(k) = 0\}, \\ T^+ &= \{k \in N \mid T(k) = 1\}. \end{aligned} \tag{4}$$

We define partial ordering $\leq$ among tripartitions: if T_1 and T_2 are two tripartitions of N, we define

$$T_1 \leq T_2 \ \Leftrightarrow_{\text{def}} \ T_1(k) \leq T_2(k) \text{ for all } k \in N.$$

By a *ternary voting game* – briefly TVG – we mean a mapping v from the set $\mathcal{T}_N := \{-1, 0, 1\}^N$ of all tripartitions of N to $\{0, 1\}$, satisfying the following three conditions:

(i) $T^+ = N \Rightarrow v(T) = 1$;
(ii) $T^- = N \Rightarrow v(T) = 0$;
(iii) Monotonicity: $T_1 \leq T_2 \Rightarrow v(T_1) \leq v(T_2)$.

We call v the outcome of T (under v).

A ternary division T is interpreted as a voting division which allows abstentions. T^- and T^+ are interpreted as the sets of 'no' and 'yes' voters, T^0 as the set of abstainers, respectively. $T(k)$ can be interpreted as a degree of support of voter k for the decision in question.

Remark 2.1. Note that Definition 2.1 does not cover all possible cases of TVGs as Felsenthal and Machover include a monotonicity condition (here, condition (iii)). In fact, there exist many constitutions where a referendum is valid only if 50% of the voters cast their ballot (Italy for example). In this case, the monotonicity condition is not fulfilled and the situation becomes more complex. For more on this subject, see Corte Real and Pereira (2004). See also Remark 3.1 of the present paper.

Analogously to SVGs, we shall model power of voter $a \in N$ in a game with abstentions as the weighted sum of his or her *contributions* $C_a(T)$ to each possible tripartition. The contributions are weighted by a factor $f_a(T)$ which can be interpreted as the probability that the specific tripartition T forms. Formally, let ξ denote a measure of voting power in TVGs, then

$$\xi_a = \sum_{T \in \mathcal{T}_N} f_a(T) C_a(T). \tag{5}$$

First, consider the contribution term $C_a(T)$. For each $a \in N$ we define an *indicator function* of a, I_a, as a function on N such that $I_a(k) = 1$ for $k = a$ and zero elsewhere. Put

$$C_a(T) := \begin{cases} v(T) - v(T - 2I_a) \text{ if } a \in T^+, \\ \\ 0 \text{ otherwise.} \end{cases} \tag{6}$$

We say that voter a is *critical* iff $C_a(T) = 1$, i.e. his or her choice *makes a practical difference to the outcome*. Note that it is not important at which level the change in the outcome occurs, i.e. whether ceteris paribus from a's switch from 'yes' to 'abstain' or from 'abstain' to 'no'. Figure 1 gives an illustration.

In Scenario I, the bill passes even with voter a switching from 'yes' to 'abstain'. But T^+ no longer has a majority when a votes 'no' instead of abstaining. In Scenario II, the change in the outcome occurs when voter a decides to abstain instead of voting 'yes'. If decreasing support of a has no effect on the

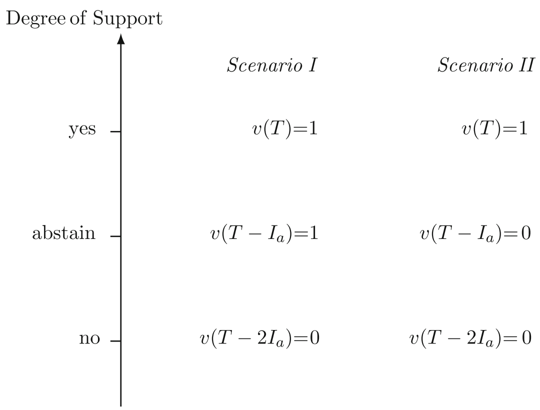

Figure 1. Contribution of a voter.

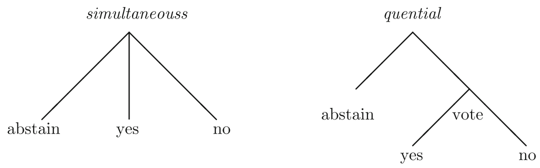

Figure 2. Nature of abstention.

outcome we have $v(T)=v(T-I_a)=v(T-2I_a)$ such that voter a has a contribution of 0 (i.e. makes no practical difference to the outcome).

Technically, C_a works as a filter. With it's binary values of either 0 or 1, it causes the sum in (5) to be taken only over the specific probabilities where a is considered to be critical. With (6) it is therefore possible to read (5) as the probability that voter a is decisive as a 'yes' voter in a random tripartition X

$$\xi_a = \text{Prob}\{X \text{ wins, } X-2I_a \text{ loses} |\ a \in T^+\}. \tag{7}$$

Equation (7) is a direct extension of the terms of a voter having power which Straffin uses for the SVG framework: the power of voter a is 'the probability that a bill passes if we assume a votes for it, but would fail if a voted against it' (Straffin, 1994, p. 1136).

The next section discusses the tripartition specific factor $f_a(T)$. The probability that a tripartition T forms hinges on two settings: the nature of the abstention decision and the behavioral assumptions about the voters.

3. NATURE OF ABSTENTION

In their 1997, Felsenthal and Machover treat abstention on a par with 'yes' and 'no' which implies that the voter decides simultaneously between the three options. In contrast Braham and Steffen (2002) propose a sequential structure in which voters first choose between participation and abstention and then between 'yes' or 'no'. Figure 2 illustrates the two approaches.

In the simultaneous approach, an abstaining voter can be thought of as being present in an assembly but indecisive about the issue to vote on. In this case the voter may feel neither affirmative nor negative about the proposal and thus chooses to declare 'I abstain' or casts an empty ballot. This is what Machover (2002) has called *active abstention*. In this case the voter is part of the quorum, even though his or her decision is neutral.

A different form of abstention takes place if the voter simply does not participate in the division. This may occur if the voter is prevented for any reason or if the issue to vote on is of minor interest to the voter, such that the costs of voting are higher than the expected pleasure of being on the winning side. Machover (2002) has termed this *abstention by default* and is reflected by the sequential approach proposed by Braham and Steffen (2002).

In general, decision rules are blind to the distinction between the two kinds of abstentions. But in some cases active and default abstentions are specified. For example, in the US Congress active abstainers are counted for purpose of a quorum. So, if a quorum is not present because too many have abstained by default, no voting can take place at all.[7] But if a quorum is present and all present actively abstain the outcome according to the ordinary majority rule in the House of Representatives is presumably negative since the number of 'yes' voters is not greater than the number of 'no' voters.

Remark 3.1. If a quorum is required for a vote to take place one could extend the binary outcomes 'accepted' or 'rejected' in Definition 2.1 by a third one, 'defer', representing a tie (see Freixas and Zwicker, 2003). In the present account we shall not discuss ties.

The Bz measure in the classical SVG setup assumes a priori that the voters vote independently and each voter votes 'yes' and 'no' with equal probability 1/2. The independence assumption is easily translated into the TVG framework. However, the spirit of a *priori* ignorance is less obvious when it comes to *assigning* probabilities to the single options in either approach of the nature of abstention. The route that Felsenthal and Machover have taken in their 1997 and 1998 is to appeal Bernoulli's Principle of Insufficient Reason[8] to justify assigning a *priori* probabilities of 1/3 for each option. In terms of (5) they put

$$f_a(T) := 3^{n-1}. \tag{8}$$

However, in TVGs the symmetric probability distribution on the option set is much less self-evident in comparison to the SVG setup although it surely is the only non-arbitrary choice. In the following we will therefore stick to a more general treatment as proposed as an alternative by the authors in their 1997 (p. 340). We assume that the *a priori* probability of any given voter abstaining is $t \in (0, 1)$ and s/he votes 'yes' and 'no' each with probability $(1-t)/2$. Hence, we put

$$f_a(T) = t^{|T^0|}((1-t)/2)^{n-1-|T^0|}, \tag{9}$$

such that (8) is given by $t = 1/3$.

In the sequential approach of Braham and Steffen (2002) any voter first decides *whether* to vote or abstain with probability $1-t$ and t, respectively. In the second stage s/he decides *how* to vote, i.e. to choose either 'yes' or 'no' with probability $1/2$ each. This provides

$$f_a(T) = t^{|T^0|}(1-t)^{n-1-|T^0|}(1/2)^{n-1-|T^0|} \tag{10}$$

which equals (9). However, an *a priori* argument appealing Bernoulli's Principle of Insufficient Reason suggests $t = 1/2$ for the sequential approach. In the next section, we will show that with an increasing size of the assembly, the particular assignment of t tends to irrelevance when it comes to measuring normalized Bz power in weighted voting games with abstentions.

With a tripartition specific factor as in (9) we will refer to the power measure as defined in (5) as the *generalized Bz measure* and denote it by $\psi(t)$. Analogously, we shall refer to it's normalized form as *generalized Bz index* $\tilde{\beta}(t)$. Hence, we put

$$\psi_a(t) = \sum_{T \in \mathcal{T}_N} t^{|T^0|}((1-t)/2)^{n-1-|T^0|} C_a(T), \tag{11}$$

$$\tilde{\beta}_a(t) = \frac{\psi_a(t)}{\sum_{k \in N} \psi_k(t)}. \tag{12}$$

Remarks 3.1.

(i) Braham and Steffen (2002) model voting games with abstentions (by default) as a whole bundle of SVGs in which each assembly consists of the non-abstaining voters. They express power as an expected value: power is the weighted sum of power in each single SVG. However, their concept is controversial in that an abstaining voter never exerts any power. Hence, with expression (5) we only partly follow their concept of power.

(ii) From (10) it is apparent that in the sequential approach the actual order of a voter's decision is not important. The order in Figure 2 is the one we observe, i.e. we see people either going to vote or not and then casting a 'yes' or 'no' ballot. However, the decision to vote 'yes' or 'no' may have been prior to the decision whether to abstain or participate in the vote.

4. PLT IN WVGS WITH ABSTENTIONS

In WVGs with abstention as a tertium quid a motion is accepted if the combined weight of affirming voters meets or exceeds some preset relative weight share *of those voting either 'yes' or 'no'*.

DEFINITION 4.1. A ternary weighted voting game – briefly, TWVG –

$$[q; w_1, w_2, \ldots, w_n]$$

is given by an assignment of a non-negative real w_k to each voter $k \in N$, and a relative Quota q such that for any tripartition T of N

$$v(T) = \begin{cases} 1 & \text{if } \sum_{k \in T^+} w_k \geq q \sum_{k \in N - T^0} w_k, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

We shall use the notation

$$< q; w_1, w_2, \ldots, w_n >$$

for a TWVG when the blunt inequality $\geq$ in (13) is replaced by the sharp inequality $>$.

Remark 4.1. The rule given in (13) may be rewritten such that a bill is passed iff the total weight of those voting for it is at least $\tilde{q} = q/(1-q)$ times the total weight of those voting against it.

We shall focus on the following problem: Under what conditions does the ratio of voting powers of any two voters, as measured by $\psi(t)$, converge to the ratio of their weights?

In order to make this problem more precise, let us introduce the following framework.

DEFINITION 4.2. Let

$$N^{(0)} \subsetneq N^{(1)} \subsetneq N^{(2)} \subsetneq \cdots$$

be an infinite increasing chain of finite non-empty sets, and let

$$N = \bigcup_{n=0}^{\infty} N^{(n)}. \tag{14}$$

Let w be a weight function that assigns to each $a \in N$ a positive real number w_a as weight; and let q be a real $\in (0, 1)$. For each $n \in \mathbb{N}$ let $\mathcal{W}^{(n)}$ be the TWVG whose assembly is $N^{(n)}$ – each voter $a \in N^{(n)}$ being endowed with the pre-assigned weight w_a – and whose relative quota is q. We shall then say that $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ is a *q-chain* of TWVGs.

Remark 4.2. In what follows, whenever we shall refer to a q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$, we shall assume that the $N^{(n)}$, N and w are as specified in Definition 4.2: $N^{(n)}$ is the assembly of $\mathcal{W}^{(n)}$, N is given by (14), and w is the weight function.

DEFINITION 4.3. We shall say that Penrose's Limit Theorem (PLT) holds for the q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ with respect to the index $\psi(t)$ and $a, b \in N$ if

$$\lim_{n\to\infty} \frac{\psi_a(t)[\mathcal{W}^{(n)}]}{\psi_b(t)[\mathcal{W}^{(n)}]} = \frac{w_a}{w_b}. \tag{15}$$

Remark 4.3. Note that $\psi_a(t)[\mathcal{W}^{(n)}]/\psi_b(t)[\mathcal{W}^{(n)}]$ in (15) is undefined if $a \notin N^{(n)}$ or $b \notin N^{(n)}$, but this does not matter because $a, b \in N^{(n)}$ for all sufficiently large n.

Let the following random variables denote the decision of every voter $k \in N$, i.e.

$$Z_k = \begin{cases} 0 & \text{if } k \text{ abstains }, \\ w_k & \text{otherwise,} \end{cases} \tag{16}$$

$$Y_k = \begin{cases} w_k & \text{if } k \text{ votes 'yes',} \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Put

$$S_{\neg a} := \left(\sum_{k\in N} Y_k\right) - Y_a\,, \quad W_{\neg a} := \left(\sum_{k\in N} Z_k\right) - Z_a\,. \tag{18}$$

Then (7) provides

$$\xi_a = Prob\{q(W_{\neg a} + w_a) - w_a \le S_{\neg a} < q(W_{\neg a} + w_a)\}. \tag{19}$$

Note that in contrast to the SVG setup the majority quota $q(W_{\neg a} + w_a)$ is random.

Let V_k denote the random variable

$$V_k := Y_k - qZ_k$$

which takes the values

$$V_k = \begin{cases} 0 & & t, \\ (1-q)w_k & \text{with probability} & (1-t)/2, \\ -qw_k & & (1-t)/2. \end{cases} \tag{20}$$

Put

$$X_{\neg a} := \left(\sum_{k\in N} V_k\right) - V_a.$$

Subtracting $qW_{\neg a}$ in (19) provides

$$\xi_a = Prob\{(q-1)w_a \leq X_{\neg a} < qw_a\}. \tag{21}$$

Given a q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ of TWVGs, we associate with it the family $\{V_k|\ k \in N\}$ of independent random variables indexed by N. We consider the chain

$$\mathfrak{W} := \left\{\{V_k|\ k \in N^{(n)}\}|\ n \in \mathbb{N}\right\}. \tag{22}$$

For any $a \in N$ we put

$$X_{\neg a}^{(n)} := \left(\sum_{k \in N^{(n)}} V_k\right) - V_a, \quad \mu_{\neg a}^{(n)} := E\left[X_{\neg a}^{(n)}\right],$$
$$\sigma_{\neg a}^{(n)} := \left(Var\left[X_{\neg a}^{(n)}\right]\right)^{\frac{1}{2}}.$$

Let $\bar{X}_{\neg a}^{(n)}$ be the 'standardized' form of $X_{\neg a}^{(n)}$, i.e.

$$\bar{X}_{\neg a}^{(n)} := \frac{X_{\neg a}^{(n)} - \mu_{\neg a}^{(n)}}{\sigma_{\neg a}^{(n)}}. \tag{23}$$

From (20) we obtain the following explicit expressions for $\mu_{\neg a}^{(n)}$ and $\sigma_{\neg a}^{(n)}$

$$\mu_{\neg a}^{(n)} = (1-t)(1-2q)\frac{\left(\sum_{k \in N^{(n)}} w_k\right) - w_a}{2}, \tag{24}$$

$$\left(\sigma_{\neg a}^{(n)}\right)^2 = (1-t)\left[(1-q)^2 + q^2 - \frac{1-t}{2}(1-2q)^2\right] \times \frac{\left(\sum_{k \in N^{(n)}} w_k^2\right) - w_a^2}{2}. \tag{25}$$

DEFINITION 4.4. We shall say that the chain $\mathfrak{W}$ satisfies the special local central limit condition (SLCL) if, for every $a \in N$,

$$\lim_{n\to\infty} \text{Prob}\left\{\bar{X}_a^{(n)} \in \left[-\frac{w_a}{2\sigma_{\neg a}^{(n)}}, \frac{w_a}{2\sigma_{\neg a}^{(n)}}\right)\right\} \frac{\sigma_{\neg a}^{(n)}}{w_a} = \frac{1}{\sqrt{2\pi}}; \tag{26}$$

and for all $a, b \in N$

$$\lim_{n\to\infty} \frac{\sigma_{\neg a}^{(n)}}{\sigma_{\neg b}^{(n)}} = 1. \tag{27}$$

Remark 4.4. The $\bar{X}_{\neg a}^{(n)}$ are evidently discrete random variables with mean 0. We shall be interested in cases where their standard deviations, $\sigma_{\neg a}^{(n)}$, tend to ∞ with n. Then equation (26) says that the average density of $\bar{X}_{\neg a}^{(n)}$ in a half-open interval around 0, whose length becomes vanishingly small, approaches the value of the standard normal density function φ at 0, namely $1/\sqrt{2\pi}$. This means that $\mathfrak{W}$ obeys a special case (namely, at 0) of the local central limit theorem of probability theory.

THEOREM 4.1. *Let $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ be a $\frac{1}{2}$-chain of TWVGs. If its associated chain $\mathfrak{W}$ satisfies the SLCL condition, then PLT holds with respect to the generalized Bz index and any $a, b \in N$.*

Proof. Let $a \in N$ and take n large enough so that $a \in N^{(n)}$. Then, by definition, the generalized Bz measure of a in $W^{(n)}$ is given by

$$\psi_a\left[W^{(n)}\right] = \text{Prob}\left\{\bar{X}_a^{(n)} \in \left[-\frac{w_a}{2\sigma_{\neg a}^{(n)}}, \frac{w_a}{2\sigma_{\neg a}^{(n)}}\right)\right\}.$$

Invoking (26) we obtain

$$\lim_{n\to\infty} \psi_a\left[W^{(n)}\right] \frac{\sigma_{\neg a}^{(n)}}{w_a} = \frac{1}{\sqrt{2\pi}}. \tag{28}$$

Hence by (27)

$$\lim_{n\to\infty} \frac{\psi_a\left[W^{(n)}\right]}{\psi_b\left[W^{(n)}\right]} = \frac{w_a}{w_b}. \tag{29}$$

Finally, using (11) and (12) we obtain

$$\lim_{n\to\infty} \frac{\tilde{\beta}_a\left[W^{(n)}\right]}{\tilde{\beta}_b\left[W^{(n)}\right]} = \frac{w_a}{w_b}. \tag{30}$$

From (28) and (25) follows □

COROLLARY 4.1. *If (26) holds, then*

$$\psi_a[\mathcal{W}^{(n)}] \approx \frac{1}{\sqrt{2\pi}} \frac{w_a}{\sigma_{\neg a}^{(n)}}.$$

For $q=1/2$ this simplifies to

$$\psi_a[\mathcal{W}^{(n)}] \approx w_a \sqrt{\frac{2}{(1-t)\pi\{(\sum_{k\in N^{(n)}} w_k^2) - w_a^2\}}}. \tag{31}$$

Table II illustrates the results for the US Presidential Electoral College. The numbers of the WVG are taken from Table I. All values are in percentage terms. Column (1) gives relative weights. Columns (2–4) refer to the simultaneous approach of abstention which puts $t=1/3$. Column (2) gives the generalized Bz index $\tilde{\beta}$. Column (3) provides the exact values of ψ whereas column (4) gives numerical approximations based on (31). Analogously, columns (5–7) refer to the sequential approach of abstention where $t=1/2$.

THEOREM 4.2. *Let $\{W^{(n)}\}_{n=0}^{\infty}$ be a $\frac{1}{2}$-chain of TWVGs such that its weight function assumes only finitely many values, all of them positive integers; and such that the greatest common divisor of those values w_a that occur infinitely often is* 1. *Then the associated chain $\mathcal{V}$ satisfies the SLCL condition. Hence, PLT holds with respect to the generalized Bz index and any $a, b \in N$. Also, (31) holds.*

Proof. To prove that (26) holds for any $a \in N$, observe that all possible values of $X_{\neg a}^{(n)}$ are integers multiplied by $1/2$ and therefore belong to a lattice whose span is $1/2$. Hence, all possible values of $\bar{X}_a^{(n)}$ belong to a lattice whose span is $1/(2\sigma_{\neg a}^{(n)})$. In the half open interval

$$\left[-w_a/(2\sigma_{\neg a}^{(n)}), w_a/(2\sigma_{\neg a}^{(n)})\right)$$

there are exactly $2w_a$ points of this lattice: say $x_i^{(n)}, i=1,2, \ldots, 2w_a$.

We invoke Pevtrov's version of the local central limit theorem (1975, p. 189, Theorem 2; see also Remark 4.1(i) of the present paper). It follows that if n is sufficiently large then for each $i=1,2,\ldots,2w_a$ the product

$$\text{Prob}\left\{\bar{X}_a^{(n)} = x_i^{(n)}\right\} 2\sigma_{\neg a}^{(n)} \tag{32}$$

TABLE II

US Presidential Electoral College (2000 Census)

No.	(1) $\bar{w}$ (%)	(2) $\tilde{\beta}$ (%) $t=1/3$	(3) ψ (%)	(4) ψ_{appr} (%)	(5) $\tilde{\beta}$ (%) $t=1/2$	(6) ψ (%)	(7) ψ_{appr} (%)
1	10.2230	11.0402	56.6192	63.1610	10.6870	63.6093	72.9321
1	6.3197	6.3875	32.7577	34.8101	6.3734	37.9347	40.1953
1	5.7621	5.7959	29.7240	31.4043	5.7920	34.4743	36.2626
1	5.0186	5.0199	25.7446	27.0174	5.0255	29.9118	31.1970
2	3.9033	3.8784	19.8901	20.7033	3.8907	23.1577	23.9061
1	3.7175	3.6903	18.9253	19.6764	3.7030	22.0407	22.7203
1	3.1599	3.1289	16.0464	16.6316	3.1421	18.7021	19.2045
3	2.7881	2.7569	14.1385	14.6281	2.7697	16.4855	16.8911
1	2.4164	2.3863	12.2382	12.6425	2.3983	14.2751	14.5983
1	2.2305	2.2016	11.2906	11.6556	2.2130	13.1719	13.4587
4	2.0446	2.0171	10.3446	10.6722	2.0279	12.0700	12.3231
4	1.8587	1.8329	9.3998	9.6919	1.8429	10.9692	11.1913
3	1.6729	1.6489	8.4564	8.7146	1.6582	9.8694	10.0628
2	1.4870	1.4652	7.5140	7.7399	1.4735	8.7705	8.9372
4	1.3011	1.2816	6.5726	6.7674	1.2890	7.6725	7.8143
3	1.1152	1.0982	5.6321	5.7969	1.1047	6.5751	6.6937

TABLE II

continued

No.	(1) $\bar{w}$ (%)	(2) $\tilde{\beta}$ (%) $t=1/3$	(3) ψ (%)	(4) ψ_{appr} (%)	(5) $\tilde{\beta}$ (%) $t=1/2$	(6) ψ (%)	(7) ψ_{appr} (%)
5	0.9294	0.9150	4.6923	4.8282	0.9204	5.4783	5.5751
5	0.7435	0.7318	3.7531	3.8609	0.7362	4.3821	4.4581
8	0.5576	0.5488	2.8144	2.8947	0.5521	3.2862	3.3425
Total 51	99.9998	100.0006			99.9995		
$D(\xi, w) * 10^3$		18.3940			11.0930		
$d(\xi, w) * 10^3$		74.0204			43.4172		

Quota: 270 = 50.19% of 538. For explanations see main text is arbitrarily close to $\varphi(x_i^{(n)})$. From (25) it is clear that $\lim_{n\to\infty} \sigma_{\neg a}^{(n)} = \infty$; thus for a sufficiently large n each of the $x_i^{(n)}$ is arbitrarily close to 0. Hence, the product (32) is arbitrarily close to $\varphi(0) = (2\pi)^{-1/2}$. The left-hand side of (26) is just the arithmetic mean of $2w_a$ many products (32) and hence tends to $(2\pi)^{-1/2}$ as required.

With $n \to \infty$ the term w_a^2 in (25) becomes negligible and (27) holds. □

Remarks 4.1. (i) Petrov's theorem deals with a sequence of independent integer-valued random variables each having finite variance, such that the set of distinct distributions of these variables is finite. The key condition is that the greatest common divisor of the maximal spans of those distributions that occur infinitely often in the sequence is 1. For details see Petrov (1975, ibid.).

(ii) The condition in Theorem 4.2 is similar to the one in Theorem 3.6 in Lindner and Machover (2004). The latter is a PLT statement for the Bz measure in SVGs.

(iii) For general $q \in (0,1)$ the proof for Theorem 4.2 shows one major difficulty: application of Petrov's version of the local central limit theorem analogously to the $q = 1/2$ case yields

$$\psi_a[\mathcal{W}^{(n)}]\sigma_{\neg a}^{(n)} = w_a\varphi(m_{\neg a}^{(n)}) + \varepsilon_{\neg a}^{(n)}, \tag{33}$$

where $m_{\neg a}^{(n)}$ is a mean value and $\varepsilon_{\neg a}^{(n)}$ is the approximation error which tends to 0 with increasing n. For $q = 1/2$, the mean value is arbitrarily close to 0 for any sufficiently large n. However, for $q \neq 1/2$ the mean value tends to $\pm\infty$ such that $w_a\varphi(m_{\neg a}^{(n)})$ also tends to zero. Hence, it has to be shown that the relative error of the approximation tends to 0.

5. CONCLUSIONS

This paper proposes a unified way to define a family of Bz indices with different nature of abstention. The probabilistic interpretation has shown that the actual difference between the symmetric and the sequential approach lies in the assignment of an *a priori* probability of the three options. Conceptually, it is still correct to assign equal *a priori* probability to 'yes' and 'no', however, the assignment of a value of the abstention probability is less evident. The research on PLT represents a relative irrelevant statement for weighted voting games: with an increasing number of voters the power ratio of any two voters converges to the ratio of the voting weights, irrespective of the particular probability of abstention.

The paper provides sufficient conditions for PLT to hold, however, both real-life and randomly generated TWVGs provide much empirical evidence that it holds as a general rule in most cases. This gives rise to conjecture that the sufficient conditions presented in this paper are too strict and PLT holds in larger classes of TWVGs. In cases where the asymptotic behavior asserted by PLT holds it begins to manifests itself at around $n = 15$ provided that the distribution of the weights is not too skewed (as in the US Presidential Electoral College). As a rule of thumb, the convergence process tends to get slower with an increasing ratio between the largest and the smallest voting weight. Also, for values of q getting closer to 1 we observe slower convergence due to a 'unanimity effect'.

PLT may be best regarded as an open-end research programme covering many related results. Lindner and Machover (2004) provide sufficient conditions for the S-S index (1954) and the Bz measure (1965) in the classical SVG setup. The present results of this paper for power measures in TVGs are yet another contribution to this programme. The theory of *a priori* voting power in games with active abstentions is a still young and under-developed part of the theory of voting power. The recent paper of Freixas and Zwicker (2003) introduces weighted (j, k) games. In these games a voter is endowed with j many voting weights and there are k many levels if output. This conceptual approach can be interpreted as modeling j many levels of approval ranging from complete enthusiasm to total opposition which covers the classical SVG setup as a $(2, 2)$ game. Voting games with active abstention can be interpreted as a special case of $(3, 2)$ games. We conjecture that PLT holds under rather general conditions, for large classes of variously defined weighted voting games, other values of the quota, and other measures of voting power.

NOTES

1. Perhaps the most famous precedent ocurred in 1950, when the USSR's boycott of the UNSC led to a resolution of sending UN forces to Korea. Although the USSR strongly opposed it, their absence – 'passive' abstention – did not prevent the passage of the motion.
2. The voting rule depends on the nature of the issue at hand. In some cases it is simple majority, in some the needed affirmative share is two-thirds.
3. An earlier step in this direction was taken by Fishburn (1973, pp. 53–55); but he considers only a special class of weighted voting games.
4. This way of modeling the Electoral College involves some over-simplification, because there may be more than two candidates, and since 1969 the Electors of Maine did not have to vote as a single bloc. Since 1993, the same applies to Nebraska.
5. Note that this observation is not concerned with *absolute* voting power. The (absolute) Bz powers of the voters typically do not sum up to one. PLT is a statement is for it's normalized version and hence refers to relative power.
6. In stating Definition 2.1.ii we are paraphrasing Felsenthal and Machover in using a more 'one-sided' model with respect to the outcome. For their own formulation see their 1998, Definition 8.2.1.
7. This is a simplifying assumption as, in fact, there must be a motion to consider the quorum in order for a count to even take place. For details see Felsenthal and Machover (1998), Chapter 4.
8. This principle claims that each of the alternatives should have equal probability if there is no known reason for assigning unequal ones (for more details see, for example, Felsenthal and Machover, 1998).

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