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DOWN TO TEN: THE ECONOMETRICS OF THE RED CARD

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Abstract

We investigate the effect of the expulsion of a player on the outcome of a football match. For that purpose we develop a probability model for the score in a football match. We propose estimators of the expulsion effect that are independent of match-specific effects that reflect the relative strength of the teams. We use the estimates to predict the expulsion effect on the outcome of a match between teams of equal strength.

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1. Introduction

Football is the most popular sport in the world. It is one of the few truly universal games with active participants in all countries. In addition, professional football is a prominent industry. Hence, it is not surprising that there is a permanent debate over the rules of the game. This discussion usually focuses on changes in the rules that would increase the number of goals scored, and, in general, would make football more attractive to spectators. One change that has been implemented is to make it easier to expel a player from a match[1]. Under the old rules a player was sent off if he hit another player intentionally in an action that did not involve the ball. Under the new rules a player can be sent off for repeated fouls and for using illegal means to stop an action that with a high probability would result in a goal. The idea behind these changes is to make it harder to rely on rough defensive play.

If the referee decides to expel a player, he shows him a red card. So in the language of football the showing of the red card and the expulsion of the player are synonymous.

In this paper we investigate the effect of a red card, i.e. the effect of the expulsion of a player, on the outcome of a match. Popular opinion holds widely different views on the effectiveness of the red card as an instrument to affect the outcome, but as far as we know there has been no empirical research on this question. We propose a model for the score in a match, that can be used to estimate the effect of the red card on the scoring intensities of the teams. This model is a probability model for the outcome that takes account of the differences in the strengths of the teams and the non-constant scoring intensity during the match. More specifically, we propose a time-inhomogeneous Poisson model with a match-specific effect for the score in a match.

We propose two estimators for the effect of the red card on the scoring intensity. The Conditional Maximum Likelihood (CML) estimator is independent of the match-specific effects. The CML estimator for the Poisson regression model was introduced in econometrics by Hausman, Hall and Grilliches (1984), building on ideas of Andersen (1973). The other estimator, the OLS estimator, depends on match-specific effects, and hence is potentially biased. Indeed we obtain rather different and uninterpretable results with this estimator. We use the CML estimates and the model to investigate the effect of a red card on the outcome of a match.

[1] If a player is sent off, he is excluded for the remainder of the match.

The paper is organized as follows. In section 2 we specify a model for the score in a match. Estimation methods are discussed in section 3, and in section 4 we give the estimates. We consider some implications of the estimates in section 5.

2. A Model for the Score in a Football Match

First we introduce some notation. The subscript i denotes a match, and ij, j=1,2 the two teams or sides playing in that match. In matches with a red card we always take it that the red card is given against the second team. Time is measured in minutes on the scale 0 to 90, which is the official duration of a football match. In a football match the clock is not stopped when a match is interrupted, but the referee can allow for stoppage time at the end of the first an second halfs, after 45 and 90 minutes respectively. Recorded time is however measured from the beginning of the match and from the resumption of play after the interval. As a result there may be some minutes when there is no play at all, while the 45-th and 90-th minute may last longer than a minute's time, but this is a minor distortion.

Let

τ_i	= minute in which a player is expelled from team 2.
N_{ij}	= total number of goals scored in match i by team j.
K_{ij}	= number of goals scored before τ_i.
M_{ij}	= number of goals scored after τ_i.
$\lambda_{ij}(t)$	= rate or intensity of scoring of team j in match i at t-th minute of play.
θ_j	= multiplicative effect on $\lambda_{ij}(t)$ of removal of player from team 2.
γ_{ij}	= relative strength of team j in match i as compared with the overall average strength or scoring rate, as defined in section 3.

We make the following assumptions

1. The two teams score according to two independent Poisson processes. As a consequence the number of goals scored by team 1 is stochastically independent of the number of goals scored by team 2. Moreover the time intervals between subsequent goals are stochastically independent. The scoring intensities are not constant over the match, so that the Poisson processes are nonhomogeneous.

2. The ratio of the scoring intensities of the two full teams is constant, i.e. $\lambda_{ij}(t)=\gamma_{ij}\lambda(t)$ for matches of eleven against eleven players, with $\lambda(t)$ the average scoring intensity at the t-th minute of play, averaged over all teams in all matches with full sides of eleven players against eleven..
3. After the red card, for $t>\tau_i$, team 2 has 10 players, and the scoring intensities are $\theta_j\gamma_{ij}\lambda(\mathrm{t})$, $j=1,2$.

In assumption 1 we describe the score in a match as a random phenomenon that is only partly predictable. Here we are only interested in three factors that affect the score. These factors are the playing time, the relative strength of the teams and the effect of the red card. Evidence cited below suggests that the scoring intensity increases with the time played. If we do not allow for this, then our estimate of the effect of the red card will be biased upward, because we confound the effect of the red card and the effect of the time played. Of course, the score is strongly affected by the relative strength of the teams. The incidence of red cards may be related to the relative strength, so that a comparison of a red card game to an average game without a red card gives a biased estimate of the effect of the red card. Not only the incidence, but also the timing of red cards may be related to the relative strength of the teams, and again this biases our estimate of the effect of the red card. The third factor is the effect of the red card, which by assumption 3 is measured by θ_1 and θ_2, where, without loss of generality, we assume that a player of team 2 is expelled.

We must stress that our intention is not to predict the outcome of football matches. For that purpose we require, among other things, an estimate of γ_{ij}. Our estimate of the effect of the red card is independent of γ_{ij}, which is of great help, because finding a good estimate of γ_{ij} is hard, as experience with the prediction of outcomes shows. The Poisson assumption is an acknowledgement of the fundamental randomness of the score in a match.

The consequences of the Poisson assumption are mentioned in assumption 1. It is not difficult to relax the Poisson assumption, at the cost of a more involved statistical analysis. For instance, one can make the scoring intensity dependent on the number of goals scored by both teams. We can also allow for variations in the relative strength during the match by subdividing the 90 minutes in time intervals. In the sequel we shall maintain the assumptions 1-3, because we doubt whether our limited number of observations will support a more specific and complete model.

3. Statistical analysis

3.1 *Estimation of the average scoring intensity*

In table 1 goals scored in 340 matches in the two professional divisions in the Netherlands[2] in the season 1991/1992 are classified by the 15-minute interval in which they were scored. This table that excludes red card matches, shows that the rate at which goals are scored increases monotonically over the match. The extra-time effect can account for some of the increase in the rate in the last 15 minutes of the first and second halfs. A monotonous increase in the scoring rate has also been observed in England (see Morris (1981) who uses data on matches in the English League and FA-cup matches in the period April 1978 until November 1980).

We estimate the average scoring intensity from the data in table 1. The average scoring intensity is specified as

(3.1) $$\lambda(t) = \alpha + \beta t$$

The expected number of goals scored by a team j in match i (each match gives two observations) in the s-th 15 minute interval is

(3.2) $$E(N_{ijs}) = \gamma_{ij}(15\alpha + 112.5\beta(2s-1)) \qquad s = 1,\dots,6$$

Hence the average number of goals scored by one side in the s-th time interval is

(3.3) $$E(\bar{N}_s) = 15\alpha + 112.5\beta(2s-1) \qquad s = 1,\dots,6$$

where we take $\bar{\gamma}=1$. Note that the average is taken over all matches and both teams in a match. By setting $\bar{\gamma}$ we implicitly define a scale for γ_{ij}, e.g. if $\gamma_{ij}=2$, then team j has a scoring intensity in match i that is two times the average scoring intensity.

OLS regression of the average number of goals scored per minute in time interval s -this average can be computed from table 1 by division by twice the number of matches (680 observations)- on $7.5(2s-1)$ gives estimates of α and β.

[2] Professional football in the Netherlands is organized in two divisions. The first division (Dutch: eredivisie) contains the stronger teams. Each year two or three teams are promoted from the second division (Dutch: eerste divisie) to the first.

We find (R^2=.95; standard errors in parentheses[3])

$$\hat{\alpha} = .01167\ (.00027) \qquad \hat{\beta} = .00008627\ (.0000080)$$

In the sequel we ignore the sampling variance of these estimates. This simplifies the computation of variances, and provides an acceptable approximation, because the sampling variances are small. The estimates imply that the 90 minute scoring intensity $90\lambda(t)$ increases from 1.05 in the first minute to 1.75 in the final minute.

3.2 *A Conditional Maximum Likelihood Estimator of the Red Card Effect*

It is likely that the incidence of red cards is related to the relative strength γ_{ij}. Hence, a comparison of red card matches with the average match may give a biased estimate of the effect of the red card. For that reason we propose an estimator that does not depend on the γ_{ij}'s or their distribution. This estimator is based on a comparison of the number of goals scored by the same team before and after the red card. More precisely, we consider the fraction of the goals scored after the red card. It is intuitively clear that this fraction is independent of the time-constant match specific effect[4].

Under assumptions 1-3 (P denotes the Poisson distribution)

$$(3.4) \qquad K_{ij} \sim P(\gamma_{ij}\int_0^{\tau_i}\lambda(t)\mathrm{d}t) \qquad\qquad M_{ij} \sim P(\theta_j\gamma_{ij}\int_{\tau_i}^{90}\lambda(t)\mathrm{d}t)$$

In the sequel we denote

$$(3.5) \qquad A_i = \int_0^{\tau_i}\lambda(t)\mathrm{d}t \qquad\qquad B_i = \int_{\tau_i}^{90}\lambda(t)\mathrm{d}t$$

The fraction of the goals scored after the red card is

$$(3.6) \qquad y_{ij} = \frac{M_{ij}}{N_{ij}}$$

[3] The reported standard errors are consistent in the presence of heteroscedasticity.

[4] This deals also with the potential home advantage that is reflected in the match-specific effects.

The conditional distribution of M_{ij} given N_{ij} is

$$(3.7) \qquad M_{ij}|N_{ij} \sim B(N_{ij}, g_{ij}(\theta))$$

where B denotes the Binomial distribution, and

$$(3.8) \qquad g_{ij}(\theta) = \frac{\theta_j B_i}{A_i + \theta_j B_i}$$

The conditional distribution is degenerate if N_{ij}=0, and y_{ij} is only defined if $N_{ij} \geq 1$. In the CML procedure we omit observations with N_{ij}=0. Our estimators of the red card effect are not biased by this restriction, as we shall see shortly.

In the conditional distribution (3.7) the match specific effects γ_{ij} cancel. Hence, the conditional likelihood based on this Binomial distribution does not depend on the match specific effects. The conditional loglikelood is, up to an additive constant that does not depend on θ_j,

$$(3.9) \qquad logL_j = \sum_{i=1}^{n_j} M_{ij} log(g_{ij}(\theta_j)) + (N_{ij}-M_{ij}) log(1-g_{ij}(\theta_j))$$

with n_1, n_2 the number of observations on teams that do not and do receive a red card, respectively. Because we condition on the total scores N_{ij}, we can treat them as non-stochastic constants in the derivation of the properties of the CML estimator. Hence, omitting observations with a given total score does not make the CML estimator asymptotically biased, and in particular we can omit observations with N_{ij}=0.

The likelihood equation is

$$(3.10) \qquad \sum_{i=1}^{n_j} N_{ij} g_{ij}(\hat{\theta}_{CMLj}) = \sum_{i=1}^{n_j} N_{ij} y_{ij}$$

This equation can be interpreted as a moment equation in which a weighted average of the y_{ij}'s is equated to a weighted average of their expectations. The weights are the total scores N_{ij}. Because the N_{ij} can be treated as known constants, our earlier remark that our estimator only depends on the fraction of goals scored after the red card applies.

In deriving the properties of the CML estimator we note that the Binomial parameter $g_{ij}(\theta_j)$ can be written in the Logit form.

$$(3.11) \qquad g_{ij}(\theta_j) = \frac{e^{log(\theta_j)+log(B_i/A_i)}}{1+e^{log(\theta_j)+log(B_i/A_i)}}$$

Hence, the loglikelihood is globally concave in $log(\theta_j)$, so that the CML estimator for θ_j is uniquely defined. The asymptotic variance of the CML estimator is

$$(3.12) \qquad V(\hat{\theta}_{CMLj}) = \frac{\theta^2}{\sum_{i=1}^{n_j} N_{ij} g_{ij}(\theta)(1-g_{ij}(\theta))}$$

3.3. *OLS Estimation of the Red Card Effect*

With an additional assumption we can estimate the effect of the red card with a linear regression estimator. From (3.4)

$$(3.13) \qquad \begin{aligned} K_{ij} &= \bar{\gamma}_j A_i + (\gamma_{ij}-\bar{\gamma}_j)A_i + (K_{ij}-E(K_{ij}|\gamma_{ij})) = \bar{\gamma}_j A_i + v_{1ij} \\ M_{ij} &= \bar{\gamma}_j \theta_j B_i + (\gamma_{ij}-\bar{\gamma}_j)\theta_j B_i + (M_{ij}-E(M_{ij}|\gamma_{ij})) = \bar{\gamma}_j \theta_j B_i + v_{2ij} \end{aligned}$$

In (3.13) we allow the average relative strength in red card games to be different from that in all games, which we have set equal to 1. The estimates of $\bar{\gamma}_1$ and $\bar{\gamma}_2$ indicate the average strengths of the teams with eleven and ten players, before a player of team 2 is expelled. The disturbances v_{1ij} and v_{2ij} are heteroscedastic and correlated. Hence, to obtain an efficient estimator of θ_j we must use an estimation method that deals with these issues. Alternatively, we can just use OLS and employ the well-known heteroscedasticity-consistent expressions for the (co)variances. Note that we require an additional assumption to ensure that OLS gives consistent estimates.

4. $\text{Cov}(\gamma_{ij}, A_i^2) = \text{Cov}(\gamma_{ij}, B_i^2) = 0$. A sufficient condition for this is that τ_i and γ_{ij} are stochastically independent.

If we denote the OLS estimates of $\bar{\gamma}_j$ and $\theta_j\bar{\gamma}_j$ in (3.13) by $\hat{\delta}_{1j}$ and $\hat{\delta}_{2j}$ respectively, we estimate θ_j by

(3.14) $$\hat{\theta}_{OLSj} = \frac{\hat{\delta}_{2j}}{\hat{\delta}_{1j}}$$

The asymptotic variance is

(3.15) $$V(\hat{\theta}_{OLSj}) = \frac{\hat{\delta}^2_{2j}}{\hat{\delta}^4_{1j}} V(\hat{\delta}_{1j}) - 2\frac{\hat{\delta}_{2j}}{\hat{\delta}^3_{1j}} Cov(\hat{\delta}_{1j},\hat{\delta}_{2j}) + \frac{1}{\hat{\delta}^2_{1j}} V(\hat{\delta}_{2j})$$

In (3.15)

(3.16) $$V(\hat{\delta}_{1j}) = \frac{\sum_{i=1}^{n} e^2_{1ij} A^2_i}{(\sum_{i=1}^{n} A^2_i)^2} \qquad V(\hat{\delta}_{2j}) = \frac{\sum_{i=1}^{n} e^2_{2ij} B^2_i}{(\sum_{i=1}^{n} B^2_i)^2} \qquad Cov(\hat{\delta}_{1j},\hat{\delta}_{2j}) = \frac{\sum_{i=1}^{n} e_{1ij} e_{2ij} A_i B_i}{(\sum_{i=1}^{n} A^2_i)(\sum_{i=1}^{n} B^2_i)}$$

with e_{1ij} and e_{2ij} the OLS residuals of the regression equations in (3.13).

4. Estimation Results

In the previous section we proposed two estimators of the effect of the red card on the scoring intensity. We apply these estimators to data on 140 red card games in the seasons 89-90, 90-91 and 91-92 in both divisions of the Dutch professional football league. Because for the CML estimator we must omit observations where a team has not scored neither before nor after the red card, the effective number of observations is 112 for teams with eleven players and 93 for teams with ten players. We obtain the following results (standard errors in parentheses)

a. CML estimator $\hat{\theta}_{CML}$

$\hat{\theta}_{CML1} = 1.78$ (.27) $\qquad \hat{\theta}_{CML2} = .68$ (.15)

b. OLS estimator $\hat{\theta}_{OLS}$

$\hat{\theta}_{OLS1} = 1.43$ (.03) $\qquad \hat{\theta}_{OLS2} = 1.14$ (.03)

According to the CML estimates the scoring intensity increases for the team with 11 players (team 1). The effect is statistically significant. The scoring intensity for the team with 10 players (team 2) decreases, and the effect is just significant. The OLS estimator gives rather different results. The estimated increase in the scoring intensity for the team with 11 players is much smaller than for the CML estimator (but highly significant). More surprisingly, the OLS estimator shows a statistically significant increase in the scoring intensity for the team that loses one player. This is an indication that the OLS estimator, that uses between-game information, is biased. Therefore, we use the CML estimates in our analysis of the effect of the red card on the outcome of a match.

We conclude that the red card increases the scoring intensity of the team with 11 players by 78%. The decrease in the scoring intensity of the team with 10 players is much smaller, 32%.

The first-stage regressions[5] of the OLS estimator show that the average team that receives the red card, has the same scoring intensity as the average team in the competition. However, the opposing team is on average much stronger than the average team in the competition. This is in keeping with the rules whereby the red card primarily punishes rough defensive play but not aggressive play. Hence, the red card usually affects the weaker team, but the scoring intensity of this team is not significantly reduced. The scoring intensity of the opposing team, that was already larger, is further increased by the red card.

5. Some Implications of the Estimates

We can use our model to estimate the effect of the red card on the outcome of a football match. Our model implies that the effect of the red card depends on the relative strengths of the teams as measured by γ_{i1} and γ_{i2}. In our calculations we set $\gamma_{i1}=\gamma_{i2}=1$, i.e. we consider a match between two teams that have the average scoring intensity of the competition. In this way we isolate the effect of the red card from the existing difference in relative strength between the teams. Let $N_1(\tau)$ and $N_2(\tau)$ denote the total number of goals scored by the teams with 11 and 10 players, respectively, with a red card given in the τ-th minute of the match. In Table 2 we give the probabilities of the three possible outcomes of the match as a function of τ.

The last row of the table shows that the probability of a draw between two teams of average strength is .25. This is an indication of the role of chance in the outcome of a football match. A red card early in the match

[5] The estimates of the average strengths of the teams with 10 and 11 players before the red card are $\bar{\gamma}_2 = 1.03$ (.09) and $\bar{\gamma}_1 = 1.33$ (.09).

increases the probability of a victory of the team with 11 players substantially. The probability of a victory of the team with 10 players decreases even more, while the change in the probability of a draw is relatively small.

When a red card is given, a player is expelled for the remainder of the match. In indoor football and ice hockey a player can be excluded for a certain period. In Table 3 we show the effect of a 15 minute time penalty on the outcome of a match between equally strong teams. Although the effect depends on the time at which the penalty is imposed, this dependence is rather weak.

As noted in the introduction, a motivation for the more frequent use of the red card is to increase the number of goals scored in a match. In Table 4 we report the expected number of goals scored in a match between equally strong teams as a function of the time of the red card. We conclude that the red card has the desired effect.

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Table 1. *Goals scored in season* 1991/1992 *by* 15 – *minute interval*

Time interval	Number of goals
0–15	128
16–30	140
31–45	147
46–60	169
61–75	170
76–90	198

Table 2. *Probabilities of the outcome of the match by minute of the red card.*

Minute of red card τ	Pr(team of 11 wins)	Pr(draw)	Pr(team of 10 wins)
0	.71	.17	.12
15	.67	.18	.15
30	.62	.20	.18
45	.57	.22	.21
60	.51	.23	.26
75	.45	.24	.31
90	.375	.25	.375

Table 3. *Probabilities of the outcome of a match with a* 15 *minute exclusion starting at* τ.

Start of penalty τ	Pr(team of 11 wins)	Pr(draw)	Pr(team of 10 wins)
0	.42	.25	.33
15	.43	.24	.33
30	.43	.25	.32
45	.44	.24	.32
60	.44	.24	.32
75	.45	.24	.31

Table 4. *Expected number of goals in match by minute of red card.*

Minute of red card τ	Expected number of goals
0	3.44
15	3.36
30	3.26
45	3.16
60	3.05
75	2.93
90	2.80

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