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Overlapping Balanced Canonical Forms for Stable Multivariable All-Pass Systems

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Overlapping Balanced Canonical Forms for Stable Multivariable All-Pass Systems

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1 Introduction

Taking the work of Ober on balanced canonical forms as a starting point, a new balanced canonical form is constructed for stable multivariable all-pass systems of fixed finite McMillan degree. The model reduction properties of this balanced canonical form are closely related to partial realization. The canonical form is extended to an atlas of overlapping balanced continuous canonical forms and induces an atlas in the sense of manifold theory of the manifold of stable multivariable all-pass i/o-systems. The results obtained can be used both for theoretical purposes, e.g. to analyse the structure of the manifold, model approximation problems, for construction of canonical forms of other classes of systems etc. and for practical purposes like optimization problems over the set of all stable all-pass systems of some fixed order. Such optimization problems arise for example in system identification (compare e.g. [1] and the references given there). The use of overlapping parametrizations for system identification has been advocated by a number of authors (see e.g. [3,5,9,10,19,7,8]). The use of balanced realizations for system identification is indicated in e.g. [15,18].

2 Balanced canonical forms, Kronecker indices and nice selections

Let us consider continuous–time multivariable systems of the form

$$\dot{x}_t = Ax_t + Bu_t, \tag{2.1}$$
$$y_t = Cx_t + Du_t \tag{2.2}$$

with $t \in \mathbf{R}, u_t \in \mathbf{R}^p, x_t \in \mathbf{R}^n, y_t \in \mathbf{R}^m, A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times p}, C \in \mathbf{R}^{m\times n}, D \in \mathbf{R}^{m\times p}$.

Let for each $n \in \{1,2,3,\cdots\}$ the set C_n be the set of all quadruples $(A,B,C,D) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p} \times \mathbf{R}^{m\times n} \times \mathbf{R}^{m\times p}$ with the properties: (a) (A,B,C,D) is a minimal realization and (b) the spectrum of A is contained in the open left half plane.
As is well-known two minimal system representations (A_1,B_1,C_1,D_1) and (A_2,B_2,C_2,D_2) have the same transfer function $G(s) = C_1(sI - A_1)^{-1}B_1 + D_1 = C_2(sI - A_2)^{-1}B_2 + D_2$, and therefore describe the same input-output behaviour, iff there exists an $n \times n$ matrix

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$T \in Gl_n(\mathbf{R})$ such that $A_1 = TA_2T^{-1}, B_1 = TB_2, C_1 = C_2T^{-1}, D_1 = D_2$. In that case we say that (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are i/o-equivalent. This is clearly an equivalence relation; write $(A_1, B_1, C_1, D_1) \sim (A_2, B_2, C_2, D_2)$. A unique representation of a linear system can be obtained by deriving a canonical form:

Definition 2.1 *A canonical form for an equivalence relation " $\sim$ " on a set X is a map*

$$\Gamma : X \to X$$

which satisfies for all $x, y \in X$:

(i) $\Gamma(x) \sim x$

(ii) $x \sim y \Longleftrightarrow \Gamma(x) = \Gamma(y)$

Equivalently a canonical form can be given by the image set $\Gamma(X)$; a subset $B \subseteq X$ describes a canonical form if for each $x \in X$ there is precisely one element $b \in B$ such that $b \sim x$. The mapping $X \to B \subseteq X, x \mapsto b$ then describes a canonical form.

Let $(A, B, C, D) \in C_n$. The controllability Grammian W_c is the positive definite matrix that is given by the integral

$$W_c = \int_0^\infty \exp(At)BB^T \exp(A^T t)\mathrm{d}t$$

As is well-known W_c can be obtained as the unique solution of the following Lyapunov equation:

$$AW_c + W_cA^T = -BB^T \tag{2.3}$$

In a dual fashion, the observability Grammian W_o is the positive definite matrix that is given by the integral

$$W_o = \int_0^\infty \exp(A^T t)C^T C \exp(At)\mathrm{d}t$$

This matrix is the unique solution of the following Lyapunov equation

$$A^T W_o + W_o A = -C^T C \tag{2.4}$$

Definition 2.2 *Let $(A, B, C, D) \in C_n$, then (A, B, C, D) is called* balanced *if the corresponding observability and controllability Grammians are equal and diagonal, i.e. there exist positive numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that*

$$W_o = W_c = diag(\sigma_1, \ldots, \sigma_n) =: \Sigma \tag{2.5}$$

The numbers $\sigma_1, \ldots, \sigma_n$ are called the (Hankel) singular values of the system.

The singular values are known to be uniquely determined by the input-output behaviour of the system.

Definition 2.3 *A* balanced canonical form *is a canonical form $\Gamma : C_n \to C_n$, such that $\Gamma(A, B, C, D)$ is balanced for each quadruple $(A, B, C, D) \in C_n$*

Definition 2.4 *A quadruple $(A, B, C, D) \in C_n$ is called* input-normal *if $W_c = I_n$. A canonical form $\Gamma : C_n \to C_n$ is called input-normal if $\Gamma(A, B, C, D)$ is input-normal for each quadruple $(A, B, C, D) \in C_n$. Similarly $(A, B, C, D) \in C_n$ is called* output-normal *if $W_o = I_n$ and a canonical form $\Gamma : C_n \to C_n$ is called output-normal if $\Gamma(A, B, C, D)$ is output-normal for each quadruple $(A, B, C, D) \in C_n$.*

It is not difficult to show that an input-normal realization is unique up to an arbitrary orthogonal state-space transformation. The same holds for an output-normal realization.

Definition 2.5 *Consider a pair (A, B) of matrices $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times p}$. Let $R_{n+1} = R_{n+1}(A, B) = [B, AB, \dots, A^n B]$ denote the corresponding reachability matrix. A selection of n columns is called a nice selection if it has the property that if the $j-th$ column of R_{n+1} is in the selection, then either $j \leq p$ or otherwise the $(j-p)th$ column is also in the selection. The corresponding square submatrix of R_{n+1} consisting of these columns will be called a nice submatrix of R_{n+1}. For each $i \in \{1, 2, \dots, p\}$ let d_i denote the smallest nonnegative value of j such that the $(jp+i)th$ column is not in the selection. Then we will call $(d_1, d_2, \dots, d_p)$ the dynamical indices (also called successor indices) corresponding to the nice selection.*

With a nice selection corresponds a sequence $p = s_0 \geq s_1 \geq s_2 \geq \dots \geq s_l > s_{l+1} = 0$ which add up to $n+p$ and a sequence of sets of indices $\{\{i_j(1), i_j(2), \dots, i_j(s_j)\} \subset \{1, 2, \dots, s_{j-1}\}, j = 1, 2, \dots, l\}$ with the property that from the s_{j-1} columns that *can* be chosen from $A^{j-1}B$ in the nice selection, the $i_j(1)-$th, the $i_j(2)-$th etc until the $i_j(s_j)-$th *are* chosen. It is clear that the sequence of sets of indices determines the nice selection completely and is in bijective correspondence with the sequence of dynamical indices $\{d_1, d_2, \dots, d_p\}$ that describes the nice selection.

Remarks.

(i) A similar definition holds for a nice selection of rows from the observability matrix of a pair (A, C) of matrices $A \in \mathbf{R}^{n\times n}$ and $C \in \mathbf{R}^{m\times n}$, and for a corresponding nice submatrix of the observability matrix.

(ii) As is well-known the first n linearly independent columns of the reachability matrix form a nice selection of columns and form a basis for $\mathbf{R}^n$. The corresponding dynamical indices are denoted by $(r_1, r_2, \dots, r_p)$. By ordering these according to magnitude, one obtains a non- decreasing sequence of p indices $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_p$ which are called the Kronecker reachability (or controllability) indices. Analogously the Kronecker observability indices are defined. See e.g. [7], section 4.4 and 2.3.5 and the references given there.

The following lemma is basic for our considerations (see e.g. [18]):

Lemma 2.6 *Let $M \in \mathbf{R}^{n\times l}, rank(M) = n \leq l$. There exists an orthogonal matrix $Q_0 \in \mathbf{R}^{n\times n}$ and natural numbers $1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq l$ such that*

$$M_0 := Q_0 M =$$

$$\begin{pmatrix} 0 & \dots & m_{1i_1} & * & \dots & * & \dots & \dots & \dots & \dots & * \\ 0 & \dots & 0 & 0 & \dots & m_{2i_2} & \dots & \dots & \dots & \dots & \dots \\ \dots & & \dots & \dots & & & \ddots & & & & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & m_{ni_n} & * \end{pmatrix},$$

with $m_{ji_j} > 0$ for all $j \in 1, 2, \dots, n$, M_0 is unique and Q_0 is unique. Such a matrix will be called positive upper triangular *with* independency indices $i_1, i_2, \dots, i_n$.

A matrix will be called *full rank upper triangular* if it is positive upper triangular up to multiplication of some (or possibly all or none) of its rows by -1.

3 The effects of truncation on a balanced realization

Consider the following well-known result.

Lemma 3.1 (Pernebo and Silverman [20]) *Suppose* $(A, B) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times p}$ *satisfies the Lyapunov equation*

$$AP + PA^T = -BB^T$$

for some $P > 0$. *Then* A *is asymptotically stable if and only if* (A, B) *is controllable.*

"Truncation of the state" is a simple form of model order reduction: Let

$$E_k := \begin{pmatrix} I_k & 0 \\ 0 & 0_{n-k} \end{pmatrix} \in \mathbf{R}^{n\times n}$$

then truncation of the state vector to its first k components means in fact that the following mapping is applied:

$$(A, B, C, D) \mapsto \Big((I_k, 0)A(I_k, 0)^T, (I_k, 0)B, C(I_k, 0)^T, D\Big)$$

Clearly this system has order $\leq k$. Note that the system (E_kAE_k, E_kB, CE_k, D) has the same input-output behaviour as $\Big((I_k, 0)A(I_k, 0)^T, (I_k, 0)B, C(I_k, 0)^T, D\Big)$. One of the advantages of balanced realizations is that if the singular values are distinct, truncation leads to a stable and minimal system. The lemma of Pernebo and Silverman stated above implies that for *any* balanced realization truncation leads to a stable system iff the truncated system is reachable iff it is observable. This follows from the fact that if P is diagonal, it commutes with E_k for any k.

We will now present an easy sufficient condition for reachability of the truncated system (by duality a sufficient condition for observability follows).

Lemma 3.2 *Consider a minimal quadruple* (A, B, C, D).
If the reachability matrix $R_{n+1}(A, B)$ *is positive upper triangular, then the truncated system* $\Big((I_k, 0)A(I_k, 0)^T, (I_k, 0)B, C(I_k, 0)^T, D\Big)$ *is reachable for any* $k \in \{1, 2, \dots, n-1\}$.

Proof. Let $k \in \{1, 2, \dots, n-1\}$ be fixed. It will be sufficient to show that $R_{n+1}(E_kAE_k, E_kB)$ has rank k, because

$$R_{n+1}(E_kAE_k, E_kB) = \begin{pmatrix} R_{n+1}((I_k, 0)A(I_k, 0)^T, (I_k, 0)B) \\ 0 \end{pmatrix}$$

Let $(jp+i)$, with $0 < i \leq p$, be the largest number such that the first $(jp+i)$ columns of the reachability matrix have their last $n-k$ entries all zero. If $j = 0$ then due to the positive upper triangular structure of B, and the fact that $(jp+i) = i \leq p$ in this case, it follows that E_kB has rank k and so in this case the statement in the lemma is clearly correct. If $j > 0$ then it easily follows by induction that $A^lB = (E_kAE_k)^lE_kB, l = 0, 1, \dots, j-1$ using the fact that $E_kA^lB = A^lB, l = 0, 1, \dots, j-1$. Because $A^{j-1}B = (E_kAE_k)^{j-1}E_kB$ and the first i columns of the matrix A^jB have their last $n-k$ entries zero it follows that $A^jBe_q = E_kA^jBe_q = E_kAA^{j-1}Be_q = E_kA(E_kAE_k)^{j-1}E_kBe_q = (E_kAE_k)^jE_kBe_q$, $q = 1, 2, \dots, i$. So the first $(jp+i)$ columns of both reachability matrices are the same. The first $(jp+i)$ columns of the reachability matrix of (A, B, C, D) contain precisely k independent columns by construction of j and i and the positive upper triangular structure of the reachability matrix. It follows that the reachability matrix $R_{n+1}(E_kAE_k, E_kB)$ has rank at least k, and therefore precisely k. □

For future reference we now present a related lemma:

Lemma 3.3 *A pair $(A,B), A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times p}$ has positive upper triangular reachability matrix $R_{n+1}(A,B)$ iff there exist integers $p = s_0 \geq s_1 \geq s_2 \geq \ldots \geq s_l > s_{l+1} = 0$ which add up to $n+p$, such that B is of the form*

$$B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

with B_1 an $s_1 \times s_0$ positive upper triangular matrix and A can be partitioned as follows:

$$A = \begin{pmatrix} A_{11} & \ldots & \ldots & \ldots & A_{1,l} \\ A_{21} & \ddots & & & \vdots \\ 0 & A_{32} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & A_{l,l-1} & A_{l,l} \end{pmatrix},$$

a block-Hessenberg matrix with $A_{i,j}$ an $s_i \times s_j$ matrix, $i,j \in \{1,2,\ldots,l\}$, $A_{i,j} = 0$ if $i-j > 1$ and $A_{i+1,i}$ a positive upper triangular matrix for each $i \in \{1,2,\ldots,l-1\}$ and all other matrices $A_{i,j}, i \geq j$, arbitrary.

The integers $s_i, i \in \{1,2,\ldots,l\}$ are related to the rank structure of the reachability matrix by the formula:

$$\sum_{i=1}^{k} s_i = rank\left[B, AB, \ldots, A^{k-1}B\right], k = 1,\ldots,l. \tag{3.1}$$

Note. Similar so-called staircase forms have been used in the literature on numerically well-behaved representations of linear dynamical systems. See e.g. [22], [23], [16] and the references given there. In Algorithm 1 of [22] a procedure is presented to bring an arbitrary pair (A,B) into a form very similar to the one used here. The difference is that here we use a finer structure, not only the block structure. In our case the reachability matrix is positive upper triangular, while in the case of [22] the reachability matrix will just be "block triangular". Formula (3.1) depends only on the block structure and is given by [22], formula (19).

Proof. First note that the product of two positive upper triangular matrices is again a positive upper triangular matrix. Furthermore if the product VU of two matrices V and U is positive upper triangular and U is positive upper triangular then V must also be positive upper triangular. (See Appendix)

If A, B are of the form given in the lemma, then the reachability matrix $R_{n+1}(A,B)$ can be partitioned as follows:

$$R_{n+1}(A,B) = \begin{pmatrix} R_{11} & * & \ldots & \ldots & * \\ 0 & R_{22} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & R_{ll} & * \end{pmatrix}, \tag{3.2}$$

where R_{jj} is an $s_j \times p$ matrix, $j = 1,2,\ldots,l$ which satisfies the equalities

$$R_{11} = B_1$$

$$R_{j+1,j+1} = A_{j+1,j}R_{j,j}, j = 1,2,\ldots,l-1$$

Because B_1 and $A_{j+1,j}, j = 1,2,\ldots,l-1$ are positive upper triangular it follows that for each $j \in \{1,2,\ldots,l\}, R_{jj}$ is positive upper triangular and therefore $R_{n+1}(A,B)$ is positive upper triangular.

If on the other hand it is given that the reachability matrix is positive upper triangular, it can be partitioned as in (3.2) *such that* for each $j \in \{1,2,\ldots,l\}, R_{jj}$ is positive upper triangular. The fact that the number of rows of $R_{11}, R_{22}, \ldots, R_{ll}$ in this partitioning form a decreasing sequence is the result of the specific structure of a reachability matrix. It then follows that

$$B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

with $B_1 = R_{11}$ positive upper triangular. Furthermore if A is partitioned as

$$A = \begin{pmatrix} A_{11} & \ldots & \ldots & \ldots & A_{1,l} \\ A_{21} & \ddots & & & \vdots \\ A_{31} & A_{32} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ A_{l1} & \ldots & \ldots & A_{l,l-1} & A_{l,l} \end{pmatrix},$$

with A_{ij} an $s_i \times s_j$ matrix for each $i,j \in \{1,2,\ldots,l\}$, then it follows easily by induction, from the equations

$$A^j B = \begin{bmatrix} * \\ \vdots \\ \\ R_{j+1,j+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = A(A^{j-1}B) = A \begin{bmatrix} * \\ \vdots \\ \\ R_{j,j} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, j = 1,2,\ldots,l-1,$$

that for all $i > j+1; i,j \in \{1,\ldots,l\}$ one has $A_{i,j} = 0$. From the same equation it then follows that $R_{j+1,j+1} = A_{j+1,j}R_{j,j}, j = 1,\ldots,l-1$. Because $R_{j,j}$ is positive upper triangular for each $j \in \{1,\ldots,l\}$ it follows from the proposition mentioned at the beginning of the proof (and shown in the Appendix) that for each $j \in \{1,\ldots,l-1\}, A_{j+1,j}$ is positive upper triangular. The formula (3.1) follows from the structure of the reachability matrix in (3.2), combined with the fact that the R_{jj} are positively upper triangular. □

Remark It is well-known that the nonincreasing sequence of nonnegative integers $s_i, i = 1,2,\ldots$ is in bijective correspondence with the reachability Kronecker indices. (See e.g. [7], section 2.3.5 and the references given there). There is a completely similar relationship between the rank structure of the rows of an observability matrix and the observability Kronecker indices of a system.

An interpretation of truncation in case of a positive upper triangular reachability matrix and a positive lower triangular observability matrix can be given in terms of partial realization:

Lemma 3.4 *Suppose (A,B,C,D) has positive upper triangular reachability matrix and positive lower triangular observability matrix and state space dimension n. For arbitrary $k \in \{1,2,\ldots,n-1\}$ the following holds. Let $j = j(k) \in \{0,1,\ldots\}, i = i(k) \in \{0,1,\ldots,p-$*

$1\}$ *be chosen such that the* $(n-k)$ *last components of the first* $jp+i$ *columns of the reachability matrix of* (A,B) *are zero and that* $jp+i$ *is the largest number for which this holds. Similarly let* $j' = j'(k) \in \{0,1,\ldots\}, i' = i'(k) \in \{0,1,\ldots,p-1\}$ *be chosen such that the* $(n-k)$ *last components of the first* $j'p+i'$ *rows of the observability matrix of* (A,C) *are zero and that* $j'p+i'$ *is the largest number for which this holds. Then the truncated system* $\Big((I_k,0)A(I_k,0)^T,(I_k,0)B,C(I_k,0)^T,D\Big)$ *is the unique (up to i/o system equivalence) minimal partial realization of the (truncated) block Hankel matrix that is obtained from the block Hankel matrix of* (A,B,C,D) *by selecting the first* $j'p+i'$ *rows of the first* $(j+1)p+i$ *columns together with the rows numbered* $j'p+i'+1, j'p+i'+2,\ldots,(j'+1)p+i'$ *of the first* $jp+i$ *columns.*

Proof. Let $k \in \{1,2,\ldots,n-1\}$ be fixed and let $j = j(k), i = i(k)$, $j' = j'(k), i' = i'(k)$ as described in the lemma. Let $R_{j,i}(A,B)$ denote the $n\times(jp+i)$ matrix that is obtained from the reachability matrix of (A,B) by selecting the first $(jp+i)$ columns. Similarly let $Q_{j',i'}(A,B)$ denote the $(j'p+i')\times n$ matrix that is obtained from the observability matrix of (A,C) by selecting the first $(j'p+i')$ rows. According to the previous lemma and the definition of j,i,j',i', one has $R_{j,i}(A,B) = E_kR_{j,i}(A,B) = R_{j,i}(E_kAE_k,E_kB) = E_kR_{j,i}(E_kAE_k,E_kB)$ and $Q_{j',i'}(A,B) = Q_{j',i'}(A,B)*E_k\ Q_{j',i'}(E_kAE_k,E_kB) = Q_{j',i'}(E_kAE_k,E_kB)*E_k$. From this it follows that $Q_{j',i'}(A,B)*R_{j,i}(A,B) = Q_{j',i'}(E_kAE_k,E_kB)*R_{j,i}(E_kAE_k,E_kB)$ and $Q_{j',i'}(A,B)*A*R_{j,i}(A,B) = Q_{j',i'}(E_kAE_k,E_kB)*E_kAE_k*R_{j,i}(E_kAE_k,E_kB)$. This shows that the selection of the block Hankel matrix of (A,B,C,D) mentioned in the lemma produces the same truncated matrix as the same selection applied to the block Hankel matrix of (E_kAE_k,E_kB,CE_k,D). Because the ranks of $Q_{j',i'}R_{j,i}$, $Q_{j'+1,i'}R_{j,i}$, $Q_{j',i'}R_{j+1,i}$, are all equal to k, which is also the rank of the block Hankel matrix $Q_n(E_kAE_k,CE_k)*R_{n+1}(E_kAE_k,E_kB)$ it follows from the partial realization lemma (see e.g. [11], see also [14], [2]) that the unique minimal partial realization of the selection of entries of the block Hankel matrix mentioned in the lemma is indeed the truncated system given by (E_kAE_k,E_kB,CE_k,D) or, equivalently, given by $\Big((I_k,0)A(I_k,0)^T,(I_k,0)B,C(I_k,0)^T,D\Big)$. □

4 A canonical form for multivariable stable all-pass systems

The following definition will be used for all-pass systems:

Definition 4.1 *A system represented by a quadruple* $(A,B,C,D)\in C_n$ *with number of inputs equal to the number of outputs:* $m=p$ *is called a stable all-pass system if its transfer matrix* $G(s) = C(sI-A)^{-1}B+D$ *has the property that for all s on the imaginary axis,* $G(s)^{-1} = G(s)^*$ *which is equivalent to the equality* $G(s)^{-1} = G(-s)^T$ *for all s for which* $\det(G(s)) \neq 0$ *and* $\det(G(-s)) \neq 0$. *The set of all minimal quadruples* (A,B,C,D) *with this property is denoted by* A_n^p.

The following result is crucial for the construction of balanced canonical forms for stable all-pass systems:

Theorem 4.2 (Glover [4]) *Let* $m=p$; *a balanced triple* (A,B,C) *has identical Hankel singular values* $\sigma = 1$ *iff there exists a matrix D such that the quadruple* (A,B,C,D) *represents a stable all-pass system. Furthermore D then has the following properties:*

$$D^{-1} = D^T; B^T = -D^TC \tag{4.1}$$

and therefore

$$C = -DB^T$$

If (A, B, C) is a balanced triple with identical Hankel singular values $\sigma = 1$ and D satisfies (4.1) then (A, B, C, D) represents a stable all-pass system.

Note that from Glover's theorem it follows that a balanced realization of a stable all-pass system has both Grammians equal to the identity matrix I_n. Using this and the equations for C and D, one gets the following characterization of a balanced realization of a stable all-pass system:

Corollary 4.3 *A quadruple (A, B, C, D) with $m = p$ and A asymptotically stable is a balanced realization of a stable all-pass system iff the following equations hold:*

$$A + A^T = -BB^T = -C^TC \tag{4.2}$$

$$C = -DB^T \tag{4.3}$$

$$DD^T = I_m \tag{4.4}$$

Note that according to the results of Pernebo and Silverman presented before, the asymptotic stability of A is guaranteed in this case if (A, B) is reachable and also if (A, C) is observable.

It follows that if (A, B, C, D) is a balanced realization of a stable all-pass system then one can decompose the matrix A into its skew-symmetric and symmetric parts as follows

$$A = \tilde{A} - \frac{1}{2}BB^T,$$

where $\tilde{A} = A + \frac{1}{2}BB^T$ and $\tilde{A} + \tilde{A}^T = 0$. This can be applied to show that for stable all-pass systems the controllability Kronecker indices are equal to the observability Kronecker indices. Consider the following linear subspaces of the state space spanned by columns of the reachability matrix:

$$V_j = \text{span}\{B, AB, \ldots, A^{j-1}B\} = \text{span}\{B, \tilde{A}B, \ldots, \tilde{A}^{j-1}B\}$$

Consider analogously the following linear subspaces of $\mathbf{R}^n$ spanned by columns of the transposed of the observability matrix:

$$\text{span}\{C^T, A^TC^T, \ldots, (A^T)^{j-1}C^T\} = \text{span}\{B, A^TB, \ldots, (A^T)^{j-1}B\} =$$

$$\text{span}\{B, \tilde{A}B, \ldots, \tilde{A}^{j-1}B\} = V_j$$

It follows that a fortiori the dimensions $t_j = \dim V_j$ of the corresponding linear subspaces of the reachability matrix and the observability matrix are equal. Let

$$s_1 := t_1 \leq p \tag{4.5}$$

$$s_j := t_j - t_{j-1}, j = 2, 3, \ldots \tag{4.6}$$

then it is known that the sequence $\{s_j, j = 1, 2, 3, \ldots\}$ is non-increasing and that the Kronecker indices are in bijective correspondence with this sequence. (See the remark after the proof of lemma 3.3) It follows that the reachability Kronecker indices and the observability Kronecker indices of stable all-pass systems are indeed equal.

In the following it will be convenient at times to use the observability matrix of the system with output

$$z_t = D^{-1}y_t = D^T y_t$$

If (A, B, C, D) is a balanced realization of the stable all-pass system with output y_t, then $(A, B, -B^T, I)$ is a balanced realization of the all-pass system with output z_t.

Suppose (A, B, C, D) is any balanced (hence minimal) realization of a stable all-pass system. An i/o-equivalent realization $(TAT^{-1}, TB, CT^{-1}, D)$ is again balanced iff T is orthogonal, because then, and only then, the Grammians remain equal to the identity matrix I_n. The idea is now to choose an orthogonal transformation of the state space such that the resulting reachability matrix is positive upper triangular. According to lemma 2.6 this determines the orthogonal transformation, and therefore the resulting quadruple, uniquely. In other words this determines a balanced canonical form for stable all-pass systems. It can be described as follows:

Theorem 4.4 *Consider a stable all-pass system with McMillan degree n. There exists a unique balanced realization $(A, B, C, D) \in A_n^p$ of the following form: There are integers $p = s_0 \geq s_1 \geq s_2 \geq \ldots \geq s_l > s_{l+1} = 0$ which add up to $n + p$, such that*

$$B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

where B_1 is an $s_1 \times s_0$ positive upper triangular matrix;

$$A = \begin{pmatrix} A_{11} & A_{12} & 0 & \ldots & 0 \\ A_{21} & A_{22} & A_{23} & \ddots & \vdots \\ 0 & A_{32} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & A_{l-1,l} \\ 0 & \ldots & 0 & A_{l,l-1} & A_{l,l} \end{pmatrix},$$

a block tridiagonal matrix with $A_{i,j}$ an $s_i \times s_j$ matrix, $i, j \in \{1, 2, \ldots, l\}$, $A_{i,j} = 0$ if $|i-j| > 1$;

$$A_{11} = \tilde{A}_{11} - \frac{1}{2} B_1 B_1^T,$$

$\tilde{A}_{11}$ an otherwise arbitrary skew symmetric $s_1 \times s_1$ matrix;
$A_{ii} = \tilde{A}_{ii}$ an otherwise arbitrary skew symmetric $s_i \times s_i$ matrix for each $i \in \{2, 3, \ldots, l\}$;
$A_{i+1,i}$ a positive upper triangular $s_{i+1} \times s_i$ matrix for each $i \in \{1, 2, \ldots, l-1\}$;
$A_{i,i+1} = -A_{i+1,i}^T$ for each $i \in \{1, 2, \ldots, l-1\}$;
D an otherwise arbitrary orthogonal $p \times p$ matrix and

$$C = -DB^T.$$

The indices $s_i, i = 1, \ldots, l$ have the same meaning as in (4.5) and so they are in bijective correspondence with the Kronecker indices. The canonical form is balanced and its reachability matrix is positively upper triangular.

Proof. We know that a quadruple (A, B, C, D) is a state space representation in the canonical form of a stable multivariable all-pass system iff (i) (A, B, C, D) is a balanced realization of a stable all-pass, i.e. the Grammians are the identity matrix and the equations for D and C from Glover's theorem hold; (ii) the reachability matrix $R_n(A, B)$ is positive upper triangular. It is clear by construction that if (A, B, C, D) is in the form as presented in the theorem, it satisfies (i) and (ii) and therefore (A, B, C, D) is in canonical form. Now suppose (A, B, C, D) satisfies (i) and (ii). From lemma 3.3 it follows that there exist integers $p = s_0 \geq s_1 \geq s_2 \geq \ldots \geq s_l > s_{l+1} = 0$ which add up to $n + p$, such that B is of the form

$$B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

with B_1 an $s_1 \times s_0$ positive upper triangular matrix and A can be partitioned as follows:

$$A = \begin{pmatrix} A_{11} & \ldots & \ldots & \ldots & A_{1,l} \\ A_{21} & \ddots & & & \vdots \\ 0 & A_{32} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & A_{l,l-1} & A_{l,l} \end{pmatrix},$$

a block-Hessenberg matrix with $A_{i,j}$ an $s_i \times s_j$ matrix, $i, j \in \{1, 2, \ldots, l\}$, $A_{i,j} = 0$ if $i - j > 1$ and $A_{i+1,i}$ a positive upper triangular matrix for each $i \in \{1, 2, \ldots, l-1\}$ and all other matrices $A_{i,j}, i \geq j$, arbitrary. Because from (i) it follows that $A + \frac{1}{2}BB^T$ is skew symmetric, A must have the form presented in the theorem. The prescriptions for D and C follow directly from Glover's theorem. □

Remarks

(i) The system with output $z = D^{-1}y$ has full rank lower triangular observability matrix $Q_n(A, -B^T)$, by which we mean that its transpose $Q_n^T(A, -B^T)$ is full rank upper triangular. Of course $Q_n^T(A, -B^T) = R_{n+1}(A^T, -B)$ which is indeed full rank upper triangular. This follows easily from the facts that $-B$ is full rank upper triangular and A^T, partitioned in the same way as A in the theorem, is also block-upper Hessenberg with for each $i \in \{1, 2, \ldots, l-1\}$ the full rank upper triangular matrix $-A(i+1, i)$ in the $(i+1, i)$–block.

It follows from this that the system with output y has full rank *block* lower triangular observability matrix (the meaning of this terminology should be obvious).

(ii) Truncation of the state in this canonical form results in a reachable system (see lemma 3.2), with an identity matrix as its reachability Grammian and therefore the resulting system is stable (according to lemma 3.1). Furthermore, if $D = I_p$ then the system is also observable because the observability matrix is full rank lower triangular (see previous remark) and so lemma 3.2 can be applied to the transpose of the observability matrix. This implies that the system with output $z = D^{-1}y$ is observable after truncation. But this implies that the original system with output $y = Dz$ results after truncation in an observable, and hence minimal and stable, system as well.

(iii) In fact if $D = I_p$, truncation corresponds to partial realization: if the matrix E_k is applied to truncate the state vector, then according to lemma 3.4 this model reduction procedure corresponds to minimal partial realization of the relevant part of the block Hankel matrix of the system. So for the system with output $z = D^{-1}y$ this property holds. It follows that for the original system with output y, the same interpretation holds *if* the number k of state components retained is chosen such that k is a partial sum of the sequence $\{s_j\}_{j=1}^{l}$. In that case the relevant part of the Hankel matrix consists of complete blocks, i.e. none of the blocks is itself truncated.

(iv) The canonical form is related to the canonical form of Bosgra-van der Weiden [2], if applied to the special class of stable all-pass systems. If $D = I$, in the canonical form presented above, the matrices $[B, A]$ and $[C^T, A^T]$ are full rank upper triangular. This implies that, according to theorem 3.6 of [2] $(\Lambda^{-1}A\Lambda, \Lambda^{-1}B, C\Lambda, D)$, with $\Lambda = \text{diag}(r_{1,i_1}, r_{2,i_2}, \ldots, r_{n,i_n})$, where $i_1, i_2, \ldots, i_n$ are the independency indices of the positive upper triangular reachability matrix $R_{n+1}(A, B)$, is in Bosgra-van der Weiden canonical form. The permutation matrix that appears in that canonical form is equal to the identity matrix in this case. Note that the Bosgra-van der Weiden canonical form is in general rather complicated, especially due to the role played by the permutation matrix and its relation to the Kronecker indices. (see especially Lemma 3.8 of [2]).

(v) The canonical form is a generalization of the one proposed for SISO stable all-pass systems by Ober [17], which is in turn related to the socalled Schwarz canonical form (see also [8]). In fact, also the realization procedure presented in [17] has a multivariable generalization: application of a Moebius transform followed by a matrix Euclidean algorithm to the transfer matrix produces the submatrix B_1 and the blocks of the block-tridiagonal matrix $\tilde{A}$. We hope to return to this aspect of the canonical form in a future publication.

5 An atlas of overlapping balanced canonical forms

In this section we present an atlas of overlapping balanced canonical forms for stable multivariable all-pass systems of fixed McMillan degree. The idea is a simple extension of the way the canonical form of the previous sections is obtained. There we made the reachability matrix positively upper triangular by taking the first n linear independent columns and applying an appropriate element of the orthogonal group. Here this is generalized as follows: take *any* nice selection of columns from the reachability matrix of a balanced realization. These columns form a square submatrix of the reachability matrix. Now apply the unique orthogonal matrix such that this submatrix is positively upper triangular. The result is a unique canonical form for all systems for which the chosen selection of columns are linearly independent. We will make use of the notation introduced after the definition of nice selections in section 2. Consider the set A_n^p. Fix a nice selection denoted by the corresponding sequence of dynamical indices $(d_1, d_2, \ldots, d_p)$. Consider the corresponding nice submatrix of the reachability matrix of a minimal realization of such a system. Whether this nice submatrix is nonsingular is of course independent of the choice of a basis for the state space of a minimal realization. The subset of all systems for which the corresponding nice submatrix of the reachability matrix has full rank will be denoted by $A^p_{d_1,d_2,\ldots,d_p}$.

Theorem 5.1 *Consider the subset $Z_{d_1,d_2,\ldots,d_p} \subset A^p_{d_1,d_2,\ldots,d_p}$ of quadruples $(A,B,C,D) \in A^p_{d_1,d_2,\ldots,d_p}$ of the following form:*

$$B = \begin{pmatrix} * & b_{1,i_1(1)} & * & * & & * & * \\ \vdots & 0 & \vdots & b_{2,i_1(2)} & & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \ldots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & * & \vdots \\ \vdots & \vdots & \vdots & \vdots & & b_{s_1,i_1(s_1)} & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & 0 & * & 0 & & 0 & * \end{pmatrix},$$

where $b_{1,i_1(1)} > 0, b_{2,i_1(2)} > 0, \ldots, b_{s_1,i_1(s_1)} > 0$, is a $n \times s_0$ matrix of which the first s_1 rows of the columns numbered $i_1(1), i_1(2), \ldots, i_1(s_1)$ form a square positive upper triangular matrix, while the the last $n - s_1$ rows of the same columns are all zero and all the other columns are arbitrary;

$$A = \tilde{A} - \frac{1}{2}BB^T,$$

where $\tilde{A}$ is a skew-symmetric matrix of the form

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \ldots & \ldots & \ldots & \tilde{A}_{1,l} \\ \tilde{A}_{21} & \ddots & & & \vdots \\ \tilde{A}_{31} & \tilde{A}_{32} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \tilde{A}_{l1} & \ldots & \ldots & \tilde{A}_{l,l-1} & \tilde{A}_{l,l} \end{pmatrix},$$

with $\tilde{A}_{ij}$ an $s_i \times s_j$ matrix for each $i,j \in \{1,2,\ldots,l\}$, and

$$\begin{bmatrix} \tilde{A}_{j,j-1} \\ \tilde{A}_{j+1,j-1} \\ \vdots \\ \tilde{A}_{l,j-1} \end{bmatrix} == \begin{pmatrix} * & a_{1,i_j(1)} & * & * & & * & * \\ \vdots & 0 & \vdots & a_{2,i_j(2)} & & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \ldots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & * & \vdots \\ \vdots & \vdots & \vdots & \vdots & & a_{s_j,i_j(s_j)} & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & 0 & * & 0 & & 0 & * \end{pmatrix},$$

for $j = 2,3,\ldots,l$, where $a_{1,i_j(1)} > 0, a_{2,i_j(2)} > 0, \ldots, a_{s_j,i_j(s_j)} > 0$, is an $(\sum_{k=j}^{l} s_k) \times s_{j-1}$ matrix of which the first s_j rows of the columns numbered $i_j(1), i_j(2), \ldots, i_j(s_j)$ form a square positive upper triangular matrix, while the the last $\sum_{k=j+1}^{l}$ rows of the same columns are all zero and all the other columns are arbitrary;
furthermore $\tilde{A}_{j,j}$ is an otherwise arbitrary skew-symmetric matrix for each $j = 1,2,\ldots,l$ and if $i > j$ then $\tilde{A}_{i,j}$ is determined by $\tilde{A}_{i,j} = -\tilde{A}^T_{j,i}$; finally D is an otherwise arbitrary orthogonal $p \times p$ matrix, i.e. $D^{-1} = D^T$, and $C = -DB^T$.

The set $Z_{d_1,d_2,\ldots,d_p}$ describes a continuous balanced canonical form on $A^p_{d_1,d_2,\ldots,d_p}$.

Proof The proof that $Z_{d_1,d_2,\ldots,d_p}$ describes a balanced canonical form on $A^p_{d_1,d_2,\ldots,d_p}$ is very similar to the proof of theorem 4.4 and is left to the reader. The continuity of the canonical form follows from the observation that the basis of the state space that is used in this canonical form can be obtained by Gram-Schmidt orthogonalization of the columns of the nice submatrix of the reachability matrix *with respect to* the inverse of the controllabilty Grammian, i.e. using as an inner product the formula $< x, y >= x^T W_c^{-1} y$. Because the controllability Grammian depends continuously on A and B and is nonsingular, and because the Gram-Schmidt orthogonalization provides a continuous mapping from the set of nonsingular matrices $Gl_n(\mathbf{R})$ to the set of orthogonal matrices O_n, the canonical form is continuous. □

The number of parameters in this canonical from can be counted as follows: B has $n \times (s_0 - s_1) + \frac{1}{2}(s_1 - 1)s_1$ completely free real parameters and s_1 parameters which are restricted to be positive, this adds up to $(s_0 - s_1)n + \frac{1}{2}s_1(s_1 + 1)$. If one adds to this the number of completely free parameters in $\tilde{A}_{1,1}$, namely $\frac{1}{2}s_1(s_1 - 1)$ one obtains a subtotal of $(s_0 - s_1)n + \frac{1}{2}s_1(s_1 - 1) = s_0(\sum_{k=1}^{l} s_k) - s_1(\sum_{k=2}^{l} s_k)$ free parameters. A similar count can be made for the number of free parameters of each matrix

$$\begin{bmatrix} \tilde{A}_{j,j-1} \\ \tilde{A}_{j+1,j-1} \\ \vdots \\ \tilde{A}_{l,j-1} \end{bmatrix},$$

together with the number of free parameters in the skew-symmetric matrix $\tilde{A}_{j,j}$ with $j \in \{2, \ldots, l\}$, giving the number $s_{j-1}(\sum_{k=j}^{l} s_k) - s_j(\sum_{k=j+1}^{l} s_k)$. Adding all these numbers we obtain a number of $s_0(\sum_{k=1}^{l} s_k) = pn$ free parameters in the matrices A, B. The matrix C has no additional free parameters. The matrix D is an orthogonal $p \times p$ matrix. The orthogonal group O_p has dimension $\frac{1}{2}p(p-1)$ as is well-known. So the total number of free parameters is $np + \frac{1}{2}(p-1)p = (n + \frac{1}{2}p - \frac{1}{2})p$.

Let $T_{d_1,d_2,\ldots,d_p} \subset \mathbf{R}^{pn}$ denote the set of values of the free parameters in (A, B) in this canonical form. Let $\phi_{d_1,d_2,\ldots,d_p}$ denote the mapping which maps each reachable pair (A, B) in canonical form to the corresponding element of $T_{d_1,d_2,\ldots,d_p}$. Let ψ_α, where α ranges over some appropriate index set, form an atlas for the orthogonal group O_p. For each α in the index set, ψ_α maps into an open subset of $\mathbf{R}^{\frac{1}{2}p(p-1)}$. (Precisely which atlas is used for the orthogonal group is not of importance here).

We can now state the main results of this section

Theorem 5.2 *The canonical forms* $A^p_{d_1,d_2,\ldots,d_p} \longrightarrow Z_{d_1,d_2,\ldots,d_p}, d_j \in \{0, 1, \ldots, n\}, j = 1, 2, \ldots, p; \sum_{j=1}^{p} d_j = n$ *form an atlas of overlapping continuous balanced canonical forms for the set of multivariable stable all-pass systems with p inputs and outputs and McMillan degree n. Each of the sets* $A^p_{d_1,d_2,\ldots,d_p}, d_j \in \{0, 1, \ldots, n\}, j = 1, 2, \ldots, p; \sum_{j=1}^{p} d_j = n$ *is an open subset of* A^p_n *and together they cover* A^p_n.

Proof Because each set $A^p_{d_1,d_2,\ldots,d_p}$ is defined as the subset of A^p_n of quadruples for which the corresponding nice submatrix of the reachability matrix is nonsingular, it follows easily that $A^p_{d_1,d_2,\ldots,d_p}$ is an open subset of A^p_n. Each quadruple in A^p_n has at least one nice selection with nonsingular nice submatrix of the reachability matrix, namely the nice selection that is obtained by choosing the first n linearly independent columns of the reachability matrix. So the sets $A^p_{d_1,d_2,\ldots,d_p}, d_j \in \{0, 1, \ldots, n\}, j = 1, 2, \ldots, p; \sum_{j=1}^{p} d_j = n$ cover A^p_n indeed.
□

Before we formulate a corollary let us note that by the very definition of a canonical form, it induces a mapping on the equivalence classes of quadruples (i.e. on input-output systems). In our case the canonical form induces a mapping which maps $A^p_{d_1,d_2,\ldots,d_p}/\sim$ to $Z_{d_1,d_2,\ldots,d_p}$.

Corollary 5.3 *Consider the mappings which are obtained by composing the canonical form mapping $A^p_{d_1,d_2,\ldots,d_p}/\sim\longrightarrow Z_{d_1,d_2,\ldots,d_p}$ with the pair of mappings*

$$\begin{pmatrix} \phi_{d_1,d_2,\ldots,d_p} \\ \psi_\alpha \end{pmatrix}$$

which map a quadruple $(A,B,C,D) \in Z_{d_1,d_2,\ldots,d_p}$ to a vector of parameter values in $\mathbf{R}^{pn+\frac{1}{2}p(p-1)}$. The set of all such mappings with $d_j \in \{0,1,\ldots,n\}, j=1,2,\ldots,p; \sum_{j=1}^p d_j = n$, forms an atlas for the manifold of stable multivariable all-pass i/o-systems with p inputs and outputs and McMillan degree n.

Remarks

(i) The results presented can also be used for discrete time stable multivariable all-pass systems by making use of a bilinear transformation which maps continuous-time stable systems bijectively into discrete time stable systems while keeping the Grammians invariant (cf. e.g. [4]).

(ii) The atlas of continuous balanced canonical forms directly gives rise, as usual in the theory of canonical forms for linear finite dimensional systems, to an atlas of the so-called state bundle and the corresponding principle fibre bundle. See e.g. [6] and the references given there.

Appendix

Here we show the propositions concerning positive upper triangular matrices that are used in the proof of lemma 3.3.

Proposition A.1 *If V is a $v \times u$ positive upper triangular matrix with dependency indices $m(1), m(2), \ldots, m(v)$ and U a $u \times w$ positive upper triangular matrix with dependency indices $l(1), l(2), \ldots, l(u)$ then VU is a $v \times w$ positive upper triangular matrix with dependency indices $l(m(1)), l(m(2)), \ldots, l(m(v))$.*

Proof. Let $D_j^N := \{x = (x_1,\ldots,x_N) | x_1 = \ldots = x_{j-1} = 0, x_j > 0\}$. Then for all $j \in \{1,2,\ldots,v\}$, for all $x \in D_j^v, xV \in D^u_{m(j)}$. For all $k \in \{1,2,\ldots,u\}$, for all $y \in D_k^u, yU \in D^w_{l(k)}$. It follows that for all $j \in \{1,2,\ldots,v\}$, for all $x \in D_j^v, xVU \in D^w_{l(m(j))}$. Because $\{m(1),\ldots,m(v)\}$ and $\{l(1),\ldots,l(u)\}$ both are a strictly increasing sequence of integers, the same holds for $l(m(1)), l(m(2)), \ldots, l(m(v))$. It follows that VU is positive upper triangular with dependency indices $l(m(1)), l(m(2)), \ldots, l(m(v))$.
□

Proposition A.2 *If V is a $v \times u$ matrix, U a $u \times w$ positive upper triangular matrix and VU positive upper triangular, then V is positive upper triangular as well.*

Proof. We will show that under the hypothesis of the proposition, V is full rank upper triangular. It then follows easily from the *positive* upper triangularity of U and VU that V is *positive* upper triangular as well. Suppose V is not full rank upper triangular. Then there exists a partition

$$V = \begin{pmatrix} V_{11} & V_{12} \\ 0 & V_{22} \end{pmatrix}$$

such that V_{11} full rank upper triangular, with possibly $V = V_{22}$, and for all $\lambda \in \mathbf{R}, V_{22}e_1 \neq \lambda e_1$. If U is positive upper triangular, then this matrix can be partitioned as

$$U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$$

conformably to the partitioning of V such that U_{11} is positively upper triangular, $U_{22}e_1 = \mu e_1 \neq 0$ and

$$VU = \begin{pmatrix} V_{11}U_{11} & V_{11}U_{12} + V_{12}U_{22} \\ 0 & V_{22}U_{22} \end{pmatrix}$$

From the previous proposition it follows that $V_{11}U_{11}$ is full rank upper triangular. Furthermore one has that for all $\lambda \in \mathbf{R}, V_{22}U_{22}e_1 = V_{22}\mu e_1 \neq \lambda\mu e_1$ and therefore $V_{22}U_{22}$ is not full rank upper triangular, so VU is not full rank upper triangular.

□

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