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A new balanced canonical form for
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1 Introduction

A new balanced canonical form is presented for stable multivariable linear systems. In [3] overlapping continuous block-balanced canonical forms were introduced for the stable SISO case, as a generalization of the balanced canonical form of [9] for the SISO case. Such a generalization of the *multivariable* balanced canonical form of [9] appears to be hard (and whether one exists is an open problem) This was the motivation to construct a different multivariable balanced canonical form, which will be presented here. The new canonical form has a number of nice properties. The integer invariants that appear in the canonical form are the multiplicities of the Hankel singular values and a number of new invariants, which are in one-to-one bijective correspondence with the Kronecker indices of subsystems. Truncation of the state vector leads to stable minimal models in canonical form, just as in the case of [9]. In the SISO case the canonical form coincides with Ober's balanced canonical form. The reachability matrix of a system in canonical form with identical singular values is positive upper triangular. A detailed treatment of the canonical form and an extension to an atlas of continuous balanced canonical forms for the class of stable multivariable all-pass systems is presented in [4].

2 Balancing, canonical forms and Kronecker indices

Let us consider continuous-time multivariable systems of the form

$$\dot{x}_t = Ax_t + Bu_t, \quad (2.1)$$

$$y_t = Cx_t + Du_t \quad (2.2)$$

with $t \in \mathbf{R}$, $u_t \in \mathbf{R}^p$, $x_t \in \mathbf{R}^n$, $y_t \in \mathbf{R}^m$, $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times p}$, $C \in \mathbf{R}^{m \times n}$, $D \in \mathbf{R}^{m \times p}$.

Let for each $n \in \{1, 2, 3, \dots\}$ the set C_n be the set of all quadruples $(A, B, C, D) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times p} \times \mathbf{R}^{m \times n} \times \mathbf{R}^{m \times p}$ with the properties: (a) (A, B, C, D) is a minimal realization and (b) the spectrum of A is contained in the open left half plane.

As is well-known two minimal system representations (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) have the same transfer function $G(s) = C_1(sI - A_1)^{-1}B_1 + D_1 = C_2(sI - A_2)^{-1}B_2 + D_2$,

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and therefore describe the same input-output behaviour, iff there exists an $n \times n$ matrix $T \in Gl_n(\mathbb{R})$ such that $A_1 = TA_2T^{-1}, B_1 = TB_2, C_1 = C_2T^{-1}, D_1 = D_2$. In that case we say that (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are i/o-equivalent. This is clearly an equivalence relation; write $(A_1, B_1, C_1, D_1) \sim (A_2, B_2, C_2, D_2)$. A unique representation of a linear system can be obtained by deriving a canonical form:

Definition 2.1 A canonical form for an equivalence relation " \sim " on a set X is a map

$$\Gamma : X \rightarrow X$$

which satisfies for all $x, y \in X$:

$$(i) \Gamma(x) \sim x$$

$$(ii) x \sim y \iff \Gamma(x) = \Gamma(y)$$

Equivalently a canonical form can be given by the image set $\Gamma(X)$; a subset $B \subseteq X$ describes a canonical form if for each $x \in X$ there is precisely one element $b \in B$ such that $b \sim x$. The mapping $X \rightarrow B \subseteq X, x \mapsto b$ then describes a canonical form.

Let $(A, B, C, D) \in C_n$. The controllability Grammian W_c is the positive definite matrix that is given by the integral

$$W_c = \int_0^\infty \exp(At)BB^T \exp(A^T t)dt$$

As is well-known W_c can be obtained as the unique solution of the following Lyapunov equation:

$$AW_c + W_cA^T = -BB^T \quad (2.3)$$

In a dual fashion, the observability Grammian W_o is the positive definite matrix that is given by the integral

$$W_o = \int_0^\infty \exp(A^T t)C^T C \exp(At)dt$$

This matrix is the unique solution of the following Lyapunov equation

$$A^T W_o + W_o A = -C^T C \quad (2.4)$$

Definition 2.2 Let $(A, B, C, D) \in C_n$, then (A, B, C, D) is called balanced if the corresponding observability and controllability Grammians are equal and diagonal, i.e. there exist positive numbers $\sigma_1, \sigma_2, \dots, \sigma_n$ such that

$$W_o = W_c = \text{diag}(\sigma_1, \dots, \sigma_n) =: \Sigma \quad (2.5)$$

The numbers $\sigma_1, \dots, \sigma_n$ are called the (Hankel) singular values of the system. It will be convenient to call an arbitrary quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times p}$ balanced if the pair of Lyapunov equations $A\Sigma + \Sigma A^T = -BB^T, A^T \Sigma + \Sigma A = -C^T C$ has a positive definite solution of the form $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_k)$ (assuming neither asymptotic stability nor minimality).

The singular values are known to be uniquely determined by the input-output behaviour of the system.

Definition 2.3 A balanced canonical form (on C_n) is a canonical form $\Gamma : C_n \rightarrow C_n$, such that $\Gamma(A, B, C, D)$ is balanced for each quadruple $(A, B, C, D) \in C_n$.

Definition 2.4 Consider a pair (A, B) of matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times p}$. Let $R_n = R_n(A, B) = [B, AB, \dots, A^n B]$ denote the corresponding reachability matrix. Suppose the pair (A, B) is reachable, i.e. the reachability matrix has rank n . The selection of the first n linearly independent columns is called the Kronecker selection. It has the property that it is a so-called nice selection, which means that if the j -th column of R_n is in the selection, then either $j \leq p$ or otherwise the $(j - p)$ th column is also in the selection. For each $i \in \{1, 2, \dots, p\}$ let d_i denote the largest value of j such that the $(jp + i)$ th column is in the selection. Then we will call (d_1, d_2, \dots, d_p) the dynamical indices (also called successor indices) corresponding to the selection. By ordering these according to magnitude, one obtains a non-decreasing sequence of m indices $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_m$ which are called the Kronecker reachability (or controllability) indices.

With any nice selection corresponds a sequence of integers $p = s_0 \geq s_1 \geq s_2 \geq \dots \geq s_l > s_{l+1} = 0$ which add up to $n + p$ and a sequence of sets of indices $\{i_j(1), i_j(2), \dots, i_j(s_j)\} \subset \{1, 2, \dots, s_{j-1}\}, j = 1, 2, \dots, l$ with the property that of the s_{j-1} columns that can be chosen from $A^{j-1}B$ in the nice selection the $i_j(1)$ -th, the $i_j(2)$ -th etc until the $i_j(s_j)$ -th are chosen. Because the Kronecker selection is also a nice selection these quantities are also defined for the Kronecker selection. It is clear that the sequence of sets of indices determines the Kronecker selection completely and is in bijective correspondence with the sequence of dynamical indices $\{d_1, d_2, \dots, d_p\}$ that describes the Kronecker selection. It is well-known and can easily be derived from the foregoing that the Kronecker indices are in one-to-one bijective correspondence with the sequence $\{s_u\}_{u=1}^l$ (cf. e.g. [2]).

Remark. A similar definition holds for the Kronecker selection of rows from the observability matrix of a pair of matrices $(A, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{m \times n}$, the corresponding Kronecker observability indices etc.

The following lemma is basic for our considerations (see e.g. [9]):

Lemma 2.5 Let $M \in \mathbb{R}^{n \times l}, \text{rank}(M) = n \leq l$. There exists an orthogonal matrix $Q_0 \in \mathbb{R}^{n \times n}$ and natural numbers $1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq l$ such that

$$M_0 := Q_0 M = \begin{pmatrix} 0 & \dots & m_{1i_1} & * & \dots & * & \dots & \dots & * \\ 0 & \dots & 0 & 0 & \dots & m_{2i_2} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & m_{ni_n} & * \end{pmatrix},$$

with $m_{ji_j} > 0$ for all $j \in 1, 2, \dots, n$, M_0 is unique and Q_0 is unique. Such a matrix will be called positive upper triangular with independency indices i_1, i_2, \dots, i_n .

A matrix will be called full rank upper triangular if it is positive upper triangular up to multiplication of some (or possibly all or none) of its rows by -1 .

3 A balanced canonical form for systems with identical singular values

The following theorem of [1] is basic for the relation between systems with all singular values equal and all-pass systems.

Theorem 3.1 Let $m = p$.

(a) If a balanced stable triple (A, B, C) has identical Hankel singular values $\sigma = 1$ then there exists an orthogonal matrix D such that $C = -DB^T$.

(b) (A, B, C, D) is a balanced realization of a stable all-pass system iff (A, B, C) is a balanced stable triple with identical singular values and D is an orthogonal matrix such that $C = -DB^T$.

It will be useful to extend the usual definition of orthogonal square matrices to rectangular matrices:

Definition 3.2 An $m \times p$ matrix U will be called orthogonal if $U^T U = I_p$ or $U U^T = I_m$.

Using this, a balanced realization (A, B, C) of a stable system with identical singular values (with possibly $m \neq p$) can be characterized as follows:

Corollary 3.3 The following three statements are equivalent:

- (i) A triple (A, B, C) is a balanced realization of a stable system with identical singular values $\sigma > 0$
- (ii) The pair (A, B) is reachable and $A + A^T = -\frac{1}{\sigma} B B^T = -\frac{1}{\sigma} C^T C$, $\sigma > 0$,
- (iii) The pair (A, B) is reachable, $A + A^T = -\frac{1}{\sigma} B B^T$, and there exists a (possibly rectangular) orthogonal matrix D such that $C = -DB^T$

Proof. Without loss of generality one can assume that $m = p$, because if $m \neq p$ than one can add a sufficient number of zero rows to C or zero columns to B to obtain a system with the same number of in- and outputs and clearly if the result holds for the square system obtained in this way, it also holds for the original system. If $m = p$ the theorem follows from the previous theorem together with a theorem of [10], (here applied to the special case where the Lyapunov equation involved has the identity matrix as a solution) which says that if (A, B) satisfies the equation $A + A^T = -B B^T$ then:

A is asymptotically stable iff (A, B) is reachable.

Now consider the following canonical form for the set of stable multivariable all-pass systems of fixed McMillan degree.

Theorem 3.4 The following two statements are equivalent: (i) A system Π a stable all-pass system with McMillan degree n . (ii) There exists a unique balanced realization $(A, B, C, D) \in C_n$ of Π of the following form: There are integers $p = s_0 \geq s_1 \geq s_2 \geq \dots \geq s_l > s_{l+1} = 0$ which add up to $n + p$, such that

$$B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

where B_1 is an $s_1 \times s_0$ positive upper triangular matrix;

$$A = \begin{pmatrix} A_{11} & A_{12} & 0 & \dots & 0 \\ A_{21} & A_{22} & A_{23} & \ddots & \vdots \\ 0 & A_{32} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & A_{l-1,l} \\ 0 & \dots & 0 & A_{l,l-1} & A_{l,l} \end{pmatrix},$$

a block tridiagonal matrix with $A_{u,v}$ an $s_u \times s_v$ matrix, $u, v \in \{1, 2, \dots, l\}$, $A_{u,v} = 0$ if $|u - v| > 1$;

$$A_{11} = \bar{A}_{11} - \frac{1}{2}B_1B_1^T,$$

\bar{A}_{11} an otherwise arbitrary skew symmetric $s_1 \times s_1$ matrix;

$A_{uu} = \bar{A}_{uu}$ an otherwise arbitrary skew symmetric $s_u \times s_u$ matrix for each $u \in \{2, 3, \dots, l\}$;

$A_{u+1,u}$ a positive upper triangular $s_{u+1} \times s_u$ matrix for each $u \in \{1, 2, \dots, l-1\}$;

$A_{u,u+1} = -A_{u+1,u}^T$ for each $u \in \{1, 2, \dots, l-1\}$;

D an otherwise arbitrary orthogonal $p \times p$ matrix and

$$C = -DB^T.$$

The indices $s_u, u = 1, \dots, l$ are in bijective correspondence with the Kronecker indices. The canonical form is balanced and its reachability matrix is positively upper triangular.

Proof. Cf. [4]. □

For triples (A, B, C) with identical singular values one obtains the same balanced canonical form with the only exception that, while in the case of all-pass systems the matrix C can be determined from B and D here instead C is an arbitrary solution of the equation

$$C^T C = B B^T = \begin{pmatrix} B_1 B_1^T & 0 \\ 0 & 0 \end{pmatrix}$$

It follows that the matrix C can be partitioned as $[C_1, 0]$, C_1 an $m \times s_1$ matrix which is a solution of $C_1^T C_1 = B_1 B_1^T$. Let $\bar{B}_1 = [(B_1 B_1^T)^{\frac{1}{2}}, 0]$, where the zero matrix is $s_1 \times (p - s_1)$ and $(B_1 B_1^T)^{\frac{1}{2}}$ is the $s_1 \times s_1$ positive definite symmetric square root of $B_1 B_1^T$. Then clearly $C_1^T C_1 = \bar{B}_1 \bar{B}_1^T$ and $C^T C = \bar{B} \bar{B}^T$ in an obvious notation. From the corollary above it follows that there exists an $m \times p$ orthogonal matrix D such that $C = -D \bar{B}^T$ and so $C_1 = -D \bar{B}_1^T$. Partition $D = [-U, V]$ where U is an $m \times s_1$ orthogonal matrix and V is an $m \times (p - s_1)$ orthogonal matrix. It follows that $C_1 = U(B_1 B_1^T)^{\frac{1}{2}}$. Because $(B_1 B_1^T)^{\frac{1}{2}}$ is positive definite, the relation between C_1 and U is bijective: $U = C_1 (B_1 B_1^T)^{-\frac{1}{2}}$ and therefore U can be used to parametrize C_1 . In this way one obtains a parametrization of the canonical form for systems with one singular value: The matrices B and A together have $s_1 + s_2 + \dots + s_l = n$ positive parameters due to the requirement that $B_1, A_{21}, A_{32}, \dots, A_{l,l-1}$ are all positive upper triangular, while all other entries in these positive upper triangular matrices are either prescribed to be zero by the structural indices, or are free to vary over the reals; furthermore the skew symmetric matrices $\bar{A}_{11}, \bar{A}_{22}, \dots, \bar{A}_{l,l}$ have $\sum_{u=1}^l \frac{1}{2} s_u (s_u - 1)$ parameters that are free to vary over the reals and finally $C_1 = U(B_1 B_1^T)^{\frac{1}{2}}$, is parametrized by the $m \times s_1$ orthogonal matrix U ; the set of all such orthogonal matrices has dimension $(s_1 - 1)(m - \frac{1}{2}s_1)$. Of course one could add a feedthrough matrix to the system, which would have mp freely varying real parameters.

4 A new balanced canonical form for stable multivariable systems

Consider a system with balanced state space representation (A, B, C, D) . The Grammians are equal and of the form

$$\Sigma = \text{diag} \left(\sigma_1 I_{n(1)}, \sigma_2 I_{n(2)}, \dots, \sigma_k I_{n(k)} \right),$$

where $\sigma_1 > \sigma_2 > \dots > \sigma_k > 0$ are the Hankel singular values of the system and $n(1), n(2), \dots, n(k)$ the corresponding multiplicities. Partition A, B, C , according to $n(1), n(2), \dots, n(k)$ to obtain

$$A = \begin{pmatrix} A(1,1) & A(1,2) & \dots & A(1,k) \\ A(2,1) & A(2,2) & \dots & A(2,k) \\ \vdots & \vdots & & \vdots \\ A(k,1) & A(k,2) & \dots & A(k,k) \end{pmatrix} \quad (4.6)$$

and

$$B = \begin{bmatrix} B(1) \\ \vdots \\ B(k) \end{bmatrix}, C = [C(1) \ C(2) \ \dots \ C(k)] \quad (4.7)$$

The following result is very important for the construction of balanced canonical forms:

Theorem 4.1 ([10],[7]) *Let (A, B, C, D) be partitioned as above, balanced in the sense of definition 2.2, but not necessarily stable and minimal. Then $(A, B, C, D) \in C_n$ iff for each $i \in \{1, 2, \dots, k\}$, $(A(i, i), B(i), C(i), D) \in C_{n(i)}$.*

The balancing (Lyapunov) equations for (A, B, C, D) are given in terms of the $A(i, j), B(i), C(j), i = 1, \dots, k, j = 1, \dots, k$ by

$$\begin{aligned} A(i, i) + A(i, i)^T &= -\frac{1}{\sigma_i} B(i) B(i)^T \\ &= -\frac{1}{\sigma_i} C(i)^T C(i) \end{aligned} \quad (4.8)$$

$$A(i, j) \sigma_j + A(j, i)^T \sigma_i = -B(i) B(j)^T \quad (4.9)$$

$$A(i, j) \sigma_i + A(j, i)^T \sigma_j = -C(i)^T C(j), \quad (4.10)$$

where $i \in \{1, \dots, k\}, j \in \{1, \dots, k\}, i \neq j$. Note that the last two equations can be solved in terms of $A(i, j)$ and $A(j, i)$ for given pair $(i, j), i \neq j$ and given $B(i), B(j), C(i), C(j)$, because $\sigma_i \neq \sigma_j$. Therefore, and because of the theorem 4.1, the construction of a balanced canonical form can be reduced to constructing a balanced canonical form for the subsystems $(A(i, i), B(i), C(i)), i = 1, 2, \dots, k$. For each $i \in \{1, \dots, k\}$ such a system is a system with identical singular values, or perhaps one should say with one singular value σ_i . For such systems the canonical form presented in section 3 can be used. This leads to the following result:

Theorem 4.2 *The following statements are equivalent:*

(i) Ξ is a stable i/o-system with p inputs and m outputs and McMillan degree n .

(ii) Ξ has a realization $(A, B, C, D) \in C_n$ of the following form: There are numbers $\sigma_1 > \sigma_2 > \dots > \sigma_k > 0$, the Hankel singular values, and integers $n(1), n(2), \dots, n(k)$, the corresponding multiplicities of the singular values, and for each $i \in \{1, 2, \dots, k\}$ there are integers $p = s_0(i) \geq s_1(i) \geq \dots \geq s_{l(i)}(i) > s_{l(i)+1}(i) = 0$ which add up to $n(i) + p$, such that

$$A = \begin{pmatrix} A(1,1) & A(1,2) & \dots & A(1,k) \\ A(2,1) & A(2,2) & \dots & A(2,k) \\ \vdots & \vdots & & \vdots \\ A(k,1) & A(k,2) & \dots & A(k,k) \end{pmatrix} \quad (4.11)$$

and

$$B = \begin{bmatrix} B(1) \\ \vdots \\ B(k) \end{bmatrix}, C = [C(1) \ C(2) \ \dots \ C(k)], \quad (4.12)$$

where $A(i, j)$ is an $n(i) \times n(j)$ matrix, $B(i)$ an $n(i) \times p$ matrix and $C(j)$ an $m \times n(j)$ matrix, $i = 1, 2, \dots, k, j = 1, 2, \dots, k$ and the triples $(A(i, i), B(i), C(i))$ have the following form:

$$B(i) = \begin{bmatrix} B_1(i) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (4.13)$$

where $B_1(i)$ is an $s_1(i) \times p$ positive upper triangular matrix and

$$A(i, i) = \begin{pmatrix} A_{1,1}(i, i) & A_{1,2}(i, i) & 0 & \dots & 0 \\ A_{2,1}(i, i) & A_{2,2}(i, i) & A_{2,3}(i, i) & \ddots & \vdots \\ 0 & A_{2,3}(i, i) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & A_{l(i)-1, l(i)}(i, i) \\ 0 & \dots & 0 & A_{l(i), l(i)-1}(i, i) & A_{l(i), l(i)}(i, i) \end{pmatrix}, \quad (4.14)$$

a block tridiagonal matrix where for each $u, v \in \{1, 2, \dots, l(i)\}$, $A_{u,v}(i, i)$ is an $s_u(i) \times s_v(i)$ matrix,

$A_{u,v}(i, i) = 0$ if $|u - v| > 1$;

$A_{11}(i, i) = \bar{A}_{11}(i, i) - \frac{1}{2}B_1(i)B_1(i)^T$, where $\bar{A}_{11}(i, i)$ is an otherwise arbitrary skew symmetric $s_1(i) \times s_1(i)$ matrix;

$A_{uu}(i, i) = \bar{A}_{uu}(i, i)$ an otherwise arbitrary skew symmetric $s_u(i) \times s_u(i)$ matrix for each $u \in \{2, 3, \dots, l(i)\}$;

$A_{u+1,u}(i, i)$ a positive upper triangular $s_{u+1}(i) \times s_u(i)$ matrix for each $u \in \{1, 2, \dots, l(i) - 1\}$;

$A_{u,u+1}(i, i) = -A_{u+1,u}(i, i)^T$ for each $u \in \{1, 2, \dots, l(i) - 1\}$, and furthermore

$$C(i) = [C_1(i), 0, \dots, 0] \quad (4.15)$$

where

$$C_1(i) = U(i) (B(i)B(i)^T)^{\frac{1}{2}}, \quad (4.16)$$

in which $U(i)$ is an $m \times s_1(i)$ orthogonal matrix, i.e. $U(i)^T U(i) = I_{s_1(i)}$;

furthermore the matrices $A(i, j), i \neq j; i, j \in \{1, \dots, k\}$ are determined as the solution of the equations 4.9, 4.10.

Proof. From the introduction to this theorem it is clear that each stable multivariable linear system has a unique representation of this form. It remains to be shown that each system of this form is indeed minimal and stable. This follows from theorem 4.1 together with the fact that the canonical form that is used for the systems with identical singular values has the same property: for each choice of the parameters that is allowed the resulting system is minimal and stable (cf. [4]).

□

Remarks

- (i) If $s_1 = p$ then one can just as well parametrize $C_1(i)$ by $C_1(i) = U(i)B(i)$; $U(i)$ a (possibly rectangular) orthogonal matrix.
- (ii) Truncation of the last $n - k$ components of the state vector corresponds to the truncation mapping

$$(A, B, C, D) \mapsto \left((I_k, 0)A(I_k, 0)^T, (I_k, 0)B, C(I_k, 0)^T, D \right) \quad (4.17)$$

The canonical form presented has the property that if truncation is applied, the result is again in canonical form. Therefore the resulting lower order system is again minimal and stable.

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