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Spectral characterization of the optional quadratic varation process
(revised version)

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Abstract

In this paper we show how the periodogram of a semimartingale can be used to characterize the optional quadratic variation process.

Keywords: semimartingale, quadratic variation, periodogram

1 introduction and notation

As is well known in the statistical analysis of time series in discrete or continuous time, the periodogram can be used for estimation problems in the frequency domain. It follows from the results of the present paper that the periodogram can also be used to estimate the variance of the innovations of a time series in continuous time. Usually in statistical problems this variance is assumed to be known, since it can be estimated with probability one, given the observations on any nonempty interval in a number of cases. (See for instance Dzhaparidze & Yaglom [5], theorem 2.1).
A fundamental result in another approach is now known as Levy's theorem, which states that the variance of a Brownian Motion can be obtained as the limit of the sum of squares of the increments by taking finer and finer partitions. This result has been generalized by Baxter [1], who showed a similar result for more arbitrary Gaussian processes (that need not to be semimartingales) and to the case where the process under consideration is a semimartingale by Doleans-Dade [3], who obtained a characterization of the quadratic variation. See also theorem VIII.20 of Dellacherie & Meyer [4] or theorem 4 on page 55 of Liptser & Shiryaev [10]. Related work on so called convergence of order p has been conducted by Lepingle [9].
In the present paper we take a different viewpoint towards the quadratic variation process (more in the spirit of theorem 2.1 of [5]) and it is our purpose to show that the periodogram of a semimartingale can be used as a statistic to estimate its quadratic variation process. We thus obtain an alternative characterization of this process as compared to, for instance, Doleans-Dade's.
The rest of this secton is devoted to the introduction of some notation.

Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be complete filtered probability space and X real valued semi martingale defined on it. X_0 is assumed to be zero. Let $(\Lambda, \mathcal{L}, Q)$ be an additional probability space. Consider the product $\Omega \times \Lambda$ and endow it with the product σ-algebra $\mathcal{F} \otimes \mathcal{L}$ and the product measure $P \otimes Q$. Identify $\mathcal{F}$ with $\mathcal{F} \otimes \{\emptyset, \Lambda\}$ as a σ-algebra on $\Omega \otimes \Lambda$.
Define for each finite stopping time T and each real number λ the *periodogram* of X evaluated at T by

$$I_T(X;\lambda) = |\int_{[0,T]} e^{i\lambda t} dX_t|^2$$

An application of Ito's formula gives

$$I_T(X;\lambda) = 2Re\int_{[0,T]}\int_{[0,t)} e^{i\lambda(t-s)} dX_s dX_t + [X]_T \tag{1.1}$$

Let $\xi : \Lambda \to \mathbb{R}$ be a real random variable with an absolutely continuous distribution (w.r.t. Lebesgue measure), that has a density G, which is assumed to be symmetric around zero and consider for any positive real number L the quantity

$$E[I_T(X;L\xi)|\mathcal{F}] = E_\xi I_T(X;L\xi) := \int_{\mathbb{R}} I_T(X;Lx)G(x)dx$$

It follows from Protter [11], pages 159, 160, that interchanging the integration order in (1.1) is allowed to obtain

$$E_\xi I_T(X; L\xi) = 2\int_{[0,T]}\int_{[0,t)} g(L(t-s))dX_s dX_t + [X]_T \tag{1.2}$$

where g is the (real) characteristic function of ξ.
Our purpose is to study the behaviour of $E_\xi I_T(X; L\xi)$ for $L \to \infty$. To that end we investigate this quantity for a number of distinguished cases in the next sections.

REMARK: As is well known, g is a continuous function and for all $s < t$, it holds that $g(L(t-s)) \to 0$ for $L \to \infty$, in view of the Riemann-Lebesgue lemma (cf. FELLER II [7]), and (of course) $|g(x)| \leq 1$ for all real x.

2 semimartingales of bounded variation

Throughout this section we assume that X is a process of bounded variation over each finite interval. Denote by $||X||_t$ the variation of the process X over the interval $[0, t]$ (t may be replaced by a finite stopping time T). In this case we obtain from (1.2)

$$|E_\xi I_T(X; L\xi) - [X]_T| \leq 2\int_{[0,T]}\int_{[0,t)} |g(L(t-s))| d||X||_s d||X||_t \tag{2.1}$$

Since $\lim_{L\to\infty} g(Lx) = \delta(x)$ (with $\delta(x) = 1$, if $x = 0$ and $\delta(x) = 0$ if $x \neq 0$), an application of Lebesgue's dominated convergence theorem yields that the right hand side of (2.1) converges (almost surely) to

$$2\int_{[0,T]}\int_{[0,t)} \delta(t-s) d||X||_s d||X||_t$$

But this is equal to zero, since $\delta(t-s) = 0$ for all $s < t$, whence the following result:

PROPOSITION 2.1 *Let X be a semimartingale of bounded variation, T a finite stopping time and ξ a real random variable, independent of $\mathcal{F}$, which has a density on the real line. Then almost surely for $L \to \infty$*

$$E_\xi I_T(X; L\xi) \to [X]_T.$$

REMARK: Notice the similarity of the above statement with formula 1.5 on page 620 of FELLER II [7], if we take the case where ξ has a uniform distribution on $[-1, +1]$, and where X is a piecewise constant process.

3 arbitrary semimartingales

In this section we assume that X is an arbitrary semimartingale. Starting point for our analysis is again equation (1.2). Consider now the process Y^L defined by

$$Y^L_{\cdot} = \int_{[0,.]} \int_{[0,t)} g(L(t-s))dX_s dX_t \tag{3.1}$$

The first thing we will do in the next subsection is to give an upper bound for the absolute value of the inner integral in (3.1).

3.1 a technical result

We state in this section a technical lemma and a corollary. Thereto we have to introduce some notation. First we need the moduli of right continuity of X:

$$W'_X[t_1, t_2) = \sup\{|X_u - X_v| : u, v \in [t_1, t_2)\},$$

$$W'_{X,\tau}(\varepsilon) = \inf\{\max_{0\leq i\leq n-1} W'_X[t_i, t_{i+1}) : 0 = t_0 < \ldots < t_n = \tau, t_{i+1} - t_i > \varepsilon\}.$$

See Billingsley [2], page 110. (τ may be a stopping time).
Furthermore we have $X^*_t = \sup\{|X_s| : s \leq t\}$. Next for any function (or process) Z, we denote by $V(Z; I)$ the total variation of Z over the interval I. (Notice that $V(X; I) = \infty$ for continuous local martingales X and for any interval I, except when $\int_I d\langle X\rangle$ is zero).

LEMMA 3.1 *Let X be a cadlag process, g a real characteristic function of an absolutely continuous distribution. Then for all $\varepsilon > 0$ and $t > 0$, the following estimate is valid almost surely*

$$|\int_{[0,t)} g(L(t-s))dX_s|$$
$$\leq W'_X[t-\varepsilon, t)(1 + V(g; [0, L\varepsilon])) + X^*_t[|g(L\varepsilon)| + V(g; [L\varepsilon, Lt])] \tag{3.2}$$

as well as the coarser estimate

$$|\int_{[0,t)} g(L(t-s))dX_s| \leq 2W'_X[t-\varepsilon, t)(V(g; [0, \infty)) + 2X^*_t V(g; [L\varepsilon, \infty)) \tag{3.3}$$

PROOF: To avoid trivialities, we can assume that both $V(g; [0, L\varepsilon])$ and $V(g; [L\varepsilon, Lt])$ are finite. Consider first $\int_{(t-\varepsilon,t)} g(L(t-s))dX_s$. (If $\varepsilon > t$, then we interpret the integral by extending the definition of X to the negative real line and setting $X_t = 0$ for $t < 0$). Integration by parts together with the fact that g is continuous yields that this integral is equal to

$$X_{t-} - g(L\varepsilon)X_{t-\varepsilon} - \int_{(t-\varepsilon,t)} (X_s - X_{t-\varepsilon})dg(L(t-s)) - X_{t-\varepsilon}\int_{(t-\varepsilon,t)} dg(L(t-s))$$

$$= X_{t-} - X_{t-\varepsilon} - \int_{(t-\varepsilon,t)} (X_s - X_{t-\varepsilon}) dg(L(t-s))$$

Hence

$$|\int_{(t-\varepsilon,t)} g(L(t-s))dX_s|$$

$$\leq |X_{t-} - X_{t-\varepsilon}| + \sup_{t-\varepsilon<s<t} |X_s - X_{t-\varepsilon}| \int_{(t-\varepsilon,t)} d||g||(L(t-s))$$

$$\leq W'_X[t-\varepsilon,t)(1 + V(g;[0,L\varepsilon])).$$

Consider now the integral over $[0, t-\varepsilon]$. Using again integration by parts, we obtain

$$|\int_{[0,t-\varepsilon]} g(L(t-s))dX_s| \leq |g(L\varepsilon)X_{t-\varepsilon}| + X^*_{t-\varepsilon} \int_{[0,t-\varepsilon]} d||g||(L(t-s))$$

$$\leq X^*_{t-\varepsilon}(|g(L\varepsilon)| + V(g;[L\varepsilon, Lt])).$$

Putting the above two estimates together, we obtain the first statement of the lemma. The second one is a simple consequence, since $V(g;[0,\infty)) \geq 1, V(g;[0,\infty)) \geq V(g;[0,L\varepsilon])$ and $V(g;[L\varepsilon,\infty)) \geq |g(L\varepsilon)|$. □

COROLLARY 3.2 *Let X be a cadlag process and let the function g be of bounded variation over $[0,\infty)$, and T a finite stopping time. Then*

$$\sup_{t\leq T} |\int_{[0,t)} g(L(t-s))dX_s| \to 0, \quad a.s.$$

for $L \to \infty$. In particular the process $\int_{[0,.)} g(L(.-s))dX_s$ is locally bounded on $[0,T]$.

PROOF: First we prove the following statement.

$$\sup\{W'_X[t-\varepsilon,t) : t \in [0,\tau]\} \leq 2W'_{X,\tau}(\varepsilon). \tag{3.4}$$

Fix $\eta > 0$ and choose a partition $0 = t_0 < \ldots < t_n = \tau$, such that

$$\max_{0\leq i\leq n-1} W'_X[t_i, t_{i+1}) \leq W'_{X,\tau} + \eta.$$

Pick the i such that $t_i \leq t < t_{i+1}$.
Then by taking suprema over three different possibilities we get

$$W'_X[t-\varepsilon,t) =$$

$$\max\{\sup_{u,v\in[t-\varepsilon,t_i)} |X_u - X_v|, \sup_{u,v\in[t_i,t)} |X_u - X_v|, \sup_{u\in[t-\varepsilon,t_i),v\in[t_i,t)} |X_u - X_v|\},$$

which is by the triangle inequality at most

$$\max\{\sup_{u,v\in[t-\varepsilon,t_i)}|X_u-X_v|,\ \sup_{u,v\in[t_i,t)}|X_u-X_v|,$$

$$\sup_{u\in[t-\varepsilon,t_i)}|X_u-X_{t_i}|+\sup_{v\in[t_i,t)}|X_v-X_{t_i}|\}.$$

By taking suprema over larger intervals, this is less than or equal to

$$\max\{\sup_{u,v\in[t_{i-1},t_i)}|X_u-X_v|,\ \sup_{u,v\in[t_i,t_{i+1})}|X_u-X_v|,$$

$$\sup_{u,v\in[t_{i-1},t_i)}|X_u-X_v|\}+\sup_{u,v\in[t_i,t_{i+1})}|X_u-X_v|\}$$

By taking now maxima over i, this is bounded above by

$$2\max_{0\leq i\leq n-1}W'_X[t_i,t_{i+1}).$$

This by the choice of the partition less than or equal to

$$2(W'_{X,T}(\varepsilon)+\eta).$$

Since η is arbitrary, inequality (3.4) has been proved.
Now we have proved this statement, the remaining part of the proof follows easily from lemma 3.1 by taking $\varepsilon=L^{-\frac{1}{2}}$, since both $|g(L^{\frac{1}{2}})|$ and $V(g;[L^{\frac{1}{2}},\infty))\to 0$, for $L\to\infty$, as well as $W_{X,T}(L^{-\frac{1}{2}})\to 0$ *a.s.*, in view of equation (14.8) in Billingsley [2]. The fact that the process $\int_{[0,.)}g(L(.-s))dX_s$ is locally bounded, is now an immediate consequence. □

3.2 convergence of the periodogram

In this subsection we prove the analog of proposition 2.1 for the case of an arbitrary semimartingale X. The main result of this section is the following:

THEOREM 3.3 *Let X be a semimartingale and let the function g be of bounded variation over $[0,\infty)$. Let T be a finite stopping time. Then*

$$E_\xi I_T(X;L\xi)\to[X]_T$$

in probability, for $L\to\infty$.

PROOF: There exist a decomposition of X as $X=Z+M$, where M is a local martingale satisfying $\sup|\Delta M_t|\leq 1$ and Z is a process of bounded variation. This follows from the decomposition theorem for local martingales. In particular M is locally square integrable (cf. Dellacherie & Meyer [4], VI.85). Use this decomposition to write Y_T^L from equation (3.1) as the sum of two terms. These two are

$$Y_T^L(X,Z)=\int_{[0,T]}\int_{[0,t)}g(L(t-s))dX_s dZ_t \tag{3.5}$$

$$Y_T^L(X,M)=\int_{[0,T]}\int_{[0,t)}g(L(t-s))dX_s dM_t \tag{3.6}$$

Since the process Z is of bounded variation, it follows that $|Y_T^L(X,Z)|$ is bounded above by $\sup_{t\leq T}|\int_{[0,t)}g(L(t-s))dX_s|V(Z;[0,T])$. The first factor tends to zero *a.s.* in view of corollary 3.2.

We proceed with the second term $Y_T^L(X,M)$. Notice that for fixed L this a locally square integrable martingale with predictable variation at T given by

$$\langle Y^L(X,M)\rangle_T = \int_{[0,T]}(\int_{[0,t)} g(L(t-s))dX_s)^2 d\langle M\rangle_t.$$

This is bounded above by

$$\sup_{t\leq T}|\int_{[0,t)} g(L(t-s))dX_s|\langle M\rangle_T.$$

Hence a simple application of Lenglart's inequality (cf. Jacod & Shiryaev [8], page 35) together with an application of corollary 3.2 yields that $Y_T^L(X,M)$ tends to zero in probability as $L\to\infty$. This completes the proof. □

Some examples of distributions for which the conditions on g in theorem 3.3 are satisfied are the triangular distribution, the double exponential distribution, the Cauchy distribution, the normal distribution (See table 1 of FELLER II [7] on page 503), or the distribution which has the Epanechnikov kernel as its density (This kernel enjoys some optimality properties in problems of kernel density estimation. See e.g. page 21 of [6]). The characteristic function of the uniform distribution on [-1,+1] is not of bounded variation over $[0,\infty)$.

REMARK: It is instructive to see that for deterministic times T in the situation where moreover X is a square integrable martingale with deterministic predictable variation, the proof of the above theorem is much simpler and that we don't need that g is of bounded variation (as well as in propostion 2.1). Indeed consider again Y^L with its quadratic variation given by

$$\langle Y^L\rangle_T = \int_{[0,T]}(\int_{[0,t)} g(L(t-s))dX_s)^2 d\langle X\rangle_t.$$

Taking expectations yields

$$E\langle Y^L\rangle_T = \int_{[0,T]}\int_{[0,t)} g(L(t-s))^2 d\langle X\rangle_s d\langle X\rangle_t.$$

Using again the dominated convergence theorem, we see that $E\langle Y^L\rangle_T$ tends to zero for $L\to\infty$. So $Y_T^L\to 0$ in probability, in view of Chebychev's inequality.

4 some consequences

As a simple consequence of theorem 3.3 we obtain a representation result for the optional quadratic covaration of two semimartingales.
Let X and Y be arbitrary real valued semimartingales, T a finite stopping time and g be of bounded variation. Define the *cross periodogram* of X and Y for each real number λ by

$$I_T(X,Y;\lambda) = \int_{[0,T]} e^{i\lambda t} dX_t \int_{[0,T]} e^{-i\lambda t} dY_t.$$

Let ξ be a real random variable as before. Then we have

COROLLARY 4.1 *Under the conditions of theorem 3.3 we have*

$$E_\xi I_T(X,Y;L\xi) \to [X,Y]_T$$

in probability.

PROOF: It is easy to verify that the following form of the polarization formula holds:

$$I_T(X,Y;\lambda) + I_T(Y,X;\lambda) = \frac{1}{2}[I_T(X+Y;\lambda) - I_T(X-Y;\lambda)].$$

Then an application of theorem 3.3 together with the known polarization formula for the square bracket process and the observation that $E_\xi I_T(X,Y;L\xi)$ is real yields the result. □

REMARK: One can define the periodogram for a multivariate semimartingale X with values in $\mathbb{R}^n$ as

$$I_T(X;\lambda) = \int_{[0,T]} e^{i\lambda t} dX_t (\int_{[0,T]} e^{i\lambda t} dX_t)^*.$$

Then the parallel statement of theorem 3.3 holds in view of corollary 4.1 with $[X]$ the $n \times n$-matrix valued optional quadratic variation process.

We end this section with a consequence of theorem 3.3 in terms of Dirac delta approximations. Return to real valued semimartingales X and use partial integration to rewrite the periodogram as

$$I_T(X;\lambda) = |e^{i\lambda T} - i\lambda \int_{[0,T]} e^{i\lambda t} X_t dt|^2$$

$$= X_T^2 + X_T \int_{[0,T]} i\lambda (e^{i\lambda(T-t)} - e^{-i\lambda(T-t)}) X_t dt + \lambda^2 |\int_{[0,T]} e^{i\lambda t} X_t dt|^2$$

Let ξ be as in the introduction, assume that $E\xi^2 < \infty$. Then g is twice continuously differentiable, so we obtain from the above equation

$$E_\xi I_T(X;L\xi) =$$

$$X_T^2 - 2X_T \int_{[0,T]} X_t \frac{\partial}{\partial t} g(L(T-t))dt + \int_{[0,T]} \int_{[0,T]} X_t X_s \frac{\partial^2}{\partial t \partial s} g(L(t-s))dtds \quad (4.1)$$

The idea is that both the two kernels in equation (4.1) behave as a Dirac distribution (although not quite). More precisely we have

PROPOSITION 4.2 *Let X be a real semimartingale, T a finite stopping time and g a twice continuously differentiable real characteristic function, which is assumed to be of bounded variation over $[0,\infty)$.*
The following statements hold almost surely, respectively in probability

$$(i) \int_{[0,T]} X_t \frac{\partial}{\partial t} g(L(T-t))dt \to X_{T-}$$

$$(ii) \int_{[0,T]} \int_{[0,T]} X_t X_s \frac{\partial^2}{\partial t \partial s} g(L(t-s))dtds \to X_{T-}^2 + [X]_{T-}$$

PROOF: (i) follows by partial integration and an application of corollary 3.3.
(ii) is then a consequence of (i) and theorem 3.3. □

REMARK: The second statement of this propostion is at first glance perhaps somewhat surprising, since one would expect for continuous X the term X_T^2 only. The extra term $[X]_T$ is due to the fact, that X is in general not of bounded variation.

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