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Overlapping block-balanced canonical forms and parametrizations: the stable SISO case

Bernard Hanzon
Raimund J. Ober

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Bernard Hanzon *
Dept. Econometrics, Free University Amsterdam†
Raimund J. Ober ‡
Center for Engineering Mathematics, University of Texas at Dallas §

Abstract

The balanced canonical form and parametrization of Ober for the case of SISO stable systems are extended to block- balanced canonical forms and related input-normal forms and parametrizations. They form an overlapping atlas of parametrizations of the manifold of stable SISO systems of given order. This extends the usefullness of these parametrizations, e.g. in gradient algorithms for system identification.

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†Address: De Boelelaan 1105, 1081 HV Amsterdam, Holland; E-mail: bhnz@sara.nl; Fax +31-20-6461449

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§Address: Programs in Mathematical Sciences, UTD, Richardson, Texas 75083-06688, USA

1 Introduction

In [16],[17] a canonical state space form was presented for the set of asymptotically stable linear systems, with the property that it is balanced, i.e. for each system represented in canonical form the corresponding observability and controllability Grammians are equal and diagonal (and positive definite). One motivation for studying balanced realizations and balanced canonical forms is their close relation to model reduction (see [17] and the references given there), which is in turn closely related to robust control theory (see e.g. [18], [2]). Another motivation mentioned in [17] is the potential usefullness of balanced realizations for system identification, as indicated by [14]. In many cases, in system identification as well as in related areas, one can reduce the problem at hand to an optimization problem in which some criterion function is optimized over a set of systems. Very often one cannot solve the optimization problem analytically and one has to use search algorithms (e.g. gradient algorithms), in which an initial point in the set of systems is adapted iteratively to give a hopefully good approximation of the optimal system. In such search algorithms one often uses a parametrization of the set of relevant systems. The balanced parametrization of [17] has the advantage that by construction, problems of identifiability are to a large extend avoided in such a search algorithm. The parametrization has the property that it contains structural indices (i.e. discrete–valued parameters), and with each possible choice of values for these indices corresponds a particular submanifold of systems, for which a parametrization in terms of real-valued parameters is given. To each system corresponds a unique set of structural indices. As the structural indices can take a large number of values, even for rather low-order systems (the number of possibilities increases fast with increasing order of the system), this means that in a search algorithm one has to either identify the structural indices by other means or one has to apply the search algorithm to a large number of parametrized submanifolds of systems. This is due to the fact that the parametrizations are disjoint.

Several authors (e.g. [3,1,9,10,19,4,7]) have investigated the possibility of using socalled overlapping parametrizations (in differential geometric terms: an atlas of coordinate charts). If one uses overlapping parametrizations, one does not have to search through each and every of the submanifolds, but instead one can search through the manifold as a whole, using the parametrizations to describe the manifold locally and changing from one parametrization to another when required. In case the search algorithm is of the gradient type, one can make sure that the decision rule for changing from one parametrization to another has little effect on the search algorithm by using a Riemannian gradient, with respect to some suitable Riemannian metric on the manifold (cf. [6,5,7,21,8]).

In view of this it would be very desirable if the balanced parametrization of [17] could be extended to give a set of overlapping parametrizations. In this paper such an extension will be presented for the case of SISO stable systems. In the extension balancedness of the realization no longer holds for all realizations. Instead (what we will call) block-balanced realizations are used and the corresponding input-normal realizations. With a block-balanced canonical form we mean a canonical form for which the observability and controllability Grammian are equal and block-diagonal (and of course positive definite).

In section 2 some basic definitions are presented including the new concept of block-balanced realizations. In section 3 we present a Schwarz-like canonical form which will be a building block in the block-balanced canonical forms and the corresponding input-normal canonical forms that are treated in section 4. In section 5 it is shown how this leads to a set of overlapping block-balanced canonical forms and a corresponding atlas for the manifold of stable SISO input-output systems of a fixed order and remarks are made as to how this atlas can be used if one wants to work with balanced and "almost balanced" realizations in search algorithms in e.g. system identification.

2 Canonical forms, balanced realizations and block-balanced realizations

In this section, as well as in the whole paper, to a large extent the set-up of the paper [17] is followed. Let us consider continuous time SISO systems of the form

$$\dot{x}_t = Ax_t + bu_t, \quad (1)$$
$$y_t = cx_t \quad (2)$$

with $t \in \mathbf{R}, u_t \in \mathbf{R}, x_t \in \mathbf{R}^n, y_t \in \mathbf{R}, A \in \mathbf{R}^{n\times n}, b \in \mathbf{R}^{n\times 1}, c \in \mathbf{R}^{n\times 1}, (A,b,c)$ a minimal triple.

Let for each $n \in \{1,2,3,\cdots\}$ the set C_n be given by $C_n = \{(A,b,c) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times 1} \times \mathbf{R}^{1\times n} | (A,b,c)$ minimal and the spectrum of A is contained in the open left half plane$\}$.

As is well-known two minimal system representations (A_1, b_1, c_1) and (A_2, b_2, c_2) have the same transfer function $g(s) = c_1(sI - A_1)^{-1}b_1 =$

$c_2(sI - A_2)^{-1}b_2$, and therefore describe the same input-output behaviour, iff there exists an $n \times n$ matrix $T \in Gl_n(\mathbf{R})$ such that $A_1 = TA_2T^{-1}, b_1 = Tb_2, c_1 = c_2T^{-1}$. In that case we say that (A_1, b_1, c_1) and (A_2, b_2, c_2) are i/o-equivalent. This is clearly an equivalence relation; write $(A_1, b_1, c_1) \sim (A_2, b_2, c_2)$. A unique representation of a linear system can be obtained by deriving a canonical form:

Definition 2.1 *A canonical form for an equivalence relation '* $\sim$*' on a set* X *is a map*

$$\Gamma : X \to X$$

which satisfies for all $x, y \in X$ *:*

(i) $\Gamma(x) \sim x$

(ii) $x \sim y \iff \Gamma(x) = \Gamma(y)$

Equivalently a canonical form can be given by the image set $\Gamma(X)$*; a subset* $B \subseteq X$ *describes a canonical form if for each* $x \in X$ *there is precisely one element* $b \in B$ *such that* $b \sim x$*. The mapping* $X \to B, x \mapsto b$ *then describes a canonical form.*

Let $(A, b, c) \in C_n$. The controllability Grammian W_c is the positive definite matrix that is given by the integral

$$W_c = \int_0^\infty \exp(At)bb^T \exp(A^Tt)\mathrm{d}t$$

As is well-known W_c can be obtained as the unique solution of the following Lyapunov equation:

$$AW_c + W_cA^T = -bb^T \tag{3}$$

In a dual fashion, the observability Grammian W_o is the positive definite matrix that is given by the integral

$$W_o = \int_0^\infty \exp(A^Tt)c^Tc\exp(At)\mathrm{d}t$$

This matrix is the unique solution of the following Lyapunov equation

$$A^TW_o + W_oA = -c^Tc \tag{4}$$

Definition 2.2 *Let* $(A,b,c) \in C_n$, *then* (A,b,c) *is called* balanced *if the corresponding observability and controllability Grammians are equal and diagonal, i.e. there exist positive numbers* $\sigma_1, \sigma_2, \ldots, \sigma_n$ *such that*

$$W_o = W_c = diag(\sigma_1, \ldots, \sigma_n) =: \Sigma \qquad (5)$$

The numbers $\sigma_1, \ldots, \sigma_n$ *are called the (Hankel) singular values of the system.*

The singular values are known to be uniquely determined by the input-output behaviour of the system.

Theorem 2.3 (Moore 1981) *Let* $(A,b,c) \in C_n$ *with*

$$\Sigma = diag(\sigma_1 I_{n(1)}, \ldots, \sigma_k I_{n(k)}), \sigma_1 > \sigma_2 > \ldots \sigma_k > 0 \text{ and } \sum_{j=1}^{k} n(j) = n.$$

Then (A,b,c) *is unique up to an orthogonal state-space transformation of the form*

$$Q = diag(Q_1, Q_2, \ldots, Q_k)$$

with orthogonal $Q_i \in \mathbf{R}^{n(i)\times n(i)}, i = 1, \ldots, k$.

Definition 2.4 *Let* $(A,b,c) \in C_n$, *then* (A,b,c) *is called* input-normal *if* $W_c = I_n$ *and will be called* σ-input-normal *if* $W_c = \sigma I_n$.

Similarly (A,b,c) *is called* output-normal *if* $W_o = I_n$ *and* σ-output-normal *if* $W_o = \sigma I_n$.

It is not difficult to show that an input-normal realization is unique up to an arbitrary orthogonal state-space transformation.

The following definition is new and basic to our considerations in this paper.

Definition 2.5 *Let* $(A,b,c) \in C_n$, *then* (A,b,c) *will be called* block- balanced, *with indices* $n(i) \in \mathbf{N}, i = 1, \ldots, k$,*adding up to* n, *if the observability Grammian and the controllability Grammian are equal and block-diagonal, i.e. there exist* $n(i) \times n(i)$ *positive definite matrices* $\Sigma_i, i = 1, \ldots, k$, *such that*

$$W_o = W_c = diag(\Sigma_1, \ldots, \Sigma_k)$$

It will be convenient to call an arbitrary system representation $(A,b,c) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times 1} \times \mathbf{R}^{1\times n}$ *block-balanced if the pair of Lyapunov equations* $A\Sigma + \Sigma A^T = -bb^T, A^T\Sigma + \Sigma A = -c^T c$ *has a positive definite solution of the form* $\Sigma = diag(\Sigma_1, \ldots, \Sigma_k)$ *(assuming neither asymptotic stability nor minimality).*

Remark. The matrices $\Sigma_i, i = 1, \ldots, k$ are in general *not* uniquely determined by the input-output behaviour of the system. However the eigenvalues $\lambda_1(\Sigma_i) \geq \lambda_2(\Sigma_i) \geq \ldots \geq \lambda_{n(i)}(\Sigma_i)$ of the matrices $\Sigma_i, i = 1, \ldots, k$ together form the set of Hankel singular values of the system, which are uniquely determined by the input- output behaviour of the system, as remarked before.

Theorem 2.6 *Suppose $(A, b, c) \in C_n$ is block-balanced with indices $n(j) \in \mathbf{N}, j = 1, \ldots, k, \sum_{j=1}^{k} n(j) = n$ and with the additional property $\lambda_1(\Sigma_1) \geq \lambda_{n(1)}(\Sigma_1) > \lambda_1(\Sigma_2) \geq \lambda_{n(2)}(\Sigma_2) > \ldots > \lambda_1(\Sigma_k) \geq \lambda_{n(k)}(\Sigma_k) > 0$.*

This uniquely determines (A, b, c) up to an orthogonal state-space transformation of the form

$$Q = diag(Q_1, \ldots, Q_k)$$

with orthogonal $Q_i \in \mathbf{R}^{n(i) \times n(i)}, i = 1, \ldots, k$

Proof. Firstly note that if an *orthogonal* state-space transformation Q is applied to the system representation, then both Grammians transform in the same way and therefore if they were equal before the orthogonal state-space transformation, then they will also be equal after the transformation.

Now consider two systems $(A_1, b_1, c_1), (A_2, b_2, c_2)$ which are both block-balanced with the same indices $n(j), j = 1, \ldots, k$, and with Grammians $W_o = W_c = \text{diag}(\Sigma_1^{(i)}, \ldots, \Sigma_k^{(i)}), i = 1, 2$ with the property that $\lambda_1(\Sigma_1^{(i)}) \geq \lambda_{n(1)}(\Sigma_1^{(i)}) > \lambda_2(\Sigma_2^{(i)}) \geq \lambda_{n(2)}(\Sigma_2^{(i)}) > \ldots > \lambda_1(\Sigma_k^{(i)}) \geq \lambda_{n(k)}(\Sigma_k^{(i)}) > 0, i = 1, 2$.

Because $\Sigma_j^{(i)}$ is symmetric positive definite for any $i = 1, 2, j = 1, \ldots, k$, there exists an orthogonal matrix $Q_j^{(i)}$ such that $Q_j^{(i)} \Sigma_j^{(i)} (Q_j^{(i)})^T$ is a diagonal matrix. Therefore the state-space transformation $Q^{(i)} := \text{diag}(Q_1^{(i)}, \ldots, Q_k^{(i)})$ applied to the system representation (A_i, b_i, c_i) brings it into *balanced form*, $i = 1, 2$. We can therefore apply Theorem 2.3 to the transformed system representations and it follows that there exists an orthogonal state-space transformation that transforms (A_1, b_1, c_1) into (A_2, b_2, c_2) (and vice versa).

□

The following theorem will be fundamental for our results.

Theorem 2.7 (Pernebo and Silverman, [22], Kabamba, [11]) *Let $(A, b, c) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times 1} \times \mathbf{R}^{1 \times n}$ be conformally partitioned as follows:*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, c = \begin{pmatrix} c_1 & c_2 \end{pmatrix},$$

with $A_{ii} \in \mathbf{R}^{n(i)\times n(i)}, i = 1,2$ *and let* (A,b,c) *be block-balanced with indices* $n(1), n(2)$ *such that* $\Sigma_1, \Sigma_2 > 0$ *have no eigenvalues in common.*

Then $(A,b,c) \in C_n \Leftrightarrow (A_{ii}, b_i, c_i) \in C_{n(i)}, i = 1,2.$

3 The case k=1: a Schwarz-like canonical form for stable SISO systems in continuous time

Theorem 3.1 *Consider the set* B_n *of all* $(A,b,c) \in C_n$ *of the following form:*

$$\begin{aligned}
A &= \begin{pmatrix} a_{11} & \alpha_1 & & 0 \\ -\alpha_1 & 0 & \ddots & \\ & \ddots & \ddots & \alpha_{n-1} \\ 0 & & -\alpha_{n-1} & 0 \end{pmatrix}, a_{11} = -\frac{b_1^2}{2} < 0, \\
\alpha_i &> 0, i = 1, \ldots, n-1, \\
b &= \begin{pmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, b_1 > 0, \\
c &= \begin{pmatrix} c_1 & \gamma_1 & \cdots & \gamma_{n-1} \end{pmatrix}, c_1 \in \mathbf{R}, \gamma_j \in \mathbf{R}, j = 1, \ldots, n-1.
\end{aligned}$$

Each triple $(A,b,c) \in B_n$ *is input-normal.*

Let S_n *be the set of values of the vector of parameters* $(b_1, \alpha_1, \ldots, \alpha_{n-1}, c_1, \gamma_1, \ldots, \gamma_{n-1})$ *such that the corresponding triple* $(A,b,c) \in B_n$, *i.e. such that* $b_1 > 0, \alpha_i > 0, i = 1, \ldots, n$ *and* $c_1, \gamma_1, \ldots, \gamma_{n-1}$ *such that the pair* (c, A) *is observable.*

The set B_n *describes a continuous canonical form and the parametrization mapping* $S_n \longrightarrow B_n$, *which maps each parameter vector to the corresponding triple* (A,b,c), *is a homeomorphism.*

If $(\gamma_1, \ldots, \gamma_{n-1}) \neq 0 \in \mathbf{R}^{n-1}, n \geq 2$, *then the system has several different singular values.*

Proof The requirement that a realization is input-normal reduces the freedom of choosing a basis of the state space to the freedom of choosing an *orthonormal* basis, i.e. to the freedom of choosing an element from the orthogonal group.

Now consider the controllability matrix of a triple $(A, b, c) \in B_n$. It is easily seen to be positive upper triangular. According to [17] there is a unique element in the orthogonal group that transforms a controllability matrix to a positive upper triangular matrix. Therefore the form presented here is canonical indeed.

Next let us show the continuity properties. The mapping $S_n \longrightarrow B_n$ which maps a parameter vector from S_n to its corresponding triple (A, b, c) is clearly continuous.

Now consider the mapping $C_n \longrightarrow S_n$ which maps *any* triple $(\tilde{A}, \tilde{b}, \tilde{c}) \in C_n$ to the corresponding parameter vector describing the *canonical form* of the system. Clearly the coefficients of the characteristic polynomial of $\tilde{A}$ depend continuously on $\tilde{A}$, and therefore the parameters $a_{11}, \alpha_1, \ldots, \alpha_{n-1}$ depend continuously on $\tilde{A}$, as they are rational functions of these characteristic polynomial coefficients (cf. [16]).

It remains to show that the parameter vector $c = (c_1, \gamma_1, \ldots, \gamma_{n-1})$ depends continuously on the entries of $(\tilde{A}, \tilde{b}, \tilde{c})$. Let (A, b, c) denote the canonical form of the system and let $g(z) := \frac{p(z)}{q(z)} := c(zI - A)^{-1}b = \tilde{c}(zI - \tilde{A})^{-1}\tilde{b}$ denote the (rational) transfer function of the system, with polynomial denominator $q(z) := \det(zI - A) = \det(zI - \tilde{A})$ and polynomial numerator $p(z)$. It is easy to see that the coefficients of $p(z)$ depend continuously on the entries of $(\tilde{A}, \tilde{b}, \tilde{c})$. Let $M(z)$ denote the polynomial matrix of cofactors of $(zI - A)$. Then one has

$$p(z) = cM(z)^T b \tag{6}$$

Consider $m_{i1}(z)$, which is $(-)^{1+i}$ times the determinant of the matrix that is obtained from $zI - A^T$ by leaving out the first row and i-th column, $i \in \{1, \ldots, n\}$:

$m_{1i}(z) = (-1)^{1+i}\times$

$$\left|\begin{array}{cccc|cccccc} -\alpha_1 & * & \dots & * & * & \dots & \dots & \dots & \dots & * \\ 0 & \ddots & & \vdots & \vdots & & & & & \vdots \\ \vdots & & \ddots & * & \vdots & & & & & \vdots \\ 0 & \dots & 0 & -\alpha_{i-1} & * & \dots & \dots & \dots & \dots & * \\ \hline 0 & \dots & \dots & 0 & z & \alpha_{i+1} & 0 & \dots & \dots & 0 \\ \vdots & & & \vdots & -\alpha_{i+1} & z & \alpha_{i+2} & 0 & \dots & 0 \\ \vdots & & & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \ddots & \ddots & \alpha_{n-1} \\ 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 & -\alpha_{n-1} & z \end{array}\right| =$$

$$= (\prod_{j=1}^{i-1} \alpha_j) z^{n-i} + \text{terms of lower degree in } z,$$

where $i \in \{1, \dots, n\}$; if $i = 1$, the product $\prod_{j=1}^{i-1} \alpha_j$ is taken to be equal to one by convention. Because $\prod_{j=1}^{i-1} \alpha_j$ is unequal to zero (and in fact positive) for each $i \in \{1, \dots, n\}$ the polynomials $m_{11}(z), \dots, m_{1n}(z)$ form a basis of the linear vector space of polynomials of degree $< n$ over $\mathbf{R}$. Therefore the equation (6), which can be rewritten as

$$c_1 m_{11}(z) + \gamma_1 m_{21}(z) + \dots + \gamma_{n-1} m_{n1}(z) = \frac{p(z)}{b_1} \tag{7}$$

has a unique solution $c = (c_1, \gamma_1, \gamma_2, \dots, \gamma_{n-1})$, which depends continuously on the entries of $(\tilde{A}, \tilde{b}, \tilde{c})$ and the parameters $b_1, \alpha_1, \dots, \alpha_{n-1}$. As these parameters themselves depend continuously on the entries of $(\tilde{A}, \tilde{b}, \tilde{c})$, the continuity of all parameters on the entries of $(\tilde{A}, \tilde{b}, \tilde{c})$ follows. This completes the proof of the continuity properties.

The remaining statements follow from the fact that for $\gamma = 0$, the form is a canonical form for systems with only *one* positive Hankel singular value (i.e. all nonzero Hankel singular values coincide), cf. [17],[16]

□

Remarks

(i) The fact that *if* the asymptotically stable matrix A can be brought into the presented form by a basis change of the state space, then the resulting matrix is unique, also follows from the fact mentioned in the proof, that for $\gamma = 0, c_1 \neq 0$ the form is a canonical form for systems with only one positive Hankel singular value, cf. [17], [16].

(ii) If $c_1 \neq 0$ we define $\sigma := \left|\frac{c_1}{b_1}\right| > 0$, which we will call a pseudo-singular value. If the vector $\gamma = (\gamma_1, \ldots, \gamma_{n-1})$ is close enough to zero the pseudo-singular value will be close to the true singular values of the system, because of continuity of the singular values as a function of γ and the fact that if $\gamma = 0$, the system has only one singular value and its value is σ. If $c_1 \neq 0$ the system can be brought simply into $\sigma-$input-normal form by multiplying c by $\sigma^{-\frac{1}{2}}$ and b by $\sigma^{\frac{1}{2}}$. The resulting $\sigma-$input -normal form is a *canonical* form locally around $\gamma = 0$, but not globally because the systems which have $c_1 = 0$ in the previous canonical form cannot be represented in this way (it would lead to $\sigma = 0$ and therefore one cannot transform back to the input-normal case etc.).

Locally around $\gamma = 0$ it takes the following form:

$$\begin{aligned}
A &= \begin{pmatrix} a_{11} & \alpha_1 & & 0 \\ -\alpha_1 & 0 & \ddots & \\ & \ddots & \ddots & \alpha_{n-1} \\ 0 & & -\alpha_{n-1} & 0 \end{pmatrix}, \\
a_{11} &= -\frac{b_1^2}{2\sigma} < 0, \\
\alpha_i &> 0, i = 1, \ldots, n-1, \\
b &= \begin{pmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, b_1 > 0 \\
c &= \begin{pmatrix} sb_1 & \gamma_1 & \ldots & \gamma_{n-1} \end{pmatrix}, s \in \{-1, 1\}, \gamma_j \in \mathbf{R}, j = 1, \ldots, n-1
\end{aligned}$$

(iii) Because the canonical form is input-normal, if one starts with an arbitrary input-normal realization $(\tilde{A}, \tilde{b}, \tilde{c})$ of the system it takes an orthogonal state-space transformation Q in order to obtain the canonical form of the system involved. The same holds for the (local) $\sigma-$input-normal canonical form.

(iv) Clearly the canonical forms presented are controllable (because they are input-normal, resp. $\sigma-$input-normal), but observability will fail for certain choices of c; the observability Grammian will be singular for such a choice of c. If $\gamma = 0, c_1 \neq 0$, the system is observable, because the observability Grammian will be $\sigma^2 I$, resp. σI. (In that case the system representation is σ^2-output -normal, resp. $\sigma-$output-normal) Therefore also in some open neighbourhood around such a system, observability will still hold (this follows from the continuity of the determinant of the observability Grammian as a function of the parameters).

(v) This canonical form is closely related to the so-called Schwarz canonical form, cf. [12], [13], [23].

4 An input-normal and a block-balanced canonical form

Let $n(1), \ldots, n(k) \in \{1, 2, \ldots, n\}, \sum_{j=1}^{k} n(j) = n$, denote a partition of n as before. Let $C_{n(1),n(2),\ldots,n(k)}$ denote the subset of all systems in C_n with the property that their n Hankel singular values (multiplicities *included*) $\sigma(1) \geq \sigma(2) \geq \ldots \geq \sigma(n) > 0$ can be partitioned into k disjoint sets of singular values (again with multiplicities included) in the following way:

$$\begin{aligned}
\sigma(1) &\geq \ldots \geq \sigma(n(1)) > \sigma(n(1)+1) \geq \\
&\geq \ldots \geq \sigma(n(1)+n(2)) > \sigma(n(1)+n(2)+1) \geq \\
&\geq \ldots \geq \sigma(\sum_{j=1}^{l} n(j)) > \sigma((\sum_{j=1}^{l} n(j)) + 1) \geq \\
&\geq \ldots > 0
\end{aligned} \tag{8}$$

So we require that $\sigma(\sum_{j=1}^{l} n(j)) > \sigma((\sum_{j=1}^{l} n(j)) + 1)$ for $l = 1, 2, \ldots, k-1$ and $\sigma(n) > 0$ of course. Note that the notation is consistent with the fact that C_n denotes the set of stable systems which have as their only "restriction" that there are n positive singular values (multiplicities included), i.e. that the order of the system is n.

The other extreme is $C_{1,1,\ldots,1}$, which denotes the set of $n-$th order stable systems with n *distinct* singular values. For this set of systems a balanced canonical form was derived in [11].

Remark. The set $C_{n(1),\ldots,n(k)}$ should not be confused with the subset of C_n consisting of the systems which have k distinct singular values $\sigma_1 >$

$\ldots > \sigma_k > 0$ with multiplicities $n(1), \ldots, n(k)$. Of course these systems are included in $C_{n(1),\ldots,n(k)}$, but they generally form only a (thin) subset.

Next we will present a canonical form on $C_{n(1),\ldots,n(k)}$.

Theorem 4.1 *Consider the set $B_{n(1),\ldots,n(k)}$ of triples (A, b, c) of the following form:*

$$\begin{aligned}
A &= (A(i,j))_{1\leq i,j\leq k}, \\
A(i,j) &\in \mathbf{R}^{n(i)\times n(j)}, i,j \in \{1,\ldots,k\} \\
b &= \begin{pmatrix} b(1) \\ b(2) \\ \vdots \\ b(k) \end{pmatrix}, b(i) \in \mathbf{R}^{n(i)}, i = 1,\ldots,k, \\
c &= (c(1),\ldots,c(k)), c(j)^T \in \mathbf{R}^{n(j)}, j = 1,\ldots,k, \\
A(i,i) &= \begin{pmatrix} a(i,i)_{11} & \alpha(i)_1 & 0 & \ldots & 0 \\ -\alpha(i)_1 & 0 & \alpha(i)_2 & \ddots & \vdots \\ 0 & -\alpha(i)_2 & & \ddots & 0 \\ \vdots & \ddots & \ddots & & \alpha(i)_{n(i)-1} \\ 0 & \ldots & 0 & -\alpha(i)_{n(i)-1} & 0 \end{pmatrix}, \\
a(i,i)_{11} &= -\frac{b_i^2}{2}, \\
\alpha(i)_j &> 0, j = 1,\ldots,n(i)-1, \\
b(i) &= \begin{pmatrix} b_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, b_i > 0, \\
c(i) &= \left(c_i, \gamma(i)_1, \ldots, \gamma(i)_{n(i)-1}\right), i = 1,\ldots,k,
\end{aligned}$$

where the parameters are to be taken such that the corresponding observability Grammians $\Sigma_i^2, i = 1,\ldots,k$, which satisfy the observability Lyapunov equations

$$\Sigma_i^2 A(i,i) + A(i,i)^T \Sigma_i^2 = -c(i)^T c(i) \tag{9}$$

are fulfilling the following matrix inequalities

$$\Sigma_1^2 > \Sigma_2^2 > \ldots > \Sigma_k^2 > 0; \tag{10}$$

for each pair $(i,j), i \neq j$, the matrices $A(i,j), A(j,i)$ are determined (uniquely!) from the following pair of linear matrix equations:

$$\begin{aligned} A(i,j) + A(j,i)^T &= -b(i)b(j)^T \\ \Sigma_i^2 A(i,j) + A(j,i)^T \Sigma_j^2 &= -c(i)^T c(j) \end{aligned} \tag{11}$$

The set $B_{n(1),\ldots,n(k)}$ describes a continuous canonical form on $C_{n(1),\ldots,n(k)}$. The $2n$ "free" parameters of the canonical form are

$$b_i, \alpha(i)_1, \ldots, \alpha(i)_{n(i)-1}, c_i, \gamma(i)_1, \ldots, \gamma(i)_{n(i)-1}, i = 1, \ldots, k.$$

Let $S_{n(1),\ldots,n(k)} \subset \mathbf{R}^{2n}$ be the set of all values of the parameter vector for which the corresponding triple $(A,b,c) \in B_{n(1),\ldots,n(k)}$, i.e. for all $i \in \{1,\ldots,k\} : b_i > 0, \alpha(i)_j > 0, j = 1,\ldots,n(i)-1$, and $c_i, \gamma(i)_1, \ldots, \gamma(i)_{n(i)-1}$ such that the matrices $\Sigma_i, i = 1,\ldots,k$, found in (9) satisfy the inequalities (10). The mapping $S_{n(1),\ldots,n(k)} \longrightarrow B_{n(1),\ldots,n(k)}$ which maps a parameter vector to the corresponding triple (A,b,c) is a homeomorphism.

The form is input-normal, i.e.

$$A + A^T = -bb^T \tag{12}$$

and has block-diagonal observability Grammian $\Sigma^2 := diag(\Sigma_1^2, \ldots, \Sigma_k^2) > 0$.

Let $\sigma(1) \geq \sigma(2) \geq \ldots \geq \sigma(n) > 0$ denote the n positive Hankel singular values of the system (with their multiplicities). If for some $i \in \{1,\ldots,k\}$ the vector $\gamma(i) = 0$, then Σ_i^2 is a scalar matrix

$$\Sigma_i^2 = \sigma^2 \left(1 + \sum_{j=1}^{i-1} n(j)\right) .I_{n(i)}, \tag{13}$$

and

$$\begin{aligned} &\sigma\left(\sum_{j=1}^{i-1} n(j)\right) > \\ &\sigma\left(1 + \sum_{j=1}^{i-1} n(j)\right) = \sigma\left(2 + \sum_{j=1}^{i-1} n(j)\right) = \ldots = \sigma\left(\sum_{j=1}^{i} n(j)\right) > \\ &\sigma\left(1 + \sum_{j=1}^{i} n(j)\right) \end{aligned}$$

If for all $i \in \{1,\ldots,k\}, \gamma(i) = 0$, then the observability Grammian is consequently diagonal.

Remark. A block-balanced realization can be obtained from the presented canonical form by applying a state-space transformation

$$T := \Sigma^{\frac{1}{2}} = \operatorname{diag}\left(\Sigma_1^{\frac{1}{2}}, \ldots, \Sigma_k^{\frac{1}{2}}\right) > 0 \tag{14}$$

The corresponding controllability and observability Grammians will both be equal to

$$\Sigma = \operatorname{diag}(\Sigma_1, \ldots, \Sigma_k) > 0$$

Proof.

(i) To start we will show that the form presented is canonical on $C_{n(1),\ldots,n(k)}$. Consider a system which can be represented by a triple in $C_{n(1),\ldots,n(k)}$. A balanced realization of the system is also in block-balanced form with partitioning indices $n(1), \ldots, n(k)$. So one can find a block-balanced realization (A, b, c) of the system, with these partitioning indices. It follows from theorem(2.6) that the requirement that (A, b, c) is block-balanced with these partitioning indices uniquely determines $(A, b.c)$ up to an orthogonal state-space transformation of the form $Q = \operatorname{diag}(Q_1, Q_2, \ldots, Q_k)$, with orthogonal matrices $Q_i \in \mathbb{R}^{n(i)\times n(i)}$. If (A, b, c) is in block-balanced form it can be brought into input-normal form with block-diagonal observability Grammian by the state-space transformation T^{-1}, where T is as defined in (14). It follows easily that if (A, b, c) is in input-normal form with block-diagonal controllability Grammian $\Sigma^2 = \operatorname{diag}(\Sigma_1^2, \ldots, \Sigma_k^2)$, with $\lambda_1(\Sigma_1^2) \geq \lambda_{n(1)}(\Sigma_1^2) > \lambda_1(\Sigma_2^2) \geq \lambda_{n(2)}(\Sigma_2^2) > \ldots > \lambda_1(\Sigma_k^2) \geq \lambda_{n(k)}(\Sigma_k^2) > 0$, $\Sigma_i^2 \in \mathbb{R}^{n(i)\times n(i)}$, then (A, b, c) is uniquely determined up to an orthogonal state-space transformation of the form $Q = \operatorname{diag}(Q_1, Q_2, \ldots, Q_k))$. If such a transformation is applied then $(A(i,i), b(i), c(i))$ is transformed to $\left(Q_i A(i,i) Q_i^T, Q_i b(i), c(i) Q_i^T\right)$. Note that $(A(i,i), b(i), c(i)) \in C_{n(i)}$ because of theorem 2.7 and therefore it follows from theorem (3.1) that there is a unique choice for Q_i which brings $\left(Q_i A(i,i) Q_i^T, Q_i b(i), c(i) Q_i^T\right)$ into the required canonical form.

It only remains to be checked that by using the solutions $A(i,j), A(j,i)$ of the equations (11) the Grammians have indeed the required block structure, which is straightforward and left to the reader.

(ii) Secondly we will show the continuity properties. Clearly the mapping $S_{n(1),\ldots,n(k)} \longrightarrow B_{n(1),\ldots,n(k)}$ which maps any parameter vector in

$S_{n(1),\ldots,n(k)}$ to the corresponding triple $(A,b,c) \in B_{n(1),\ldots,n(k)}$, is continuous.

Now consider the mapping $C_{n(1),\ldots,n(k)} \longrightarrow S_{n(1),\ldots,n(k)}$ which maps a triple $(\tilde{A},\tilde{b},\tilde{c})$ to the parametervector of the corresponding canonical form. Clearly the mapping

$$(\tilde{A},\tilde{b},\tilde{c}) \longmapsto (\sigma(1),\ldots,\sigma(n)) \in \mathbf{R}^n_+$$

which maps $(\tilde{A},\tilde{b},\tilde{c})$ to its $n-$vector of singular values (multiplicities included) is continuous (for the continuity of the zeroes of a polynomial in dependence of the coefficients of that polynomial, see [15]).

Now consider the polynomial

$$a(z) = \prod_{j=n(1)+1}^{n} (z - \sigma(j)^2)$$

Let $\Sigma^2 = W_c^{\frac{1}{2}} W_o W_c^{\frac{1}{2}}$, where W_c, W_o are the controllability and observability Grammian resp. of $(\tilde{A},\tilde{b},\tilde{c})$; W_c, W_o depend continuously on $(\tilde{A},\tilde{b},\tilde{c})$. The matrix $a(\Sigma^2)$ has as its range space an $n(1)-$dimensional linear subspace of $\mathbf{R}^n$ which clearly depends continuously on $(\tilde{A},\tilde{b},\tilde{c})$. The corresponding orthogonal projection matrix Π_1, which maps an arbitrary vector $x \in \mathbf{R}^n$ to its orthogonal projection in the linear subspace spanned by the columns of $a(\Sigma^2)$ (i.e. the linear subspace which is obtained by taking the direct sum of the eigenspaces of the largest $n(1)$ eigenvalues $\sigma(1)^2,\ldots,\sigma(n(1))^2$ of Σ^2), depends continuously on $a(\Sigma^2)$.

Now consider $\left(\Pi_1 W_c^{-\frac{1}{2}} \tilde{A} W_c^{\frac{1}{2}} \Pi_1, \Pi_1 W_c^{-\frac{1}{2}} \tilde{b}, \tilde{c} W_c^{\frac{1}{2}} \Pi_1\right)$ with corresponding controllability Grammian Π_1 and observability Grammian $\Pi_1 \Sigma^2 \Pi_1 = \Pi_1 \Sigma^2 = \Sigma^2 \Pi_1$ (because of the way Π_1 is constructed, it commutes with Σ^2) We can now apply the canonical form of theorem (3.1) to find a basis for the range space of Π_1, (which corresponds to the state-space here) depending continuously on $(\tilde{A},\tilde{b},\tilde{c})$; the first basis vector is $\frac{\Pi_1 W_c^{-\frac{1}{2}} \tilde{b}}{\|\Pi_1 W_c^{-\frac{1}{2}} \tilde{b}\|}$; the second one (Gram-Schmidt orthonormalization) is obtained by normalization of the following vector:

$$\Pi_1 W_c^{-\frac{1}{2}} \tilde{A} W_c^{\frac{1}{2}} \Pi_1 \quad W_c^{-\frac{1}{2}} \tilde{b} +$$

$$- \frac{\left(\tilde{b}^T W_c^{-\frac{1}{2}} \Pi_1 W_c^{\frac{1}{2}} \tilde{A}^T W_c^{-\frac{1}{2}} \Pi_1 W_c^{-\frac{1}{2}} \tilde{b}\right)}{\left(\tilde{b}^T W_c^{-\frac{1}{2}} \Pi_1 W_c^{-\frac{1}{2}} \tilde{b}\right)} \times \Pi_1 W_c^{-\frac{1}{2}} \tilde{b};$$

etc. Clearly this choice of basis of the range space of Π_1 is continuous. Now multiply the first basis vector with 1, the second one with -1, the third one with $+1$ etc. With respect to the resulting basis of the $n(1)$−dimensional state space the triple

$$\left(\Pi_1 W_c^{-\frac{1}{2}} \tilde{A} W_c^{\frac{1}{2}} \Pi_1, \Pi_1 W_c^{-\frac{1}{2}} \tilde{b}, \tilde{c} W_c^{\frac{1}{2}} \Pi_1\right)$$

takes the form $\left(\tilde{A}(1,1), \tilde{b}(1), \tilde{c}(1)\right)$ as described in theorem (3.1):

$$\begin{aligned}
\tilde{A}(1,1) &= \begin{pmatrix} a(1,1)_{11} & \alpha(1)_1 & 0 & \dots & 0 \\ -\alpha(1)_1 & 0 & \alpha(1)_2 & \ddots & \vdots \\ 0 & -\alpha(1)_2 & & \ddots & 0 \\ \vdots & \ddots & \ddots & & \alpha(1)_{n(1)-1} \\ 0 & \dots & 0 & -\alpha(1)_{n(1)-1} & 0 \end{pmatrix}, \\
a(1,1)_{11} &= -\frac{b_1^2}{2}, \\
\alpha(1)_j &> 0, j = 1, \dots, n(1)-1, \\
\tilde{b}(1) &= \begin{pmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, b_1 > 0, \\
\tilde{c}(1) &= \left(c_1, \gamma(1)_1, \dots, \gamma(1)_{n(1)-1}\right),
\end{aligned}$$

and therefore this triple and the parameters describing it , depend continuously on $(\tilde{A}, \tilde{b}, \tilde{c})$. Similarly for any $i \in \{1, \dots, k\}$ the matrix triple and the parameters describing it depend continuously on $(\tilde{A}, \tilde{b}, \tilde{c})$. This proves the continuity of the mapping which maps $(\tilde{A}, \tilde{b}, \tilde{c})$ to the parameters of the canonical form.

(iii) The remaining statements follow from the results in [17].

□

5 An atlas of overlapping block-balanced canonical forms

Theorem 5.1 *Let the state space dimension n be fixed. The continuous canonical forms $C_{n(1),\dots,n(k)} \longrightarrow B_{n(1),\dots,n(k)}$, $n(j) \in \{1,\dots,n\}; j = 1,\dots,k$; $\sum_{j=1}^{k} n(j) = n$; $k \in \{1,\dots,n\}$, form an overlapping set of continuous canonical forms covering C_n. Each of the sets $C_{n(1),\dots,n(k)}$, $\sum_{j=1}^{k} n(j) = n$, is an open subset of C_n and together they cover C_n.*

Proof. Let $P(n;k) := \{(n(1),\dots,n(k)) | n(j) \in \{1,\dots,n\}; j = 1,\dots,k; \sum_{j=1}^{k} n(j) = n\}$ the set of partitions of n into k parts. It is trivial to show that

$$\bigcup_{k=1}^{n} \bigcup_{(n(1),\dots,n(k))\in P(n;k)} C_{n(1),\dots,n(k)} = C_n, \qquad (15)$$

because $C_{n(1),\dots,n(k)} \subset C_n$ for each partition $(n(1),\dots,n(k))$ of n and for $k = 1$ one has $n(1) = n$ and $C_{n(1)} = C_n$. Clearly for each partition $(n(1),\dots,n(k))$ of n the set $C_{n(1),\dots,n(k)}$ is an open subset of C_n.

□

Corollary 5.2 *The set of mappings*

$$C_{n(1),\dots,n(k)}/\sim \longrightarrow S_{n(1),\dots,n(k)} \subset \mathbf{R}^{2n},$$

$$(n(1),\dots,n(k)) \in P(n;k), k = 1,\dots,n$$

which map each equivalence class of triples to the corresponding parameter vector in the canonical form, forms an atlas for the manifold of stable SISO i/o-systems of order n.

Proof. Any i/o-system has a minimal state space realization which is unique up to choice of basis of the state space. Therefore the equivalence classes of (minimal!) triples in C_n can be identified with stable SISO i/o-systems and the result follows from the theorem.

□

Remark. A motivation for using this atlas rather than e.g. just the Schwarz-like canonical form B_n is the following. Suppose one wants to use *balanced realizations.* Then one can use the balanced canonical form of [17]. However this form is discontinuous at all points of $C_{n(1),\dots,n(k)} \setminus C_{1,\dots,1}$, i.e. in all triples $(\tilde{A}, \tilde{b}, \tilde{c})$ which have two or more coinciding singular values. And the complement $C_{1,\dots,1}$, of the set of discontinuity points consists of 2^n

topological components, one component for each sign pattern; this should be compared to C_n which has only $n+1$ topological components (the Brockett components). It appears that this is a serious disadvantage if one wants to use balanced realizations and canonical forms in e.g. search algorithms for system identification.

In order to overcome these difficulties one could use the overlapping block-balanced canonical forms as follows. If $(\tilde{A}, \tilde{b}, \tilde{c})$ has k distinct Hankel singular values $\sigma_1 > \sigma_2 > \ldots > \sigma_k > 0$ with multiplicities resp. $n(1), \ldots, n(k)$, then one can use the block-balanced continuous canonical form on $C_{n(1),\ldots,n(k)}$ *locally around* $(\tilde{A}, \tilde{b}, \tilde{c})$. If one is moving away from $(\tilde{A}, \tilde{b}, \tilde{c})$ in a search algorithm for example, one has to decide whether the canonical form corresponding to a different partition should be used: if the largest $n(1)$ singular values differ sufficiently from each other one could use e.g. $C_{1,\ldots,1,n(2),\ldots,n(k)}$ (where there are $n(1)$ ones in the subindex before $n(2)$) etc. In this way one would use balanced realizations and "almost- balanced" realizations while moving around in the set of n−th order systems, without encountering discontinuity points.

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