

Serie Research Memoranda

Computational Probability: Old Ideas Never Die

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COMPUTATIONAL PROBABILITY: OLD IDEAS NEVER DIE

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This paper puts on the stage old ideas that are still very useful in computational probability problems arising in telecommunication and teletraffic applications. Basic computational problems to be discussed are the calculation of equilibrium and transient solutions for Markovian systems and the calculation of overflow probabilities in finite-buffer systems.

1. Introduction

Basic problems in computational probability are:

1. Solving the equilibrium equations of an *infinite* Markov chain.
2. Approximating overflow probabilities in finite-buffer queues.
3. Calculating transient solutions for Markovian systems.

Computational probability is more than getting numerical answers. The essence of computational probability is to have probabilistic ideas which make the computations transparent and natural. This paper puts on the stage old probabilistic ideas that are not very known to the teletraffic community but are often very useful for computational purposes.

2. Equilibrium distributions for an infinite Markov chain

A frequently arising problem is the computation of the (unique) equilibrium distribution $\{\pi_j\}$ of an *infinite* Markov chain with state space $I=\{0,1,\dots\}$:

$$\pi_j = \sum_{i=0}^{\infty} \pi_i p_{ij}, \quad j=0,1,\dots$$

$$\sum_{j=0}^{\infty} \pi_j = 1,$$

where the p_{ij} 's denote the one-step transition probabilities of the Markov chain. The generally used approach is to choose a sufficiently large integer M so that (hopefully) $\sum_{j>M} \pi_j \leq \epsilon$ with ϵ a very small number and next to solve the truncated system of linear equations:

$$\pi_j = \sum_{i=0}^M \pi_i p_{ij}, \quad j=0,1,\dots,M-1,$$

$$\sum_{j=0}^M \pi_j = 1.$$

The drawbacks of this approach are:

- a. M may be very large.
- b. Convergence problems may arise for the iterative methods that have to be used when M is large.

Also, it is somewhat disconcerting that we need a brute-force approximation to solve numerically the infinite-state model. Usually we introduce infinite-state models to obtain mathematical simplification, and now in its numerical analysis using a brute-force truncation we are proceeding in the reverse direction.

However, in most practical situations the infinite Markov chain satisfies regularity conditions that enables us to apply an old idea going at least back to Everett [2]:

Use that the state probabilities π_j drop geometrically fast to zero as j increases.

That is, for some constant $0 < \tau < 1$,

$$\frac{\pi_j}{\pi_{j-1}} \approx \tau \quad \text{for } j \text{ large enough.} \quad (1)$$

Conditions under which this asymptotic expansion holds are discussed in Tijms and Van de Coevering [5]. Roughly, it is required that $\Pi(z) = \sum_{j=0}^{\infty} \pi_j z^j$ can be represented as $A(z)/B(z)$ with $B(z)$ analytic outside the unit circle. Then $1/\tau$ is the smallest zero of $B(z)$ outside the unit circle. The asymptotic result (1) implies that, for a sufficiently large integer N ,

$$\pi_j \approx \pi_N \tau^{j-N} \quad \text{for all } j \geq N.$$

Substituting this into the equilibrium equations of the infinite Markov chain leads to the following finite system of linear equations:

$$\pi_j = \sum_{i=0}^{N-1} \pi_i p_{ij} + \pi_N \bar{p}_{Nj}, \quad j=0,1,\dots,N-1$$

$$\sum_{j=0}^{N-1} \pi_j + \frac{\pi_N}{1-\tau} = 1,$$

where $\bar{p}_{Nj} = \sum_{i=N}^{\infty} \tau^{i-N} p_{ij}$. This reduction of the infinite system of equilibrium equations to a finite system of linear equations is not based on brute force, but makes essential use of properties of the infinite-state model. From a practical point of view the question is of course: Is the size of this finite system smaller than that of the finite system obtained by brute-force truncation? The answer is in the affirmative. It is an empirical finding that for practical purposes the asymptotic expansion (1) can be used already for remarkably small values of j . How large N should be chosen has to be determined experimentally and depends of course on the specific application and the required accuracy in the state probabilities. It is our experience that an appropriate value for N is typically in the order of 1-200. Consequently, the resulting system of linear equations can routinely be solved by Gaussian elimination methods for which reliable and fast codes are widely available.

It is hoped that the approach using the geometric tail behavior of the state probabilities will be used more in practice. It is a simple numerical approach that can be easily used by the nonspecialist using standard software. We successfully applied the approach to many basic queueing models including the M/D/c queue and the D/G/1 queue.

3. Overflow probabilities in finite-buffer queues

A practical problem of considerable interest is the calculation of (very small) overflow probabilities in finite-buffer queues. Let us first consider this problem for a queueing system in which customers arrive singly. A common heuristic for obtaining the overflow probability in the finite-buffer queue with a maximum number of K customers allowed is:

$$\Pi_{\text{overflow}} \approx \sum_{j=K}^{\infty} \pi_j^{(\infty)},$$

where $\pi_j^{(\infty)}$ is the equilibrium probability that in the corresponding in-

finite-buffer queue a customer finds upon arrival j other customers present. Here it is assumed that the offered traffic intensity is less than 1. That is, the overflow probability is approximated by the steady-state probability that in the infinite-buffer queue a customer finds upon arrival K or more other customers present. This approximation may perform rather unsatisfactorily. A much improved approximation was derived in Tijms [6] by using the following approximation idea:

Assume that the first $K-1$ state probabilities (of the interior states) in the finite-buffer queue are proportional to the first $K-1$ state probabilities in the corresponding infinite-buffer queue.

This approximation idea results from an old question of J.W. Cohen (private communication) whether there are other queueing systems than the $M/G/1/K$ queue and the $M/M/c/K$ queue for which the proportionality result for the state probabilities holds. The improved heuristic for the overflow probability in the finite-buffer queue is given by:

$$\Pi_{\text{overflow}} \approx \frac{(1-\rho) \sum_{j=K}^{\infty} \pi_j^{(\infty)}}{1 - \rho \sum_{j=K}^{\infty} \pi_j^{(\infty)}}$$

where ρ is the offered traffic intensity. Numerical investigations in Tijms [6] indicate that the new heuristic performs very well for practical purposes. The heuristic provides an estimate that is typically of the same order of magnitude as the exact value of the overflow probability. This is what is needed for a heuristic used for dimensioning the buffer size.

Also, an extension of the heuristic to finite-buffer queues with batch arrivals was suggested in Tijms [6]. Denoting by $\{\beta_k\}$ the batch-size distribution, the extended heuristic is:

$$\Pi_{\text{overflow}} = \frac{(1-\rho) \sum_{j=K}^{\infty} q_j^{(\infty)}}{1 - \rho \sum_{j=K}^{\infty} q_j^{(\infty)}}$$

with

$$q_j^{(\infty)} = \sum_{k=0}^j \pi_k^{(\infty)} \sum_{s>j-k} \beta_s$$

where $\pi_j^{(\infty)}$ now denotes the steady-state probability that in the corresponding infinite-buffer queue an arriving batch sees j other customers.

4. Transient solutions for Markovian systems

How to compute transient solutions for Markovian systems is a very important question that arises in numerous problems in teletraffic, telecommunication and reliability. Supposing a continuous-time Markov chain $(X(t), t \geq 0)$ with discrete state space I , basic problems are:

a. Compute the transient probabilities

$$p_{ij}(t) = P\{X(t)=j | X(0)=i\}.$$

b. Compute first-passage time probabilities

$$f_{iA}(t) = P \{ \text{system reaches the set A of states before time } t | X(0)=i \}.$$

c. Compute cumulative expected rewards

$$E \left[\int_0^t r(X(u)) du | X(0)=i \right]$$

when a reward at rate $r(j)$ is incurred whenever the process is in state j .

In fact, the problems b and c can be reduced to problem a (e.g. for problem b make the state of the set A absorbing).

The usual approach to answer above questions is to set up a system of linear differential equations and to solve them using Runge-Kutta methods. However, there is a better approach called Jensen's method. This method is also known under the names randomization method and uniformization method. The name Jensen's method was recently proposed in Grassmann [3]. This method that was already introduced in 1953 by Jensen [4] has recently become very popular in operations research and computer science and has seen many interesting applications. Remarkably, the use of Jensen's method is not widespread in the field of teletraffic theory, though it is a powerful and beautiful method. What is the idea behind the method? Suppose for the moment that the leaving rates ν_i of the states of the continuous-time Markov chain are identical, say $\nu_i = \nu$ for all i . Then, using that the number of transitions in a time interval of length t is Poisson distributed with mean νt , the transient probabilities are easy to compute from:

$$p_{ij}(t) = \sum_{n=0}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} p_{ij}^{(n)},$$

where $p_{ij}^{(n)}$ is the probability that in n state transitions the process goes from state i to state j . The n -step transition probabilities can be recursively computed. However, in general the leaving rates ν_i are not identical. To overcome this difficulty, the old idea of Jensen [4] is:

Make the leaving rates identical by introducing *fictitious* transitions leaving the state of the system unchanged.

That is, take a uniform leaving rate

$$\nu = \max_i \nu_i$$

and modify the one-step transition probabilities p_{ij} (with the convention $p_{ii}=0$) accordingly

$$\bar{p}_{ij} = \begin{cases} \frac{\nu_i}{\nu} p_{ij}, & j \neq i \\ 1 - \frac{\nu_i}{\nu}, & j = i. \end{cases}$$

The method of Jensen has as advantages:

- insightful and easy to program
- numerically stable (roundoff errors can be easily controlled)
- infinite or very large state spaces can often be handled by a dynamic implementation of the method.

The latter point needs some explanation. Notice first that the infinite series for $p_{ij}(t)$ can be truncated on beforehand for a given value of t by choosing M so that

$$\sum_{n=M}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} p_{ij}^{(n)} \leq \sum_{n=M}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} \leq \epsilon$$

for some precision ϵ . Next this bound M may be used in dynamically adjusting the set of states that can be reached in $n(<M)$ transitions from the given initial state i .

It is our experience that Jensen's method tends to outperform Runge-Kutta methods, see also Grassmann [3]. The method is not only superior from a computational point of view, but its probabilistic transparency also allows to attack problems that are otherwise be difficult to handle. For example, the paper of De Sousa e Silva and Gail [1] uses Jensen's method to give a very nice algorithm to compute the probability distribution of the sojourn time in a given set of states of a continuous-time Markov chain.

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