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## On the Approximation of the Durbin-Watson Statistic in O(n) Operations

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# ON THE APPROXIMATION OF THE DURBIN-WATSON STATISTIC IN O(n) OPERATIONS

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## SUMMARY

In this paper we propose an approximation for the computation of tail probabilities of the Durbin-Watson and related statistics. For the Durbin-Watson statistic the procedure has been (apart from a minor detail) suggested by DURBIN AND WATSON (1951, 1971), but here we use formulae for the moments of the statistic that require only $\mathcal{O}(n)$ arithmetic operations and we show that the results are usable for more general statistics. The approximations are accurate enough as to replace exact tail probabilities.

## 1. INTRODUCTION AND THEORY

The Durbin-Watson test (see DURBIN and WATSON (1950, 1951, 1971)) is often used to test against serial correlation in regression models, but a drawback of the test is the inconclusive region in the tables. If one wants to test against 4-th order serial correlation one may use the test proposed by KING (1984), but in that case one has a similar inconclusive region. A further problem with the tables is that reasonable upper and lower bounds depend on the inclusion of a constant or the inclusion of seasonal dummies.

In all these cases the exact tail probabilities can be computed using the procedure in IMHOF (1961). To introduce notation we review the necessary steps.

For $n$ observations the DW statistic $d$ to test against first order serial correlationcan be written as

$$d = \frac{\epsilon' M A M \epsilon}{\epsilon' M \epsilon} \tag{1}$$

where $A = A_n$ and $A_n$ is the $n$ by $n$ tridiagonal matrix

$$A_n = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}.$$

If one tests against $k$-th order serial correlation one can write $A = A_{[(n+k-1)/k]} \otimes I_k$ and delete some rows and columns if $n$ is not a multiple of $k$ ([] denoting rounding downwards and $I_k$ denoting the $k \times k$ unit matrix). We do not exclude the possibility of testing against even more general alternatives, see e.g. KING (1980). For $M$ in (1) we mean the matrix $M = I - X(X'X)^{-1}X'$ if $X$ is the matrix containing the regressors. The probability $P(d \leq c)$ can be rewritten as

$$P(\varepsilon'(MAM - cM)\varepsilon \leq 0), \tag{2}$$

i.e. a tail probability involving a quadratic form. Finding the probabilities now proceeds by computing the eigenvalues of the matrix $(MAM - cM)$ and applying the procedure of IMHOF. Unfortunately, the computation of eigenvalues is an $O(n^3)$ process if $n$ is the number of rows of the matrix $X$. Upper and lower bounds can be found using the eigenvalues $\{\lambda_j\}$ of the matrix $A$, in case of the Durbin-Watson statistic known to be equal to $\lambda_j = 2(1 - \cos(\pi \frac{j-1}{n}))$, $j = 1,..,n$ (see the papers of DURBIN and WATSON).

In DURBIN and WATSON (1971) several approximations are compared and it is found that the approximation of HENSHAW (1966) and approximations based on 4-th order Jacobi polynomials yield excellent results, unless perhaps the number of degrees of freedom is very low (in their *Pears* data they have $n$=16 and 5 regressors). Both approximations make use of the first four moments of $d$ to fit a Beta distribution. As the result are very similar we only review the Jacobi expansion.

A Beta($p$,$q$) distribution is defined on the interval [0,1] and its density function at the point $x$ is given by

$$\beta(p,q;x) = \frac{x^{p-1}(1-x)^{q-1}}{\beta(p,q)},$$

where $p$ and $q$ are the two parameters of the distribution. Denoting its moments around zero by $\mu_j^{p,q}$ one has

$$\mu_1^{p,q} = p/(p+q)$$

$$\mu_2^{p,q} = p(p+1)/\{(p+q)(p+q+1)\}$$

$$\mu_j^{p,q} = p(p+1)\ldots(p+j-1)/\{(p+q)(p+q+1)\ldots(p+q+j-1)\}.$$

Suppressing the superscripts $p$ and $q$, the inverse relations for $p$ and $q$ expressed in terms of $\mu_1$ and $\mu_2$ are given by

$$p = \mu_1(\mu_1-\mu_2)/(\mu_2-\mu_1^2)$$

$$q = (1-\mu_1)(\mu_1-\mu_2)/(\mu_2-\mu_1^2).$$

The Jacobi polynomials are a set of polynomials that are orthogonal over the interval [0,1] with respect to the $\beta(p,q)$ density. In ABRAMOWITZ and STEGUN (1965) these polynomials are listed, but with redefined values for $p$ and $q$, a different proportionality factor in the weight function and not normalized to an orthonormal system. Given a set of polynomials $\{f_j\}$ define $<f_i,f_j>$ by

$$<f_i,f_j> = \int_0^1 f_i(x)f_j(x)\beta(p,q;x)dx. \tag{3}$$

Starting with powers in $x$ functions $f_n(x)$ can be found through Gram-Schmidt orthogonalization such that $<f_i,f_j>=0$ for $i\neq j$. Writing $f_n(x)=\Sigma_{m=0}^n d_{nm}x^{n-m}$ one finds

$$d_{nm} = (-1)^{n-m}\binom{n}{m}\frac{\Gamma(p+q-1+n+n-m)\ \Gamma(p)}{\Gamma(p+q-1+n)\ \Gamma(p+n-m)} \tag{4}$$

(note the binomial coefficient that is missing in DURBIN and WATSON (1951)). For e.g. $n=3$ this specializes to

$$f_3(x) = 1-3\frac{p+q+2}{p}x+3\frac{(p+q+2)(p+q+3)}{p(p+1)}x^2-\frac{(p+q+2)(p+q+3)(p+q+4)}{p(p+1)(p+2)}x^3.$$

Finally, to obtain an orthonormal system one may deduce from ABRAMOWITZ and STEGUN (1965) that

$$<f_n,f_n> = \|f_n\|^2 = \frac{\Gamma(n+1)\ \Gamma(n+q)\ \Gamma(p)\ \Gamma(p+q)}{(2n+p+q-1)\ \Gamma(p+q-1+n)\ \Gamma(p+n)\ \Gamma(q)}$$

so dividing the coefficients $d_{nm}$ by $\|f_n\|$ yields an orthonormal system which again we denote by $\{f_n\}$. We note that errors in formulae or programming errors are most easily detected by verifying $<f_i,f_j>=\delta_{ij}$ directly through numerical integration, where $\delta_{ij}$ is the Kronecker delta. For the remaining part of this section we define the coefficients $c_{ki}$ such that the $k$-th normalized polynomial equals

$$f_k(x) = \Sigma_0^k c_{ki}x^i.$$

Let the variable $d/4$ have density $g(x)$, then as $0 \leq d/4 \leq 1$ one may write

$$g(x) \approx f(x) = \left(1+\Sigma_{j=1}^{m}\alpha_j f_j(x)\right)\beta(p,q;x), \tag{5}$$

where $f(x)$ is the expansion of $g(x)$ up to the order $m$ with respect to the system $\{f_j\}$. From completeness of the orthonormal system it follows that if $m$ goes to infinity, $\|g-f\|$ goes to zero and because one integrates over a finite range also that $\int|g-f|\beta(p,q;x)dx$ goes to zero. This implies that the approximation of $g$ by $f$ can be made accurate if $m$ is taken large enough. Fortunately there is the additional property that less terms are needed in (5) if the number of observations $n$ goes to infinity. This can be expected because if the statistic $d$ in (1) approaches normality, $p$ and $q$ can be selected such that the weight function $\beta(p,q;x)$ is close to $g(x)$, so little correction is needed (if $p \approx q$ is large, then a Taylor expansion of the Beta density around $x = \frac{1}{2}$ shows that it is approximately normal).

Expanding $g(x)$ in terms of $\{f_j\}$ involves the coefficients $\alpha_j$ to be chosen such that the first $m$ moments $\mu_j$ of $d/4$ are corrected towards the first $m$ moments of $f(x)$ as follows. First $p$ and $q$ are selected such that $\mu_1 = \mu_1^{p,q}$ and $\mu_2 = \mu_2^{p,q}$. Next, because $\{f_j\}$ is an orthonormal system with respect to $\beta(p,q;x)$, one has taking expectations with respect to $f(x)$ for $k \geq 1$

$$Ef_k = \int f_k(x)\left(1+\Sigma\alpha_j f_j(x)\right)\beta(p,q;x)dx = \alpha_k = \Sigma_{i=0}^{k} c_{ki}\Delta\mu_i, \tag{6}$$

where the lower triangular matrix $C=(c_{ki})$ is such that $f_k(x) = \Sigma c_{ki}x^i$ and where $\Delta\mu_i = \mu_i - \mu_i^{p,q}$. Note that the matrix $C$ has already been obtained during the construction of the Jacobian polynomials $\{f_k\}$. Obviously it follows that $\alpha_1 = \alpha_2 = 0$. The bulk of the computation in (6) is the determination of the moments of the statistic $d$, a problem we discuss later. Writing $\alpha = (1\ \alpha_1\ \alpha_2\ ..\ \alpha_m)'$ the function $f(x)$ can be expressed from (5) as

$$f(x) = \left(\Sigma_0^m b_i x^i\right)\beta(p,q;x)$$

where the coefficients $b_i$ are the components of the vector $b = C'\alpha$.

For the probability $P(d \leq 4c)$ one now obtains

$$P(d \leq 4c) \approx \int_0^c \left(\Sigma b_i x^i\right)\beta(p,q;x)dx = \sum\left[b_i\beta^{ic}(p+i,q;c)\frac{\beta(p+i,q)}{\beta(p,q)}\right], \tag{7}$$

where $\beta^{ic}$ denotes the incomplete Beta function. The incomplete Beta function can be very conveniently evaluated through a continued fraction expansion (see e.g. PRESS e.a. (1988)). The function $\beta^{ic}$ has to be evaluated only once because of the relationship

$$\beta^{ic}(p+1,q;c) = \beta^{ic}(p,q;c) - \frac{x^p(1-x)^q}{(p+q)\beta(p+1,q)}.$$

To serve as an example we have divided the statistic $d$ by 4 to map its range into [0,1]. DURBIN and WATSON (1951) used the least and greatest eigenvalues of $A$ in (1) to map the statistics $d_L$ and $d_U$ exactly onto [0,1]. As we will see in the next section such a mapping does not necessarily lead to the best approximations if beta densities are used. We have tried the following factors $f$ in $d/f$ to map $d$ into [0,1]:

1) $f=4$, the easiest way, independent of the data.
2) $f=\lambda^{A}_{max}$, the maximum eigenvalue of the matrix $A$. If $A$ is designed to test against $k$-th order serial correlation then $\lambda^{A}_{max}$ can be determined easily, but in general it depends on $n$.
3) $f=\lambda_{max}$, the maximum eigenvalue of $MAM$. The disadvantage is that $\lambda_{max}$ depends on the data, although in SNEEK and SMITS (1990) a sort of generalized power method is described that requires only about ten iterations from $MAMy^{(t)}=y^{(t+1)}$, clearly an $\mathcal{O}(n)$ process.
4) $f$ is such that $\alpha_3^2+\alpha_4^2$ is minimized. Although this depends on the data it is an $\mathcal{O}(1)$ procedure once the moments of $d$ are computed.

In the next section we choose a strategy based on numerical experiments. We note that the accuracy is so good for the expansions that, even for small values of $n$, in case there is difficulty in finding the eigenvalues of $A$ one may use the value 4. Also, strategy 4) may theoretically lead to a small value for $f$ such that $d$ is no longer mapped into [0,1]. In that case the Jacobi expansion can not converge to the density function of $d/f$, but in practice we always ended up with a value for $f$ between those of strategies 1) and 3).

The bulk of the computation is the determination of the moments of $d$. In the following we show how to compute the moments in $\mathcal{O}(n)$ arithmetic operations.
It is well known that, as the denominator is independent of the statistic itself, the moments of $d$ can be obtained from the moments of numerator and denominator. Therefore the main problem is the determination of the moments of the numerator and for that one needs sums of powers of the eigenvalues $\Sigma_{i=1}^{n}\lambda_i^m$ of the matrix $B=MAM$, $m=1,2,\ldots$.
Let $N=X(X'X)^{-1}X'$, so $M=I-N$, and let $W_m=X'A^mX(X'X)^{-1}$. It is not difficult to show that the following trace expressions are valid:

$$\begin{aligned}
\mathrm{tr}\{A^{m_1}NA^{m_2}\ldots A^{m_k}N\} &= \mathrm{tr}\{W_{m_1}W_{m_2}\ldots W_{m_k}\}\\
\mathrm{tr}\{A^{m_1}NA^{m_2}\ldots A^{m_k}\} &= \mathrm{tr}\{W_{m_1+m_k}W_{m_2}\ldots W_{m_{k-1}}\}\\
\mathrm{tr}\{W_{m_1}W_{m_2}W_{m_3}\} &= \mathrm{tr}\{W_{m_1}W_{m_3}W_{m_2}\}.
\end{aligned}$$

The last equality we do not use explicitly below in order not to obscure how the expressions are obtained, but it can be used to further simplify the results. As we nowhere need two-digit subscripts we further abbreviate e.g. $W_1W_2W_3$ to $W_{123}$. By straightforward expansion of $\mathrm{tr}\{\left((I-N)A(I-N)\right)^m\}$ and using rules for traces and the fact that $N^2=N$ one finds

$$\begin{aligned}
\mathrm{tr}\{B\} &= \mathrm{tr}\{A-W_1\}\\
\mathrm{tr}\{B^2\} &= \mathrm{tr}\{A^2-2W_2+W_{11}\}\\
\mathrm{tr}\{B^3\} &= \mathrm{tr}\{A^3-3W_3+3W_{12}-W_{111}\}\\
\mathrm{tr}\{B^4\} &= \mathrm{tr}\{A^4-4W_4+4W_{13}+2W_{22}-4W_{112}+W_{1111}\}\\
\mathrm{tr}\{B^5\} &= \mathrm{tr}\{A^5-5W_5+5W_{14}+5W_{23}-5W_{113}-5W_{122}+5W_{1112}-W_{11111}\} \qquad (8)\\
\mathrm{tr}\{B^6\} &= \mathrm{tr}\{A^6-6W_6+6W_{15}+6W_{24}+3W_{33}-6W_{114}-6\left(W_{123}+W_{132}\right)+\\
&\quad -2W_{222}+6W_{1113}+6W_{1122}+3W_{1212}-6W_{11112}+W_{111111}\}\\
\mathrm{tr}\{B^7\} &= \mathrm{tr}\{A^7-7W_7+7W_{16}+7W_{25}+7W_{34}-7W_{115}-7\left(W_{124}+W_{142}\right)+\\
&\quad -7W_{133}-7W_{223}+7W_{1114}+7\left(W_{1123}+W_{1132}\right)+7W_{1222}+\\
&\quad +7W_{1213}-7W_{11113}-7W_{11122}-7W_{11212}+7W_{111112}-W_{1111111}\}.
\end{aligned}$$

The formulas above are valid irrespective of how the matrix $A$ has been defined. The major part of the computation is due to the factor $X'A^mX$ in $W_m$, but if $A$ is very nearly equal to a Toeplitz matrix with very many zeros (like the matrices in (1)) then the computing load is roughly equal to an additional regression each time $m$ is increased by one. The many products of $W$'s should not bother too much as the size of these matrices only depends on the number of regressors. When testing the constancy of regression relationships over time using the cumulative sum of squares of residuals, see McCabe and Harrison (1980), it is even simpler to compute the quantities $W_m$ as the extra computations then involve only $O(1)$ operations.

In (8) one needs the values of $\mathrm{tr}(A^m)$; for convenience we list them below for the case that $A$ is the matrix $A=A_{[(n+k-1)/k]}\otimes I_k$ corresponding to the testing of k-th order autocorrelation.

$$
\begin{aligned}
\mathrm{tr}\{A\} &= 2n-2k \\
\mathrm{tr}\{A^2\} &= 6n-8k \\
\mathrm{tr}\{A^3\} &= 20n-32k \\
\mathrm{tr}\{A^4\} &= 70n-128k+4c_1 \\
\mathrm{tr}\{A^5\} &= 252n-512k+40c_1 \\
\mathrm{tr}\{A^6\} &= 924n-2048k+252c_1+6c_2 \\
\mathrm{tr}\{A^7\} &= 3432n-8192k+1288c_1+84c_2
\end{aligned} \tag{9}
$$

These formulas are valid for $n \geq 2k$ if $c_1 = \min\{0, 3k-n\}$ and $c_2 = \min\{0, 4k-n\}$. The same formulas can be used as well for the matrix in (2), i.e. $A$ replaced by $A-cI$, though in that case one requires an additional step of expressing $\mathrm{tr}\{(A-cI)^m\}$ in terms of $\mathrm{tr}\{A^j\}$, $j=1,..,m$. We actually used these formulas to compute Cornish-Fisher expansions (FISHER and CORNISH (1960)) for the tail probabilities in (2). Although quite reasonable the resulting approximations are not as accurate as the ones considered in this paper.
Note that the formulas above do not depend on the regressor matrix $X$.

## 3. SIMULATION EVIDENCE

In this section we report some simulation results to give evidence on the accuracy of the approximations.
For the regressor matrix $X$ we have generated a 'typical economic behaviour' for the columns of $X$, see DUBBELMAN, LOUTER and ABRAHAMSE (1978). According to their equation (1) we generate one or two columns in the $X$ matrix using

$$
\begin{aligned}
x_i &= it + e_i \\
e_i &= \lambda e_{i-1} + \eta_i \qquad i=1,\ldots,n,
\end{aligned}
$$

where $t=0.4$ or $t=0.8$ and $\lambda=0.7$. Furthermore, the inclusion of an additional constant and additional seasonal dummies is tried in the experiments as well. After having determined the strategy for choosing $f$ in $d/f$ to map $d$ into [0,1] we have taken $X$ as a set of eigenvectors of the matrix $A$ corresponding to the least or greatest eigenvalues of $A$ to see how the resulting approximations are in these extreme cases. For the exact tail probabilities we used numerical inversion of the characteristic function of the quadratic form in (2); the truncation and integration error were both set at $2.5\times10^{-6}$.

Based on the numerical evidence we have selected the factor $f$ in $d/f$ to map $d$ into [0,1] as follows.

For small values of $n$ (say $n \leq 20$) it seems not advisable to use $f=4$, especially if one is testing against higher order autocorrelation because a substantial loss in accuracy (with respect to other choices) may be the result while gains in accuracy are usually slight.

Using $\lambda_{max}^{A}$ instead of $f=4$ may lead to a considerable improvement if one is testing against higher order serial correlation. However, if one is considering more general matrices $A$ in (1) then the value $\lambda_{max}^{A}$ may not be easy to obtain, although it is a function of $n$ only.

Using $\lambda_{max}$ seems unnecessary despite the fact that an efficient $O(n)$ procedure is available because other choices lead to similar or better accuracy if one uses the Jacobi coefficients $\alpha_i$ as indicators.

Based on the numerical evidence we advocate the use of $f=f^*$ that is selected such that $\|\alpha_{3,4}\|^2=\alpha_3^2+\alpha_4^2$ is minimized. Furthermore, if $\|\alpha_{3,4}\|<0.01$ then a 4-th order expansion is used, otherwise a 6-th order expansion is used. In the latter case we report the value of $\|\alpha_{5,6}\|$ because in our experience small values (i.e. small 'corrections') indicate high accuracy. Results can further be improved by using either $f=\lambda_{max}^{A}$ or $f=f^*$ according to what leads to the smallest value of $\|\alpha_{5,6}\|$ in case a 6-th order Jacobi expansion is selected, but to keep the procedure simple and most generally applicable we always use $f=f^*$. We note that for the minimization to find $f^*$ a tolerance of 0.1 is used as the value of $f$ is not very critical. In our experience the value of $f^*$ is always such that $d/f^*$ is well within the interval [0,1].

In table 1 we show results for (very) small values of $n$ only as in all cases the accuracy of the approximations improves very rapidly if $n$ gets bigger. In the table we report for each column the number of observations $n$, the degree of the Jacobi expansion that is used for that particular case and the norm $\|\alpha_{3,4}\|$ or $\|\alpha_{5,6}\|$ according to whether a 4-th or 6-th order expansion is selected. In each column itself we report exact minus approximated probabilities (multiplied by 100 to save space) in points at which the approximated probabilities equal the p-values in the first columns, i.e. the nominal significance levels if the approximations are used for testing. Note that if in practice the approximations indicate a particular significance level, then one would be interested in the actual significance level as a measure of the accuracy of the approximation.

For the model with two regressors and no constant the error that is made is always less than 0.1% even for $n$ as low as 8. E.g., for $n=8$ at nominal significance level $p=0.050$, selecting by our criterion a 6-th degree Jacobi

polynomial, the actual significance level using the exact distribution equals (0.050+0.00089).

If a constant is included, the approximation is perhaps too inaccurate for $n=8$, but for $n=10$ or $n=12$ is already good enough for most practical purposes. For the model with four seasonal dummies we start the table at $n=12$ (i.e. only 6 degrees of freedom), but $n$ needs to be a little bigger to get accuracies better than 0.1%.

The right half of the table shows similar results when one wants to test against 4-th order serial correlation; to our opinion also in that case the approximations are remarkably accurate and for practical work good enough to replace exact tail probabilities. It is also quite clear that the results improve rapidly if $n$ gets larger. Similar tables were obtained for other numbers of regressors, different types of regressors and when testing against different orders of serial correlations.

In table 2 we report some results if $X$ is a matrix of eigenvectors of $A$. If one takes eigenvectors belonging to a set of smallest eigenvalues of $A$ (this corresponds to low frequency data, usually encountered in economic time series), then not unexpectedly the degree of the Jacobi expansion is higher, i.e. more 'correction' terms are needed to fit a beta distribution. This is exactly the case in which $d$ can not obtain the value zero. We note however that the transformation $(d-\lambda_{min})/(\lambda_{max}-\lambda_{min})$, where $\lambda_{min}$ is the smallest eigenvalue of $MAM$, maps $d$ exactly onto [0,1] but leads to less accurate approximations in all data sets we generated. If $X$ is taken as a set of eigenvectors belonging to largest eigenvalues (high frequency data), then quite rapidly the procedure automatically seeks 4-th order approximations. To our opinion even for very small numbers of degrees of freedom the approximations are in these 'extreme' cases good enough for most practical purposes.

In table 2 we rather arbitrarily show the results for the 3 smallest and largest and the 6 smallest and largest eigenvalues; similar tables are obtained however when using different numbers of regressors.

Because of the excellent results even in table 2, we decided not to try transformations of the type $(d-f_1)/(f_2-f_1)$ and minimizing with respect to $\|\alpha_{3,4}\|$. The monitoring quality of $\|\alpha_{3,4}\|$ would disappear in this case because the first 4 moments can be fitted exactly.

As in table 2 essentially the accuracy of the approximations are reported for the statistics $d_L$ and $d_U$, one may use in practice the following procedure.

Given the value of of the statistic $d$ in (1) from the sample, first compute the tail probabilities of the statistics $d_L$ and $d_U$ and if possible draw the conclusions about serial correlation. Secondly, if there is no conclusion yet, compute the 'exact' tail probability using the procedure above. In this way one would even avoid quite often the $\mathcal{O}(n)$ procedure of computing moments of $d$. Of course for doing so it is assumed that the matrix $A$ is simple enough to allow specific knowledge about its eigenvalues in order to compute upper and lower bounds, which is e.g. the case if one tests against $k$-th order serial correlation.

## 4. CONCLUSIONS

In this paper we have given evidence that the distribution of the Durbin-Watson and related statistics can be approximated in $\mathcal{O}(n)$ arithmetic operations and can easily be implemented in statistical computer programs. One would need the incomplete beta function, but that function is usually already available if the distributions of $t$- and $F$-statistics are computable. The approximations are accurate enough for most applications and are not restricted to the 'standard' Durbin-Watson statistic. To our opinion there is no reason why the distribution should not be incorporated in regression programs even on small and relatively slow computers.

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TABLE 1 TAILPROBABILITIES: (EXACT-APPROXIMATED)×100

| | 1-th autocorrelation | | | | 4-th autocorrelation | | | |
|---|---|---|---|---|---|---|---|---|
| **TWO regressors, NO constant** | | | | | | | | |
| n | 8 | 10 | 12 | 16 | 10 | 12 | 14 | 18 |
| k | 6 | 6 | 4 | 4 | 6 | 6 | 4 | 6 |
| $\|\alpha_{..}\|$ | 0.044 | 0.007 | 0.002 | 0.005 | 0.018 | 0.026 | 0.007 | 0.005 |
| 0.005* | -0.043 | -0.002 | -0.006 | -0.003 | 0.003 | 0.000 | 0.023 | -0.003 |
| 0.010 | -0.041 | -0.025 | 0.005 | 0.001 | -0.011 | 0.008 | 0.015 | -0.005 |
| 0.025 | 0.035 | -0.049 | 0.030 | 0.011 | -0.022 | 0.020 | -0.018 | -0.003 |
| 0.050 | 0.089 | -0.013 | 0.045 | 0.018 | -0.009 | 0.006 | -0.051 | 0.003 |
| 0.100 | 0.040 | 0.062 | 0.022 | 0.014 | 0.016 | -0.048 | -0.055 | 0.008 |
| 0.900 | -0.044 | -0.014 | -0.002 | 0.002 | 0.028 | 0.044 | -0.018 | 0.003 |
| 0.950 | 0.036 | -0.031 | -0.016 | 0.000 | 0.001 | 0.007 | -0.015 | -0.001 |
| 0.975 | 0.088 | -0.020 | -0.016 | 0.001 | -0.012 | -0.021 | -0.007 | -0.002 |
| 0.990 | 0.020 | -0.009 | -0.008 | 0.000 | -0.007 | -0.022 | 0.003 | -0.002 |
| 0.995 | -0.020 | 0.001 | -0.002 | -0.001 | -0.001 | -0.010 | 0.007 | -0.000 |
| **TWO regressors, ONE constant** | | | | | | | | |
| n | 8 | 10 | 12 | 16 | 10 | 12 | 14 | 18 |
| k | 6 | 6 | 4 | 4 | 6 | 6 | 6 | 6 |
| $\|\alpha_{..}\|$ | 0.084 | 0.019 | 0.002 | 0.001 | 0.014 | 0.024 | 0.012 | 0.004 |
| 0.005 | 0.005 | -0.069 | 0.005 | -0.003 | 0.014 | 0.003 | 0.002 | -0.003 |
| 0.010 | 0.090 | -0.053 | 0.016 | 0.002 | 0.009 | 0.012 | -0.001 | -0.010 |
| 0.025 | 0.163 | 0.030 | 0.031 | 0.014 | -0.024 | 0.019 | -0.008 | -0.014 |
| 0.050 | 0.064 | 0.108 | 0.016 | 0.023 | -0.034 | 0.002 | -0.013 | -0.004 |
| 0.100 | -0.251 | 0.095 | -0.031 | 0.017 | 0.008 | -0.044 | -0.005 | 0.018 |
| 0.900 | 0.209 | -0.088 | -0.047 | 0.012 | -0.068 | 0.052 | 0.028 | 0.015 |
| 0.950 | 0.349 | -0.020 | -0.033 | 0.021 | -0.025 | -0.003 | 0.001 | -0.003 |
| 0.975 | 0.109 | 0.007 | -0.008 | 0.017 | 0.026 | -0.033 | -0.013 | -0.010 |
| 0.990 | -0.098 | 0.019 | 0.017 | 0.003 | 0.028 | -0.024 | -0.010 | -0.006 |
| 0.995 | -0.154 | 0.020 | 0.021 | -0.005 | 0.006 | -0.006 | -0.004 | -0.002 |
| **TWO regressors, FOUR seasonal dummies** | | | | | | | | |
| n | 12 | 14 | 16 | 20 | 12 | 14 | 16 | 20 |
| k | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 4 |
| $\|\alpha_{..}\|$ | 0.080 | 0.018 | 0.017 | 0.001 | 0.097 | 0.019 | 0.037 | 0.001 |
| 0.005 | 0.045 | -0.029 | 0.010 | 0.001 | -0.332 | -0.063 | -0.022 | 0.005 |
| 0.010 | 0.076 | -0.044 | 0.016 | 0.003 | -0.268 | -0.071 | -0.028 | 0.012 |
| 0.025 | 0.079 | -0.003 | 0.016 | 0.004 | 0.079 | -0.026 | -0.014 | 0.012 |
| 0.050 | 0.048 | 0.066 | 0.010 | 0.002 | 0.417 | 0.080 | 0.037 | -0.001 |
| 0.100 | -0.119 | 0.075 | -0.015 | -0.004 | 0.483 | 0.179 | 0.100 | -0.009 |
| 0.900 | 0.104 | -0.070 | -0.069 | -0.010 | -0.603 | -0.076 | -0.051 | 0.045 |
| 0.950 | 0.427 | -0.002 | 0.015 | 0.001 | -0.524 | -0.103 | -0.046 | 0.007 |
| 0.975 | 0.313 | 0.049 | 0.061 | 0.006 | -0.088 | -0.045 | -0.021 | -0.021 |
| 0.990 | -0.021 | 0.038 | 0.048 | 0.006 | 0.420 | 0.015 | 0.008 | -0.023 |
| 0.995 | -0.167 | 0.013 | 0.019 | 0.003 | 0.475 | 0.032 | 0.020 | -0.013 |

n is number of observations; k is degree of selected Jacobi polynomial
$\|\alpha_{..}\|$ is the norm of $\alpha_{k-1,k}$; *first column contains target p-values

TABLE 2 TAILPROBABILITIES: (EXACT-APPROXIMATED)×100

| | 3 smallest eigenvalues+ | | | 3 largest eigenvalues+ | | |
|---|---|---|---|---|---|---|
| n | 8 | 10 | 12 | 8 | 10 | 12 |
| k | 6 | 6 | 6 | 6 | 4 | 4 |
| $\|\alpha_{..}\|$ | 0.023 | 0.012 | 0.012 | 0.024 | 0.008 | 0.004 |
| 0.005* | -0.091 | 0.001 | 0.001 | 0.017 | 0.010 | 0.005 |
| 0.010 | 0.010 | 0.006 | 0.007 | -0.028 | 0.019 | 0.008 |
| 0.025 | 0.170 | 0.009 | 0.011 | -0.112 | 0.019 | 0.011 |
| 0.050 | 0.151 | 0.032 | 0.003 | 0.017 | 0.006 | 0.005 |
| 0.100 | -0.151 | 0.011 | -0.011 | 0.107 | 0.016 | -0.005 |
| 0.900 | -0.080 | -0.003 | -0.014 | 0.137 | -0.012 | -0.001 |
| 0.950 | 0.092 | -0.005 | 0.007 | -0.141 | -0.024 | -0.016 |
| 0.975 | -0.025 | -0.002 | 0.018 | -0.147 | -0.027 | -0.021 |
| 0.990 | -0.049 | 0.010 | 0.010 | -0.014 | -0.037 | -0.009 |
| 0.995 | -0.017 | 0.005 | 0.002 | 0.068 | -0.017 | -0.002 |
| | 6 smallest eigenvalues+ | | | 6 largest eigenvalues+ | | |
| n | 12 | 14 | 16 | 12 | 14 | 16 |
| k | 6 | 6 | 6 | 6 | 6 | 4 |
| $\|\alpha_{..}\|$ | 0.043 | 0.023 | 0.019 | 0.026 | 0.008 | 0.006 |
| 0.005* | -0.060 | 0.002 | 0.006 | 0.014 | 0.002 | 0.004 |
| 0.010 | -0.047 | -0.005 | 0.007 | 0.017 | -0.003 | 0.007 |
| 0.025 | -0.037 | -0.011 | -0.001 | 0.010 | -0.005 | 0.009 |
| 0.050 | -0.086 | -0.011 | -0.016 | -0.039 | -0.001 | 0.009 |
| 0.100 | 0.022 | -0.025 | -0.028 | -0.099 | -0.003 | -0.001 |
| 0.900 | 0.202 | 0.017 | -0.008 | 0.086 | 0.011 | 0.004 |
| 0.950 | 0.164 | 0.062 | 0.041 | 0.002 | -0.003 | -0.013 |
| 0.975 | 0.069 | 0.046 | 0.042 | -0.072 | -0.001 | -0.018 |
| 0.990 | -0.058 | 0.001 | 0.014 | -0.027 | -0.008 | -0.011 |
| 0.995 | -0.103 | -0.015 | -0.003 | 0.020 | -0.008 | -0.004 |

+REGRESSORS are eigenvectors corresponding to smallest or largest eigenvalues of A when testing for first order serial correlation

*first column contains target p-values

n is number of observations;

k is degree of selected Jacobi polynomial

$\|\alpha_{..}\|$ is the norm of $\alpha_{k-1,k}$

| | | |
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