Faculteit der Economische Wetenschappen en Econometrie

Serie Research Memoranda

Impacts of Multiple Period Lags in Dynamic Logic Models

P. Nijkamp
A. Reggiani

Research Memorandum 1991-95
december 1991

vrije Universiteit *amsterdam*

Abstract

This paper will provide an introduction to a new field of research, viz the sensitivity of the solution trajectory of a dynamic logit model (belonging to the class of discrete choice models) in the light of a multi-period lag structure. It is well known from recent advances in the area of chaos and turbulence theory that the stability of a dynamic system is critically dependent on various factors, such as threshold values of parameters, initial conditions and also the lag structure. This paper aims to identify the consequences of different lag structures in dynamic logit models (including also dynamic spatial interaction models). Various simulation experiments will be used to show that the onset of instability of the solution trajectory tends to decrease as the number of time lags increases (depending also on the growth rate of the system).

Acknowledgement

The second author would like to thank Andrea Gerali of the University of Bergamo for computer assistance. Also the CNR grant n.90.01247.CT11 is greatfully acknowledged.

1. Theory of Turbulence in Social Sciences

Modelling tradition in the social sciences was usually based on linear static systems models. Sometimes also dynamic linear models were used in order to describe the growth or decline of certain phenomena, but non-linear dynamic models were rather an exception. Although linear dynamic models are not necessarily very restrictive for well defined and regular movements of phenomena, they have severe shortcomings in case of highly irregular movements (e.g., in case of non-periodic evolution; see Brock 1986). It is in this context that the theory of chaos or turbulence has recently become an important analytical tool.

An important feature of chaos theory is that it is essentially concerned with **deterministic, non-linear** dynamic systems which are able to produce complex motions of such a nature that they are sometimes seemingly random. In particular, they incorporate the feature that small uncertainties may grow exponentially (although all time paths are bound), leading to a broad spectrum of different trajectories in the long run, so that precise or plausible predictions are - under certain conditions - very unlikely.

In this context, a very important characteristic of non-linear models which can generate chaotic evolutions is that such models exhibit strong sensitivity to initial conditions. Points which are initially close will diverge exponentially over time. Hence, even if we knew the underlying structure exactly, our evaluation of the current state of the system is subject to measurement error and, hence it is impossible to predict with confidence beyond the very short run. Similarly, if we knew the current state with perfect precision, but the underlying structure only approximately, the future evolution of the system would also be unpredictable. The equivalence of the two situations has been demonstrated by e.g. Crutchfield et al. (1982).

After a series of interesting studies on chaotic features of complex systems in physics, chemistry, biology, meteorology and ecology, chaos theory has also been introduced and investigated in the social sciences. The main purpose of the use of this theory in the social sciences was to obtain better insight into the underlying causes of unforeseeable evolutions of complex dynamic social systems.

Informative surveys of chaos theory and its relevance for the social sciences can among others be found in contributions by Andersen (1988), Benhabib and Day (1981, 1982), Boldrin (1988), Devaney (1986), Guckenheimer and Holmes (1983), Kelsey (1988), Lasota and Mackey (1985), Lung (1988, 1989), Pohjola (1981), Prigogine and Stengers (1985), Stewart (1989), and Stutzer (1980).

Interesting applications of chaos theory can in particular be found in geography and regional science (see also Nijkamp and Reggiani, 1990a). Examples are:

- regional industrial evolution (White 1985)
- urban macro dynamics (Dendrinos 1984)
- spatial employment growth (Dendrinos 1986)
- relative population dynamics (Dendrinos and Sonis 1987)
- spatial competition and innovation diffusion (Sonis 1986, 1988)
- migration systems (Haag and Weidlich 1983, Reiner et al. 1986)
- urban evolution (Nijkamp and Reggiani 1990b)
- transport systems (Reggiani 1990)

It is interesting to observe that most applications of chaos theory in economics (and in general the social sciences) lack empirical content. While empirical research on chaos went hand in hand with theoretical developments in the natural sciences in the early 1980s, attempts to detect chaos in financial and economic data are more recent. The results in this area are so far disappointing. Brock (1989) claims that as yet no class of structural economic models has been estimated which allows for chaotic behaviour and in which the estimated model parameters are indeed in the chaotic range. Moreover, statistical tests which have been designed to detect chaos in time series without a priori specification of the nature of the data generating process, have not provided as yet unambiguous empirical support for the presence of chaos in observable economic processes. Some interesting attempts at linking chaos to empirical evidence in economics can be found in Barnett et al., (1990).

Besides the need to provide a solid basis for justifying and interpreting turbulent motions in complex dynamic systems (e.g., by providing a behavioural foundation), there is also the question of specifying the time delay structure (or the lag pattern). This is usually done in an ad hoc way, for instance, by introducing a simple one-period lag structure. In this paper we will focus attention on the implications of varying lag structures in a dynamic choice model whose behavioural foundation is linked to the class of discrete choice models (notably logit models) or -in a macro context - the class of spatial interaction models.

2. Seemingly Simple Dynamic Systems Models

In most recent literature dynamic systems models are often based on **logistic growth**. For nearly two decades first order-difference equations of a logistic type have been a subject of interest to many researchers. In particular the seminal work by May (1976), notably his study on population dynamics, has opened a rich research field on the complex behaviour of difference equations. He has shown that a first-order difference equation, even in a seemingly simple and deterministic form, can exhibit a surprising array of dynamical behaviour, from stable points to a bifurcating hierarchy of stable cycles or even a 'chaotic' fluctuation.

Let us recall the **logistic map** of May, which can be represented as follows:

$$X_{t+1} = N X_t (1-X_t) \qquad (2.1)$$

where X is the 'population variable' and N is its growth parameter. Equation (2.1) requires for its existence that $0 \leq X \leq 1$ and $0<N<4$.

It is well known that the **bifurcation diagrams** (in which the values of the parameters are plotted against the population values) related to (2.1) show a rich spectrum of unstable behaviour for $-3 < N < 4$. In particular, for $N > 3.824$ a cycle of period 3 appears, beyond which there are cycles with every integer period, as well as an uncountable number of aperiodic trajectories; in other words, this is a typical example of a chaotic region. It has recently been shown (see Nijkamp and Reggiani, 1990a) that a 'binary' logit model can belong to the family of May models under particular assumptions of its utility function (in particular, the assumption that utility increases linearly with time through a fixed parameter, denoted by α). For the sake of simplicity we present here the standard form of a dynamic multinomial logit model, i.e.:

$$P_j = \exp(u_j) / \sum_\ell \exp(u_\ell) \tag{2.2}$$

where u_j represents the utility of choosing alternative j (j=1,...ℓ...,J) at time t and P_j is the dynamic probability of choosing j.

It can be shown that expression (2.2) may emerge as a solution of an optimal control model maximizing a cumulative entropy (see Nijkamp and Reggiani, 1988); moreover, it is formally equivalent to a production constrained spatial interaction model of the following form:

$$P_{ij} = \frac{T_{ij}}{O_i} = \frac{W_j \exp(-\beta c_{ij})}{\sum_\ell W_\ell \exp(-\beta c_{ij})} \tag{2.3}$$

where P_{ij} represents the dynamic probability of choice from i to j and c_{ij} is the interaction cost between i and j with β as a friction parameter (see also Reggiani, 1990). Consequently, our subsequent analysis which is associated with a dynamic logit model can also be considered in the context of a spatial interaction system.

Then, if we consider the rate of change of P_j with respect to time (i.e. dP_j/dt), we get (see Nijkamp and Reggiani, 1990a) the following expression:

$$\frac{dP_j}{dt} = \dot{P}_j = \dot{u}_j P_j (1 - P_j) - P_j \sum_{j \neq \ell} \dot{u}_\ell P_\ell \tag{2.4}$$

where $\dot{u}_j = du_j/dt$ represents the time rate of change of u_j. The latter part of (2.4) represents essentially interaction effects.

If we now approximate equation (2.4) in difference equation form by considering discrete time periods (see also Wilson and Bennet,

1985) and if we assume a constant utility change (i.e., $u_{j,t+1} - u_{j,t} = \alpha_j$), we find the following final expression for (2.4) in dis crete terms:

$$P_{j,t+1} = (\alpha_j + 1) P_{j,t} - \alpha_j P_{j,t}^2 - P_{j,t} \Sigma_{\ell \neq j} \alpha_\ell P_{\ell,t} \qquad (2.5)$$

It is now clear that system (2.5) represents in discrete terms a prey-predator system with limited prey (P_j); P_ℓ can be interpreted as a predator whose influence will be the reduction of population P_j through the parameter α_ℓ.

The dynamic patterns of system (2.5) for different values of α, and the initial conditions (considering also the additivity condition $\sum_\ell P_\ell = 1$) have been illustrated and discussed in Nijkamp and Reggiani (1990a). Here we will focus our attention on the analysis of a particular case of (2.5). In fact it is interesting to notice that if we delete for the time being the last term of (2.5) (i.e., the interaction term), we find a degenerate case (i.e., a binary choice) clearly belonging to the family of May equations illustrated in (2.1).

In particular if we simply put:

$$N = \alpha_j + 1 \qquad (2.6)$$

we have the following expression for a dynamic (degenerated) logit model:

$$P_{j,t+1} = N P_{j,t} [1 - P_{j,t} (N-1)/N] \qquad (2.7)$$

It is evident that if we make the transformation:

$$X_{j,t} = P_{j,t} (N-1)/N \qquad (2.8)$$

equation (2.7) can be written in the canonical form (2.9) (see also Wilson, 1981):

$$X_{j,t+1} = NX_j (1-X_j) \qquad (2.9)$$

Equation (2.9) is exactly May's law introduced in (2.1). Here X_j assumes the meaning of the probability of choosing the first alternative in a binary choice situation; therefore we can deduce the complementary probability as $(1-X_j)$. Thus we can apply May's results on stability (see the bifurcation diagram in Fig. 1) to degenerate dynamic logit models.

This interesting result is novel in social science research, since it opens the possibility of having chaotic behaviour also in decision processes of a logit type (i.e., discrete choice problems). Obviously in this case the initial conditions and the values of the parameter N appear to be critical for the emergence of unstable behaviour.

A further step, which will be the subject of the present paper is the analysis of delay effects in such logit-choice models of a May type. In particular we will analyze here the model of type (2.9) which is equivalent to model (2.1), by considering its theoretical derivations based on utility theory which could offer new interpretations in behavioral choice processes.

Time delays in the growth dynamics of populations have been considered by many authors, especially in mathematical ecology. We may refer here to the first work by Hutchinson (1948) and Maynard Smith (1974) who argue that if the duration of delays are longer than the natural period of the system (i.e.1/N) divergent oscillations will result. Of course in mathematical ecology the study of time-delay effects is essential, since in a real ecosystem resources are self-renewing. Consequently the actual level of resources available at any time depends on the density of the regulating species at a time in the past. Nevertheless also in human decision processes, time lag effects are fundamental, since the decision process is often governed by a delayed response (i.e., the choice process is often not instantaneous) in a choice model, for instance, because of complicated learning and feedback effects. Moreover the study of delay effects in dynamic logit models allows one also to consider the interactions between people in space and time; these interacting choices cannot be incorporated in a static logit model owing to the well-known IID hypothesis (i.e., the random parts of the individual are Independent and Identically Distributed).

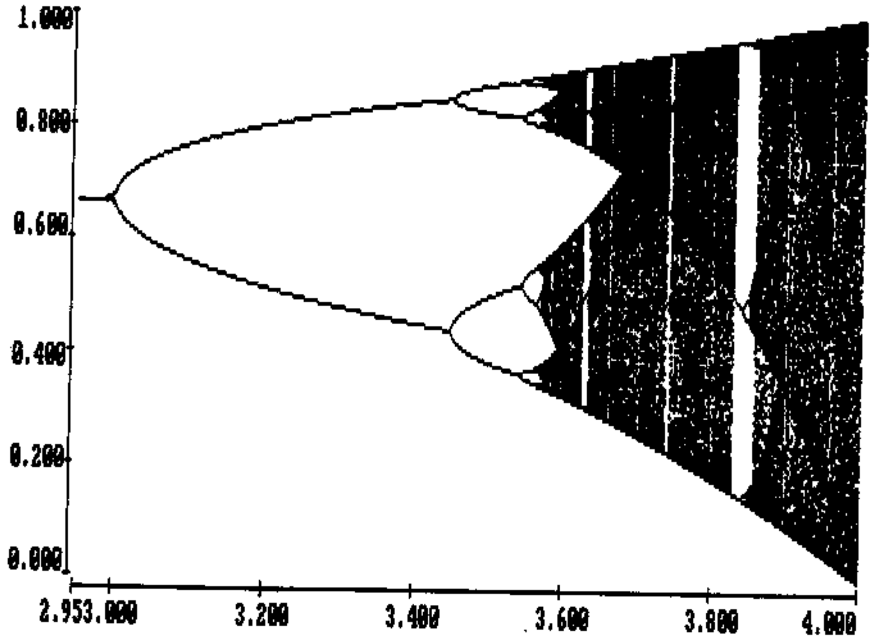

Fig.1. The logit map related to a dynamic binary logit model
y-axis: P_j; x-axis:N

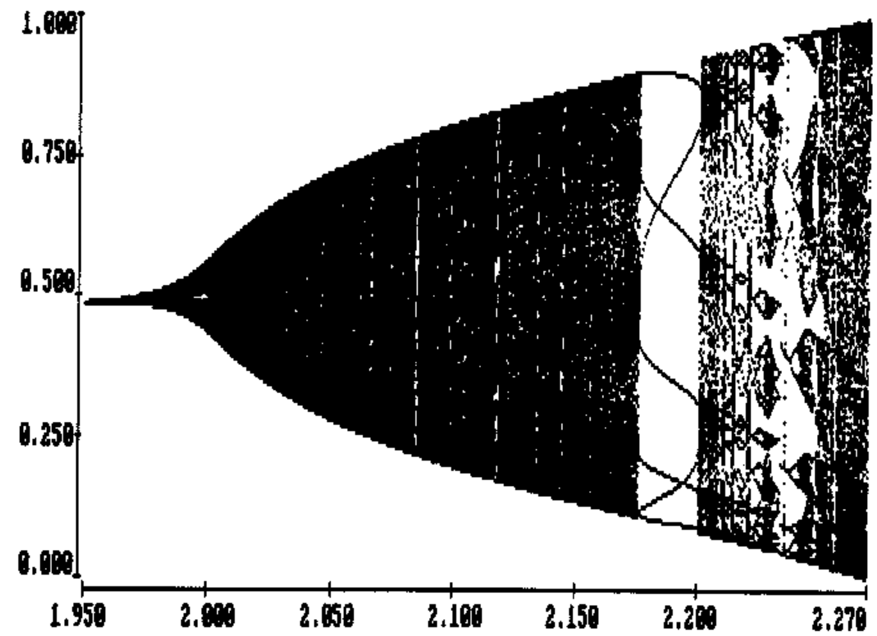

Fig.2. Bifurcation diagram for the logit map with s=1
y-axis: P_j; x-axis:N

In the next section we will present a more detailed analysis of delay effects in growth models of a combined May/Logit type, by using computer experiments based on bifurcation diagrams.

3. Delay Effects in Dynamic (Binary) Logit Models

In the ecological literature one frequently finds statements that time delays lead to destabilizing effects (see for example. Maynard Smith, 1974; McDonald, 1976 and Rose, 1987).
However, it is also increasingly realized that time delays are not necessarily destabilizing (see, e.g., Cushing and Saleem, 1982 and Hastings, 1983). Additionally, Saleem et al. (1987) confirmed this previous finding by arguing that increasing (decreasing) time delays are not necessarily destabilizing (stabilizing). It should be noted that these analysis have been mostly carried out by reference to a two - or three - dimensional continuous system, such as the Lotka-Volterra type of equations.

In this paper we will analyze delay effects in first- order difference equations of a May type, which constitute nowadays the basis of a large series of logistic growth models, commonly used in many disciplines, such as economics, social sciences, geography, etc.
An illustration of such a type of modelling in the area of urban dynamic systems can be found in Johansson and Nijkamp (1987), where the urban development follows a logistic discrete growth process characterized by a multi-episode history. In this context an event, i.e. the transition from one episode to another, is able to switch from stability to instability.

Another interesting analysis of the impact of the past has been carried out by Cugno and Montrucchio (1984) in the context of adaptive expectations. Their results lead to the conclusion that chaos seems to vanish by increasing the weight of the past in rational expectations; however, even though the parameter related to delayed effects appear to be very high, there is always the possibility of unstable behaviour. In order to investigate the impact of delay effects in a growth model of a May type, we will here formalize the following model, based on the dynamic logit model (2.9), on the basis of an ecological specification introduced by Maynard Smith (1968):

$$P_{j,t+1} = N\, P_{j,t} \quad (1 - P_{j,t-s}) \tag{3.1}$$

where t-s indicates delay effects involving s time lags. In other words, we model the growth of the population share P_j choosing option j whose ability to grow in any given time span is governed by the population in the previous time span. In general, dynamic models are from a behavioural viewpoint very rigid in terms of delay effects, although in reality behaviour may be sensitive to the lag structure in view of learning effects, different multi-period data bases etc.

It is now clear that when s=0 we get the first differential equation of a May type (see equation (2.9)), described in the previous section. Let us now consider the case of s=1, i.e. the following equation:

$$P_{j,t+1} = N\, P_{j,t} (1 - P_{j,t-1}) \tag{3.2}$$

A standard procedure in studying complex or chaotic behaviour is to draw a "bifurcation diagram" in which the value of the variable is plotted against the parameter value. Consequently, if we then examine the bifurcation diagram emerging from (3.2) (see Fig. 2), we observe a new shape, collecting cycles, and a cascade of bifurcations (see the relative blow-up in Fig.3).

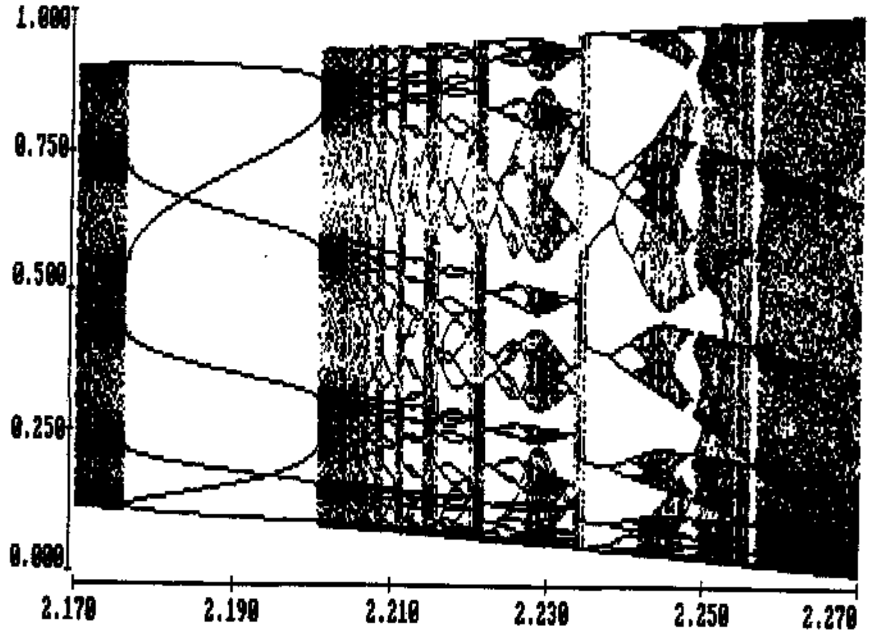

Fig.3. A blow-up of Figure 2

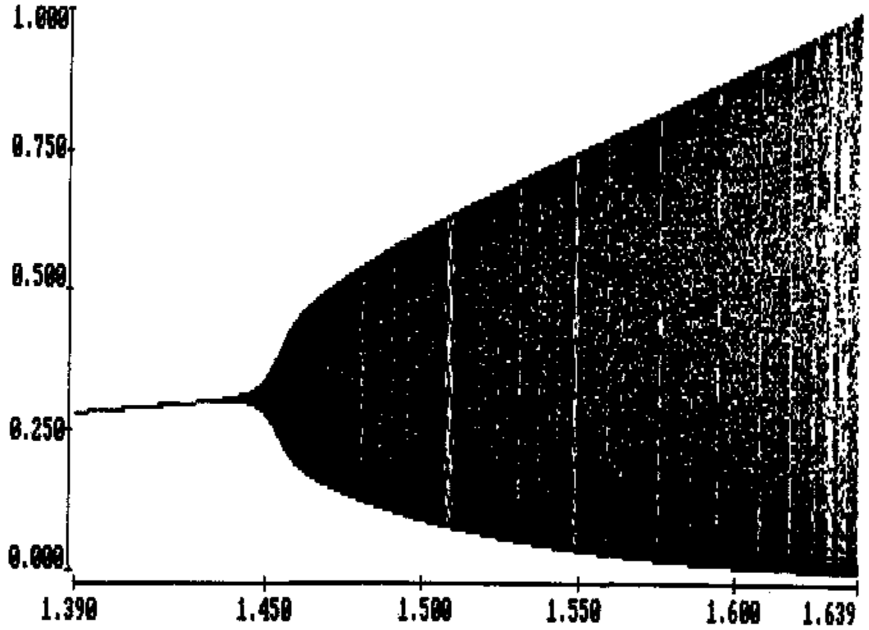

Fig.4. Bifurcation diagram for the logit map with s=3
y-axis: P_j; x-axis:N

It should also be noted that the model of type (3.2) has been analytically investigated by several authors, such as Aronson et al. (1982), Lauwerier (1986) and Pounder and Rogers (1980). By using a stability analysis in a two equation system, these authors find instability for $\approx 2 < N < 2.27$, just confirming the results from our bifurcation diagram. Moreover they show for N=2.27 the existence of a strange attractor of the Hénon type (see Hénon, 1976).

Our next step will be the analysis of further delays, by considering the following values for s (s=2; s=3; s=7; s=20).

By observing the bifurcation diagrams related to the above time delays, it is easy to see that the bell shape tends to shrink by increasing the time delays. Figures 4, 5 and 6, illustrate the evolution of the shape of the chaotic region for some values of s.

It is interesting to observe the type of irregular behaviour inside these complex regions. For example, in Figures 7 and 8 we have illustrated some enlargements of the bifurcation diagram related to Figure 5 (with s=7).

The first part of the diagram shows an irregular shape, while it is difficult to read, on this scale, the movements of cycles (see Fig. 7). However, in the second part of the diagram the windows are more clear and we can see the emergence of a large number of cycles giving rise to a sequence of bifurcations, probably leading to a strange attractor of 7 dimensions around the value N= 1.345 (see Fig. 8). It seems evident that Fig. 8 represents, in a more compact form, the unstable behaviour that is well illustrated in Fig. 3.

Having seen the existence of an impact of an specification of a time-delayed model on the results, it is now interesting to analyze in a thorough way the above results. This will be the subject of subsection 3.2.

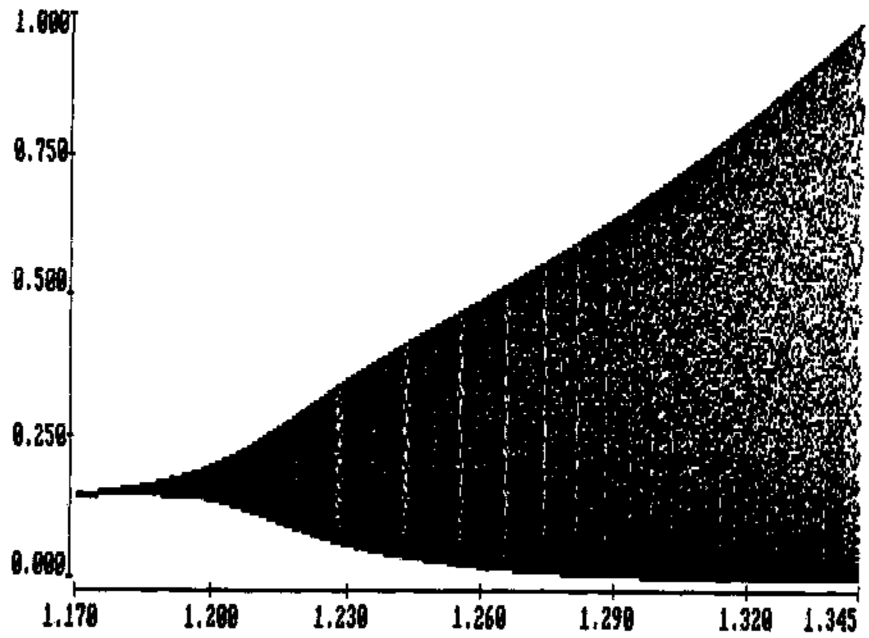

Fig.5. Bifurcation diagram for the logit map with s=7
y-axis: P_j; x-axis:N

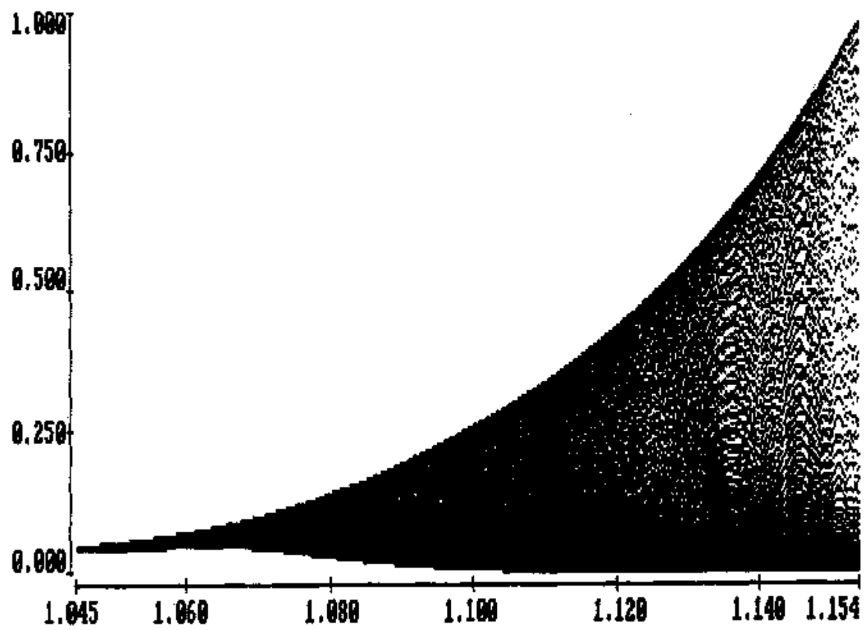

Fig.6. Bifurcation diagram for the logit map sith s=20
y-axis: P_j; x-axis:N

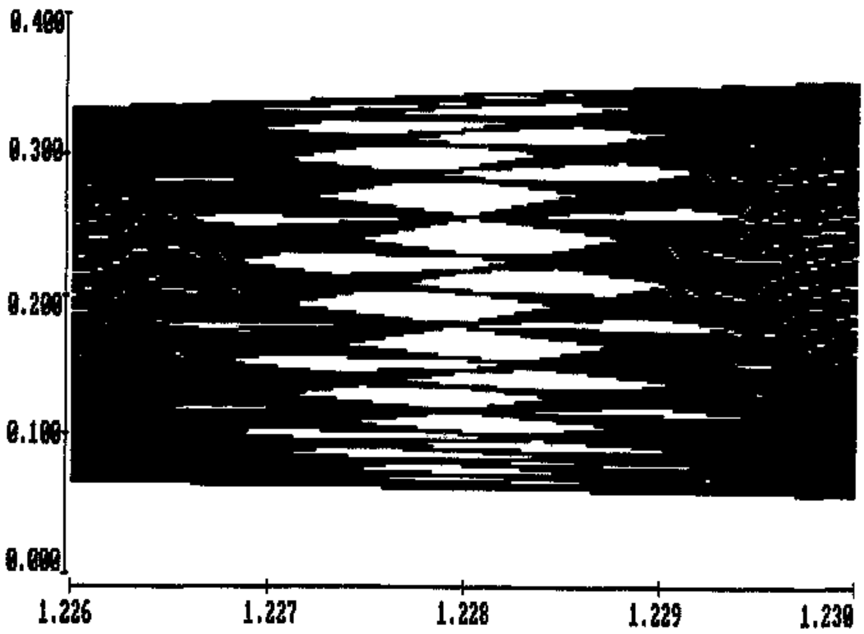

Fig.7. A blow-up of the diagram illustrated in Figure 5

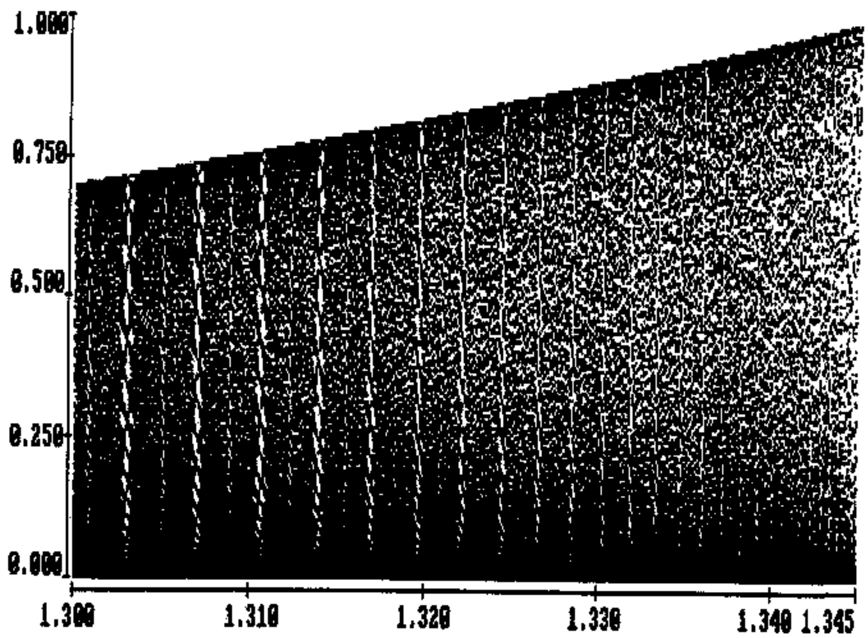

Fig.8. A second blow-up of the diagram illustrated in Figure 5

3.2 Stability and Instability

The above experimental examples appear to lead to results which are in a succinct way illustrated in Table 1.

From Table 1 and from Figures 2, 4, 5, 6 it is clear that by increasing the value of s (i.e., the number of delays) the onset of instability decreases, depending on the N values. However, it also appears that as the delays are growing, the speed of change of N is lower.

This confirms some previous statements by Cartwright (1984) who claims that several variations on the logistical difference model still produce a chaotic regime, although at an earlier point (see also May, 1976). Similarly, also the upper limit of instability decreases less strongly after an increase of the values of s.

As regards the field of stability, we can see that also the stability region decreases by increasing the values of s. Obviously the upper limit of this region is equal to the inferior limit of the unstable region, so that it follows the same decreasing pattern.

In order to get a clearer representation of the shrinking stable-unstable areas, we will plot in Figure 9 the values of s against the values of $\alpha = N-1$, according to the values represented in table 1. It should be noted that here we prefer to deal with the parameter α, since this is more meaningful in a behavioural process of a logit type where α represents the marginal utility function (see equation 2.3).

The first observation emerging from Figure 9 is that after a great number of delays (approximately 20) the region of stability and instability is monotonically approaching the horizontal axis. This means that after many delays both stable and unstable regions are very thin; for example, for $s=20$ the stable region exists for $0<\alpha<0.05$ and the unstable for $0.05 < \alpha < 0.15$. Consequently, it is very easy in these areas to switch from stability to instability, for very small changes in the parameter.

We can transfer this result from population dynamics, by linking it to the Johansson and Nijkamp model (1987). In other words, _if_ there is a regulatory effect due to a large influence of the past, a new event (i.e., a small change) can transform the growth process

Table 1. Stability and Instability in logit maps with delay effects

s= number of delays	stability or periodic behaviour	complex behaviour with chaos
0	$0 < N < 3$	$\approx 3 < N < 4$
1	$0 < N < 2$	$\approx 2 < N < 2.27$
2	$0 < N < 1.6$	$\approx 1.6 < N < 1.84$
3	$0 < N < 1.4$	$\approx 1.4 < N < 1.64$
7	$0 < N < 1.2$	$\approx 1.2 < N < 1.34$
20	$0 < N < 1.05$	$\approx 1.05 < N < 1.15$

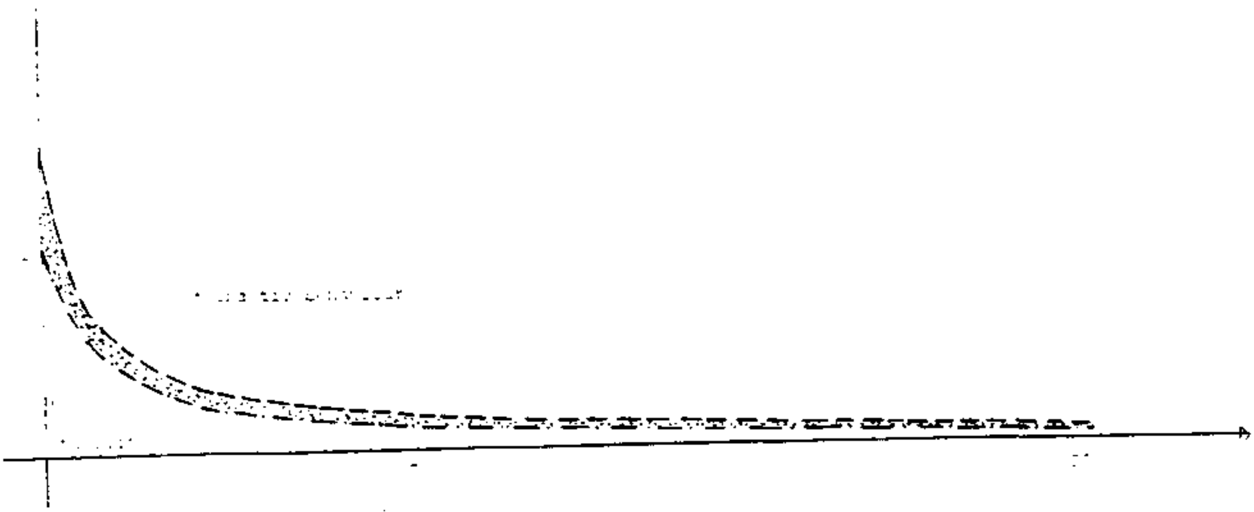

Fig.9. Representation of stable and unstable areas for increasing delay effects

from stability to instability. Similarly, in the context of choice processes, we may conclude that <u>if</u> the influence of the past is high, a small change in the utility function may lead to a switch from stable to unstable behaviour and vice versa. This result is also interesting since it reinforces the conclusion reached by Hastings (1983) and Saleem et al (1987) in a Lotka-Volterra ecological problem and by Cugno and Montrucchio (1984) in adaptive behaviour, as already introduced in Section 3.1.

A further interesting observation is also the following. When the influence of the past is high, the conditions for the existence of a growth process of a logit type are also very restrictive, since the interval value of α is shrinking more and more. For example, for s=20 the interval value necessary for feasibility is $0<\alpha<0.15$!
This also means that, since α represents the variation of the utility function, by increasing the influence of the past the utility function tends to approach asymptotically a constant value K (see Fig. 10).

In a certain sense this also means that many delay effects tend to provide a sort of 'capacity level' K in the utility function, even though small changes can produce either stability or instability.

Moreover, after a certain number of generations (when u->K) the past has no impact anymore on the utility function. Consequently, it is then plausible to assume a utility function $u=\alpha t$, which results as follows in case of a series of influences from the past:

$$u = \alpha t + K$$

where K represents the maximum utility level due to the impact of the past. Therefore, in the initial phase of a choice problem (i.e., t=0), the decision process contains an 'a priori' utility level based on previous histories (see Fig. 11).

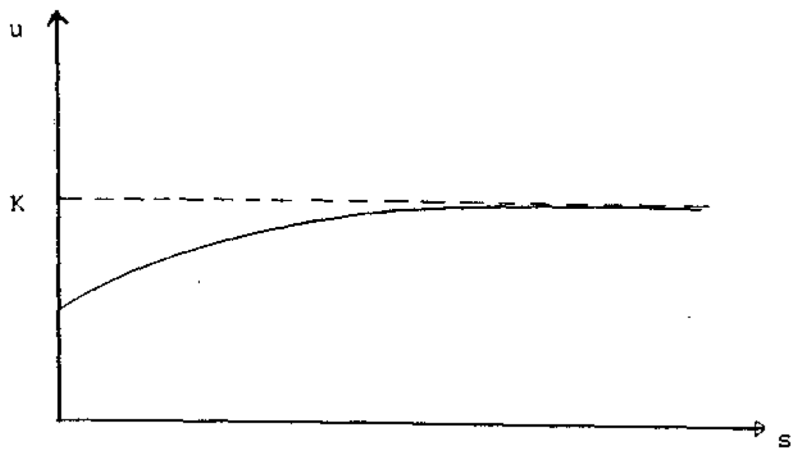

Fig. 10 The growth process of the utility function
as a function of the number of delays

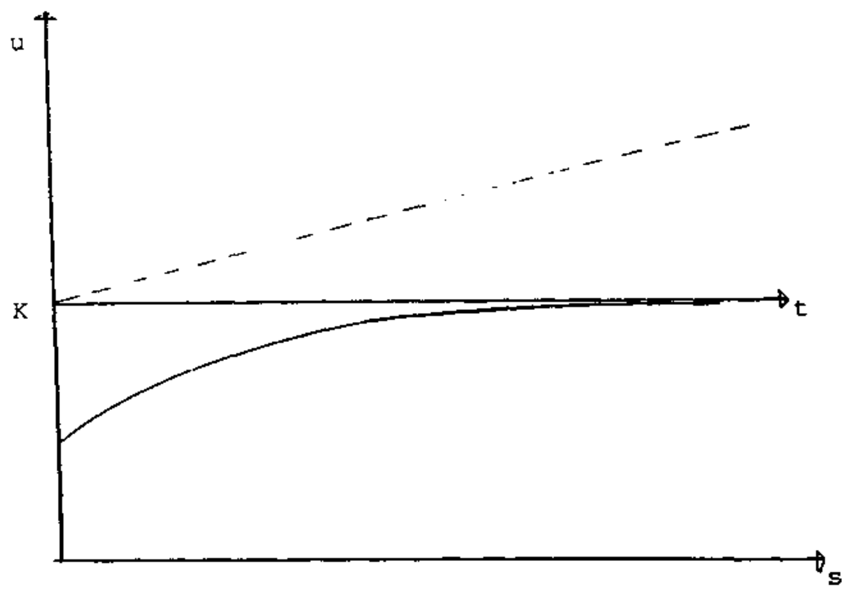

Fig. 11 Impact of the past in the dynamic utility function

4. Concluding Remarks

The previous experiments have demonstrated the impact of time specifications in spatial models; such models tend to exhibit declining irregular behaviour, depending of course on the growth rates and initial conditions.

Another point concerns the specification of non-linear dynamic models. It may be important to stress that the foundations of specifying a dynamic model would have to be firmly rooted in social science theory, as otherwise we run the danger of ad hoc and mis-specifications, which may create chaotic behaviour that is not based on plausible behavioural grounds.

Finally, it is important to call attention to the fact that often a dynamic system with multiple period delays is not chaotic on the trajectory as a whole, but has only a few areas which under certain conditions may exhibit chaotic behaviour. The question whether chaotic behaviour in a certain limited span may be dampened by the dominance of stable behaviour in another range, or whether it will exert an explosive influence upon the whole dynamic trajectory needs further investigation.

Although non-linear models for spatial behaviour may provide an interesting explanatory framework for the dynamics of spatial systems, it is also evident from the above experiences that the theory of chaos desperately needs more rigorous empirical research work.

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