Faculteit der Economische Wetenschappen en Econometrie

Serie Research Memoranda

A Note On Pareto Laws

A.H.Q.M. Merkies & I.J. Steyn

September 1991
Research Memorandum 1991-56

vrije Universiteit *amsterdam*

Faculteit der Economische Wetenschappen en Econometrie

Serie Research Memoranda

A NOTE ON PARETO LAWS

A.H.Q.M. Merkies & I.J. Steyn

Research Memorandum 1991-56

September 1991

vrije Universiteit amsterdam

A NOTE ON PARETO LAWS

A.H.Q.M. Merkies & I.J. Steyn

1. ESTEBAN [1986] launches the Weak-Weak Pareto Law (WWPL) for a set $\mathcal{F}$ of densities f characterized by, among other things, finite means. He claims that a.) for all $f \in \mathcal{F}$ this law is weaker than the Weak Pareto Law of MANDELBROT [1960] and b.) the subset $\mathcal{F}_1$ of $\mathcal{F}$ of densities satisfying this law together with two additional conditions, viz. mode existence and a constant declining rate of income elasticity, coincides with $\mathcal{G}$, the class of generalized gamma densities.

This note shows that :

(I) There exists a class of functions $\mathcal{Z}$ largely overlapping with $\mathcal{F}$ in which WWPL is stronger than WPL : $(\mathcal{Z} \cap \text{WWPL}) \subset \text{WPL}$.
$\mathcal{Z}$ includes both functions with and functions without a finite mean.

(II) There are no densities with a constant declining rate of elasticity. As a result, $\mathcal{F}_1$ is empty.

2. The Pareto distribution defined as:

(1) $$F(x) = 1 - (x/x_0)^{-\alpha} \qquad x \geq x_0,\ \alpha > 0$$

has lead to the <u>Strong</u> <u>Pareto</u> <u>Law</u> (SPL):

(2) $$\lim_{x\to\infty} \frac{(x/x_0)^{-\alpha}}{1 - F(x)} = 1 \qquad x \geq x_0,\ \alpha > 0$$

A weaker version is the <u>Weak</u> <u>Pareto</u> <u>Law</u> (WPL):

(3) $$\lim_{x\to\infty} \frac{(x/x_0)^{-\alpha}}{1 - F(x)} = 1 \qquad x \geq x_0,\ \alpha > 0$$

This weaker law, but in a reciprocal form, originates with MANDELBROT [1960]. It is weaker because SPL ⇒ WPL and WPL does not imply SPL as numerous examples may show.

KAKWANI [1980, p.15] defines the elasticity:

(4) $$K(x) = \frac{x\ f(x)}{1 - F(x)}$$

As in (2) $(x/x_0)^{-\alpha} = \frac{x\ f(x)}{\alpha}$, an alternative formulation of the laws is:

(5) SPL : $$\frac{1}{\alpha} K(x) = 1 \quad \text{or} \quad K(x) = \alpha \quad \alpha > 0.$$

(6) WPL : $$\lim_{x \to \infty} \frac{1}{\alpha} K(x) = 1 \quad \text{or} \quad \lim_{x \to \infty} K(x) = \alpha \quad \alpha > 0.$$

KAKWANI [1980, p.16] writes the derivative of K(x) as:

(7) $$K'(x) = \frac{K(x)}{x} \left[1 + k(x) + K(x) \right],$$

where $k(x) = \frac{xf'(x)}{f(x)}$ is called the elasticity of the density by Kakwani.

Elsewhere this term is reserved for $e(x) = -k(x)$, see ESTEBAN [1978], or for $\pi(x) = 1 + k(x)$, ESTEBAN [1986], assuming that his definition, see ESTEBAN [1986, p.441] is meant to be:

(8) $$\frac{1 - \pi(x)}{x} = - \frac{f'(x)}{f(x)}$$

hence with a minus sign on the right hand side.

ESTEBAN [1986] formulates his Weak Weak Pareto Law (WWPL) as

(9) WWPL : $$\lim_{x \to \infty} \pi(x) = -\alpha$$

and remarks that both MANDELBROT [1960] and MIRRLEES [1971] take (9) as equivalent to (3).
Esteban himself is of the opinion that (9) is weaker than (3).

3. A link between WPL and WWPL is provided by l'Hospital's Rule, stating (see e.g. RUDIN[1976, p.109]) that if :

A. lim $p'(x)/q'(x)$ exists

B. $q'(x) \neq 0$

C. lim $p(x) = 0$

D. lim $q(x) = 0$

lim $p(x)/q(x)$ exists and is equal to lim $p'(x)/q'(x)$. Substituting

$$p(x) = xf(x) \quad \Rightarrow \quad p'(x) = f(x) + xf'(x)$$
$$q(x) = 1 - F(x) \quad \Rightarrow \quad q'(x) = - f(x)$$

then gives that if:

a. $\lim_{x\to\infty} [f(x) + xf'(x)]/[-f(x)]$ exists (WWPL holds)

b. $f(x) \neq 0$

c. $\lim_{x\to\infty} xf(x) = 0$

d. $\lim_{x\to\infty} [1 - F(x)] = 0$

then

$$\text{(10)} \qquad \lim_{x\to\infty} K(x) = \lim_{x\to\infty} \frac{x\ f(x)}{1 - F(x)} = \lim_{x\to\infty} \frac{f(x) + xf'(x)}{- f(x)} = - \lim_{x\to\infty} \pi(x).$$

Of the four conditions listed above, condition d. is automatically satisfied, while condition b. is one of the characteristics assumed by Esteban for his densities.

Hence, if $\mathcal{Z}$ is defined as $\mathcal{Z} = \{f \mid \lim_{x\to\infty} xf(x) = 0\}$, then for all $f \in \mathcal{Z}$ satisfying WWPL (condition a.) according to (10) WPL also applies. In other words, for all $f \in \mathcal{Z}$ we have WWPL $\Rightarrow$ WPL. As there exists at least one $f \in \mathcal{Z}$ for which WPL applies but WWPL does not (see Lemma 1 in the Appendix), the reverse implication does not hold. This proves the first part of statement I.
For the second part, we note that the class $\mathcal{Z}$ includes Pareto densities with $0 < \alpha \leq 1$, which do not have a finite mean, and for which both WWPL and WPL hold. Conversely, there are many functions with a finite mean which are also in $\mathcal{Z}$. This proves the second part of statement I.

4. ESTEBAN [1986] confines his analysis to $\mathcal{F}$, defined as the class of density functions f with support [a,b], $0 \le a < b = \infty$, such that: i) f has finite mean, ii) f is C^1 in (a,b), and iii) $f(x) > 0$ for all $x \in (a,b)$, and $f(x) = 0$ otherwise.

We have not used this class for several reasons. In the first place, the requirement that the density has a finite mean is not used explicitly by Esteban in his analysis, and is not a natural property to require of a continuous income distribution function. Second, we note with interest that if a function satisfies WPL it must be a member of our class $\mathcal{Z}$. This can be seen by noting that if $\lim_{x\to\infty} xf(x)/[1 - F(x)]$ is finite, then $\lim_{x\to\infty} xf(x)$ must be 0. In other words, WPL $\subset \mathcal{Z}$ and $\mathcal{Z}$ could thus be considered a better candidate for a "Weak Weak Pareto Law" than Esteban's WWPL, for which a similar property does not hold, see Lemma 1 in the Appendix.

We note in passing that $\mathcal{Z}$ is not a subset of $\mathcal{F}$ (as there are functions with a finite mean for which xf(x) does not approach 0, see Lemma 2 in the Appendix) and that $\mathcal{F}$ is not a subset of $\mathcal{Z}$ (as e.g. the aforementioned Pareto densities with $0<\alpha\le1$ show). We still doubt if there is *any* density which satisfies WWPL but which does not satisfy WPL. The above result shows that such a density must be sought outside $\mathcal{Z}$, and we might conjecture that no such density exists.

5. There are many densities in $\mathcal{F}$ that do not satisfy either WPL or WWPL. A spicy example is the two parameter Weibull (or Extreme-value) distribution,

(11) $$F(x) = 1 - e^{-\gamma x^p} \qquad x \ge 0,\ p > 0,\ \gamma > 0$$

the simplest member of the family discovered by WEIBULL [1951]. Its density is

(12) $$f(x) = p\gamma x^{p-1}e^{-\gamma x^p}.$$

As the mean $\mu = [\frac{1}{\gamma}]^{1/p}\ \Gamma(1/p + 1)$ is finite within the given domain of γ and p, this density is clearly in $\mathcal{F}$. For this density we have from (4):

(13) $$K(x) = \gamma p x^p,$$

which for $x \to \infty$ approaches infinity for $p > 0$. Hence WPL does not apply.

Acknowledging that

(14) $$\pi(x) = \frac{x\ K'(x)}{K(x)} - K(x) = p - K(x)$$

we get :

(15) $$\pi(x) = p - \gamma p x^{p},$$

which for $x \to \infty$ and $p > 0$ does not converge to a finite value. Hence for the two parameter Weibull with $p > 0$ neither WPL nor WWPL applies.

The Weibull distribution has a mirror image in the Inverse Weibull :

(16) $$F(x) = e^{-\gamma x^{p}} \qquad x \geq 0,\ p < 0,\ \gamma > 0$$

(17) $$f(x) = -p\gamma x^{p-1} e^{-\gamma x^{p}}$$

Using (4) we get:

(18) $$K(x) = -p\gamma x^{p-1} e^{-\gamma x^{p}} \left[1 - e^{-\gamma x^{p}}\right]^{-1} \longrightarrow -p \text{ as } x \to \infty \text{ (l'Hospital)}$$

while (8) gives:

(19) $$\pi(x) = p - \gamma p x^{p} \longrightarrow p \text{ as } x \to \infty.$$

The Inverse Weibull therefore satisfies both WWPL and WPL. Note that we could have confined ourselves to proving WWPL, as the Inverse Weibull is in $\mathcal{Z}$.
The two parameter Weibull, considered above, is a special case (q=1) of the generalized gamma density, discovered by STACY [1962]:

(20) $$f(x) = |p|\gamma^{q} x^{pq-1} e^{-\gamma x^{p}} / \Gamma(q) \qquad x \geq 0,\ p \neq 0,\ q > 0,\ \gamma > 0$$

If Esteban's statement were true the class $\mathcal{G}$ would be characterised by (20). These densities would satisfy WWPL and give rise to a declining elasticity function $\pi(x)$, whatever the value of q and p. But, as the example above shows, this is clearly not true for values of $p > 0$ and $q = 1$.
This gives an indication that Esteban's statement that $\mathcal{F}_1$ coincides with $\mathcal{G}$ is incorrect.

6. ESTEBAN [1986] searches for distributions satisfying the following three hypotheses:

Hypothesis 1 : WWPL

Hypothesis 2 : The mode m exists.

Hypothesis 3 : $\pi(x)$ has a constant declining rate.

Distributions satisfy hypothesis 3, if

$$(21) \qquad \frac{d \ln \pi'(x)}{d \ln x} = \delta \qquad \delta \le 0.$$

or

$$(22) \qquad d \ln \pi'(x) = \delta \, d \ln x \qquad \delta \le 0.$$

Integration gives:

$$\ln \pi'(x) = \delta \ln x + \ln \mu \qquad \mu > 0, \quad \delta \le 0$$

$$\pi'(x) = \mu x^{\delta} \qquad \mu > 0, \quad \delta \le 0$$

$$d\, \pi(x) = \mu x^{\delta} \, dx \qquad \mu > 0, \quad \delta \le 0$$

Further integration gives

$$(23) \qquad \pi(x) = -\alpha + \frac{\mu}{\delta+1} x^{\delta+1} \qquad \text{if } \delta \le 0 \text{ but } \ne -1, \ \alpha \in \mathbb{R}, \ \mu > 0$$

$$(24) \qquad = -\alpha + \mu \ln x \qquad \text{if } \delta = -1, \ \alpha \in \mathbb{R}, \ \mu > 0.$$

(and not $\pi(x) = -\alpha + \mu/\ln x$, see ESTEBAN [1986 formula (5) on p.443])

As

$$(25) \qquad \pi(x) = 1 + \frac{d \ln f(x)}{d \ln x}$$

the concomitant densities are obtained by integration. For (24) this gives:

$$d \ln f(x) = (-1 - \alpha + \mu \ln x) \, d \ln x$$

or

$$(26) \qquad f(x) = \lambda x^{-1-\alpha} \exp(\tfrac{1}{2}\mu \ln^2 x) \qquad \mu > 0, \ \lambda > 0$$

which cannot be a density as $\lim_{x \to \infty} f(x) = \infty$.

For (23) we have:

$$d \ln f(x) = (-1 - \alpha + \frac{\mu}{\delta+1} x^{\delta+1}) \, d \ln x \qquad \delta \le 0 \text{ but } \ne -1, \ \alpha \in \mathbb{R}, \ \mu > 0$$

$$= (-1 - \alpha) d \ln x + \frac{\mu}{\delta+1} x^{\delta+1} d \ln x$$

and thus:

$$\text{(27)} \qquad f(x) = \lambda x^{-1-\alpha} \exp(\frac{\mu}{(\delta+1)^2} x^{\delta+1}) \qquad \delta \le 0, \ \delta \ne -1, \ \mu > 0, \ \lambda > 0$$

(Note that $\alpha \in \mathbb{R}$ and not $\alpha > 0$ and $\varepsilon = -(\delta + 1) > -1$ and not $\varepsilon > 0$, see ESTEBAN [1986, p.444])

Comparison with (20) shows:

	$\delta + 1$	=	p	$\delta \le 0, \ \delta \ne -1$	$\Rightarrow$	$p \le 1, \ p \ne 0$
(28)	μ	=	$-\gamma p^2$	$\mu > 0$	$\Rightarrow$	$\gamma < 0$
	α	=	$-qp$	$\alpha \in \mathbb{R}$	$\Rightarrow$	$q \in \mathbb{R}$
	λ	=	$\lvert p \rvert \gamma^q / \Gamma(q)$	$\lambda > 0$	$\Rightarrow$	$q \in \mathbb{N}$

However, for $\gamma < 0$ (20) does not represent a density. Hence there is no density with a constant declining rate of elasticity. The conclusion is that the set $\mathcal{F}_1$ of densities satisfying all of ESTEBAN's requirements is empty.
This proves statement II.

7. Since Esteban's third demand is impossible to satisfy, we can drop it. We are then left with two demands for densities to satisfy : WWPL and the existence of a mode. Obviously there are many densities satisfying these two demands, the Inverse Weibull being an example. In fact, all generalised gamma densities with $p < 0$ fall into this category. These functions are usually called Inverse Gamma Functions, or Pearson's Type V, see RAIFFA & SCHLAIFER [1961, chapter 7]. These densities also lie in $\mathcal{F}$ as their mean is $[\frac{1}{\gamma}]^{1/p}\Gamma(\frac{p+1}{p})/\Gamma(q)$ which is finite in the relevant domain, and also in $\mathcal{Z}$. The validity of WPL for the set of inverse gamma functions had already been established by KLOEK and VAN DIJK [1976], see also KLOEK and VAN DIJK [1978].

8. Finally it must be added that WPL is an observed phenomenon and not a requirement from theory, see MERKIES [1987]. This means that if some density is a good approximation of an empirical frequency distribution, it is irrelevant whether it also satisfies WPL (or WWPL for that matter).

Appendix : Two counterexamples

NOTE : unless otherwise stated, all densities considered here are defined as follows : $f(x) > 0 \ \forall\ x \in [0,\infty)$, $f(x) = 0 \ \forall\ x < 0$. $f \in C^1$.

We will find the Weibull-representation of densities (see WEIBULL [1951]) useful :

$$F(x) = 1 - \exp[-\varphi(x)]$$

with $\varphi(x)$ satisfying the following four conditions :

1. $\varphi(x) > 0$
2. $\varphi(x)$ nondescending
3. $\varphi(0) = 0$
4. $\varphi(\infty) = \infty$

A sufficient condition for $f \in C^1$ is that $\varphi(x) \in C^2$. Combined with condition 2. above we then get as corrolary that $\varphi'(x) \geq 0$.

We find that

$$f(x) = \varphi'(x)\exp[-\varphi(x)]$$

$$K(x) = xf(x)[1-F(x)]^{-1} = x\varphi'(x)$$

$$\pi(x) = 1 + xf'(x)/f(x) = 1 - x\varphi'(x) + x\varphi''(x)/\varphi'(x)$$

We can now reformulate in terms of this representation :

Weak Pareto Law : (Mandelbrot) (notation : $f \in W_1$)

$$K(x) \rightarrow \alpha > 0 \quad \Leftrightarrow$$

$$x\varphi'(x) \rightarrow \alpha \quad \Rightarrow$$

$$\varphi'(x) \rightarrow 0$$

Weak Weak Pareto Law : (Esteban) (notation : $f \in W_2$)

$$\pi(x) \rightarrow \beta < 0 \quad \Leftrightarrow$$

$$1 - x\varphi'(x) + x\varphi''(x)/\varphi'(x) \rightarrow \beta$$

$\mathcal{F}$-class : (Esteban) (notation : $f \in \mathcal{F}$)

f(x) has a finite mean

$\mathcal{Z}$-class : (Merkies & Steyn) (notation : $f \in \mathcal{Z}$)

$$xf(x) \rightarrow 0 \quad \Leftrightarrow$$

$$x\varphi'(x)\exp[-\varphi(x)] \rightarrow 0$$

Lemma 1 : $W_1 \setminus W_2 \neq \emptyset$

Proof

We can rephrase this Lemma as $W_1 \setminus (W_2 \cap W_1) \neq \emptyset$. Note that for $f \in W_2 \cap W_1$, $1 - x\varphi'(x) + x\varphi''(x)/\varphi'(x) \to \beta$ and $x\varphi'(x) \to \alpha$. This leads to the following useful result :

$$f \in (W_1 \cap W_2) \Leftrightarrow x^2\varphi''(x) \to \alpha(\alpha + \beta - 1) \stackrel{d}{=} \gamma$$

We will therefore construct a density f with the following properties :

$$(x\varphi'(x) \to \alpha > 0) \wedge \neg(x^2\varphi''(x) \to \gamma)$$

This is, of course, in addition to the four properties for the Weibull representation listed at the start of the Appendix.

A function $\varphi'(x)$ satisfying these requirements is

$$\varphi'(x) = \alpha x^{-1} + (1 + \sin x)x^{-2}$$

with

$$\varphi''(x) = -\alpha x^{-2} + x^{-2}\cos x - 2(1 + \sin x)x^{-3}$$

since

$$x\varphi'(x) = \alpha + (1 + \sin x)x^{-1} \to \alpha$$

$$x^2\varphi''(x) = -\alpha + \cos x - 2(1 + \sin x)x^{-1} \to -\alpha + \cos x$$

We therefore have the required counterexample by taking $f = \varphi'(x)\exp[-\varphi(x)]$, with $\varphi'(x)$ as above. f can not be written explicitly, and we must still check the other demands placed on the function $\varphi(x)$. Note that $\varphi'(x) > 0 \ \forall\ x$, so $\varphi(x)$ is nondecreasing. To prove that $\varphi(\infty) = \infty$, we note that :

$$\alpha \ln x \leq \alpha \ln x + \int(1 + \sin x)x^{-2}dx$$

or

$$\alpha \ln x \leq \varphi(x)$$

proving that $\varphi(x)$ approaches infinity as x increases.

The exact behaviour of $\varphi(x)$ near 0 is not crucial. Since it is a nondecreasing function there are three possiblities :

1. $\varphi(x) \to 0$. In which case all is well without further ado.
2. $\varphi(x) \to C \neq 0$, in which case we use $\varphi^*(x) = \varphi(x) - C$ instead
3. $\varphi(x) \to -\infty$. In that case, there is a $x_o > 0$ with $\varphi(x_o) = 0$. We therefore use $\varphi^*(x) = \varphi(x-x_o)$ instead.

Through this construction, the condition that $\varphi(x) > 0$ is also satisfied. Finally, we note that $\varphi(x) \in C^2$ and that the associated density $f(x) > 0\ \forall x$.

The end result is a function $\varphi(x)$ and an associated density f(x) which satisfies all our requirements. ■

Lemma 2 : $\mathcal{F} \setminus \mathcal{Z} \neq \emptyset$

Proof

We construct a density with a finite mean for which xf(x) does not approach zero. Initally we will construct this density on $[1,\infty)$, a density on $[0,\infty)$ then follows easily.

We will use the following functions as building blocks :

$$b(x;a,n) = (2\pi)^{-1}\{1 - \cos(x-n)/a\}\ I(x,[n,n+2\pi a]) \qquad a,n > 0$$

where I(x,A) = 1 if $x \in A$, and 0 otherwise.

This function has the following properties :

$$b(x;a,n) \geq 0 \qquad \int b(x;a,n)dx = a \qquad \max b(x;a,n) = 1/\pi$$

As first intermediate result we construct a function

$$h(x) = \sum_{i=1}^{\infty} b(x;a_i,n_i) \qquad \text{with } \sum_{i=1}^{\infty} a_i = S < \infty \qquad \text{and } \lim_{i\to\infty} n_i = \infty$$

To avoid the individual building blocks overlapping we also require that

$$n_{i+1} \geq n_i + 2\pi a_i \qquad n_1 = 1$$

Note that $\int h(x)dx = S$.

As our next intermediate we take

$$g(x) = S^{-1}h(x)$$

which is a density function. The density function we seek for our counter-example then follows as :

$$f(x) = Ag(x)/x \qquad \text{with } A^{-1} = \int_1^{\infty} g(u)/u\ du$$

The latter integral exists, as $0 < g(u)/u < g(u)$ for $u>1$, and $\int g(u)du$ exists.

We note that :

$$xf(x) = Ag(x) = AS^{-1}h(x)$$

This function continues to vary between 0 and $AS^{-1}\pi^{-1}$ and therefore does not approach 0. Therefore, $f \notin \mathcal{Z}$. However,

$$\mathbb{E}x = \int xf(x)dx = AS^{-1}\int h(x)dx = A < \infty$$

Therefore $f \in \mathcal{F}$. Hence $\mathcal{F} \setminus \mathcal{Z} \neq \emptyset$. ∎

References:

Esteban, J.M., "A General Density Function for the Distribution of Income", Revised version(1978), Facultad de Ciencias Economicas y Empresariales, Universidad Autonoma de Barcelona, Barcelona.

Esteban, J.M., "Income-Share Elasticity, Density Functions and the Size Distribution of Income(1981). Mimeographed manuscript, University of Barcelona.

Esteban, J.M., "Income-Share Elasticity and the Size Distribution of Income", International Economic Review, 27-2 (1986), pp. 439-444.

Kakwani,N.C., "Income Inequality and Poverty. Methods of Estimation and Policy Applications", (1980), A World Bank Publication, Oxford University Press.

Kloek, T and H.K. van Dijk, "Efficient Estimation of Income Distribution Parameters", (1976), Report 76/16 Econometric Institute, Erasmus University Rotterdam.

Kloek, T and H.K. van Dijk, "Efficient Estimation of Income Distribution Parameters", Journal of Econometrics, 8(1978), pp 61-74

Mandelbrot, B., "The Pareto-Lévy Law and the Distribution of Income", International Economic Review, 1 (1960), pp. 79-106.

Merkies, A.H.Q.M., "Inductive analysis from empirical income distributions", in: Heijmans R. and H. Neudecker "The Practice of Econometrics", published in honor of J.S.Cramer, (1987), Martinus Nijhoff Publishers, Dordrecht.

Mirrlees, J.A., "An Exploration in the Theory of Optimum Income Taxation", Review of Economic Studies, 38 (1971), pp. 175-208.

Raiffa, H. and R. Schlaifer, "Applied statistical decision theory", (1961), Graduate School of Business Administration, Harvard University, Cambridge MA.

Rudin, W.,"Principles of Mathematical Analysis", 3rd ed., (1976), McGraw-Hill, Tokyo

Stacy, E.W., " A Generalization of the Gamma Distribution", Annals of Mathematical Statistics, 33 (1962), pp. 1187-1192.

Weibull, W. "A Statistical Distribution of Wide Applicability", Journal of Applied Mechanics, 18 (1951), pp. 293-297.

1991-1	N.M. van Dijk	On the Effect of Small Loss Probabilities in Input/Output Transmission Delay Systems
1991-2	N.M. van Dijk	Letters to the Editor: On a Simple Proof of Uniformization for Continious and Discrete-State Continious-Time Markov Chains
1991-3	N.M. van Dijk P.G. Taylor	An Error Bound for Approximating Discrete Time Servicing by a Processor Sharing Modification
1991-4	W. Henderson C.E.M. Pearce P.G. Taylor N.M. van Dijk	Insensitivity in Discrete Time Generalized Semi-Markov Processes
1991-5	N.M. van Dijk	On Error Bound Analysis for Transient Continuous-Time Markov Reward Structures
1991-6	N.M. van Dijk	On Uniformization for Nonhomogeneous Markov Chains
1991-7	N.M. van Dijk	Product Forms for Metropolitan Area Networks
1991-8	N.M. van Dijk	A Product Form Extension for Discrete-Time Communication Protocols
1991-9	N.M. van Dijk	A Note on Monotonicity in Multicasting
1991-10	N.M. van Dijk	An Exact Solution for a Finite Slotted Server Model
1991-11	N.M. van Dijk	On Product Form Approximations for Communication Networks with Losses: Error Bounds
1991-12	N.M. van Dijk	Simple Performability Bounds for Communication Networks
1991-13	N.M. van Dijk	Product Forms for Queueing Networks with Limited Clusters
1991-14	F.A.G. den Butter	Technische Ontwikkeling, Groei en Arbeidsproduktiviteit
1991-15	J.C.J.M. van den Bergh, P. Nijkamp	Operationalizing Sustainable Development: Dynamic Economic-Ecological Models
1991-16	J.C.J.M. van den Bergh	Sustainable Economic Development: An Overview
1991-17	J. Barendregt	Het mededingingsbeleid in Nederland: Konjunktuurgevoeligheid en effektiviteit
1991-18	B. Hanzon	On the Closure of Several Sets of ARMA and Linear State Space Models with a given Structure
1991-19	S. Eijffinger A. van Rixtel	The Japanese Financial System and Monetary Policy: a Descriptive Review
1991-20	L.J.G. van Wissen F. Bonnerman	A Dynamic Model of Simultaneous Migration and Labour Market Behaviour
1991-21	J.M. Sneek	On the Approximation of the Durbin-Watson Statistic in O(n) Operations
1991-22	J.M. Sneek	Approximating the Distribution of Sample Autocorrelations of