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Abstract

In this paper the boundaries of several families of (time-invariant) ARMA models and corresponding linear state space models are described. The topology of pointwise convergence of the Markov parameters is used.

1 Introduction

In this paper we want to draw attention to certain results in system theory which are applicable to ARMA models; we will also treat the corresponding case of linear state space models. It is partly a survey and partly consists of new results. In this introduction we want to put the closure problems that are treated here in a somewhat broader perspective. We regard a model as an abstract object, which can be represented in different ways. In the case of linear dynamical models one can think of the fact that the same model can be represented by an ARMA model and by a linear state space model. And both for ARMA models and for linear state space models there are many different parametrizations. The question arises what are "good" or "well-conditioned" parametrizations and how they can be constructed. Such well-conditioned parametrizations can play an important role in the optimization problems of system identification (recursive and non-recursive), model reduction, etc. It also can play an important role in our understanding of the structure of the model space and vice versa: the structure of the

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model space may give us more insight in the parametrization problem. An important aspect of a model space is its topological structure. As has been argued before, one of the important issues in an investigation of the topological structure is to find the closure of the model set if this is embedded in some larger topological space in some "natural" way (cf. e.g. [7,2] and the references given there). In finding the closure of a set of models with some given structure a *parametrization-independent* characterization of the structure will play an important role as we shall see. In showing that all models of a certain type are in the boundary however, we will use a *specific parametrization*, which differs from case to case. In the present paper we will investigate the closure in the pointwise topology of the Markov parameters (one could also say: the coefficients of the MA(∞)-representation of the model) of all ARMA-models resp. linear state space models with a given structure, namely:

(i) All models with given order (i.e. McMillan degree).

(ii) All multivariable models with given observability indices; for ARMA-models this can be expressed in terms of the maximal delays in the ARMA-equations if these are put in a certain form, namely the so-called minimal base form (see section 2);

(iii) All multivariable ARMA(p, q)-models with given AR order p and MA order q.

(iv) All scalar input/scalar output models with a given Cauchy index (for an explanation see section 2).

We will treat the deterministic case here; most of the results have a counterpart in the stochastic case, however the results and the proofs are somewhat more complicated due to the special role played by the stability properties of stochastic models. We hope to return to the stochastic case elsewhere.

2 The structure of a linear system in terms of its Hankel matrix

Consider a time-invariant linear dynamical model of finite order either in state-space form

$$
\begin{aligned}
x_{t+1} &= Fx_t + Gu_t, \qquad x_t \in \mathbf{R}^n, u_t \in \mathbf{R}^{m'} \\
y_t &= Hx_t + Ju_t, \qquad y_t \in \mathbf{R}^m,
\end{aligned} \tag{1}
$$

$$
F \in \mathbf{R}^{n\times n}, G \in \mathbf{R}^{n\times m'}, H \in \mathbf{R}^{m\times n}, J \in \mathbf{R}^{m\times m'},
$$

with (F, G, H, J) minimal, i.e. (F, G) reachable and (H, F) observable; or in ARMA-form

$$
A(L)y_t = B(L)u_t, \tag{2}
$$

with $A(L) = \sum_{k=0}^{p} A_k L^k$ and $B(L) = \sum_{k=0}^{q} B_k L^k$,
A_0 nonsingular, L the lag operator and $A(z), B(z)$ left coprime matrices (over the ring $\mathbf{R}[z]$ of real polynomials in the variable z).

More generally we will still speak of an ARMA representation if the conditions A_0 nonsingular and $A(z), B(z)$ left coprime are *not* met, as long as $A(z)$ is nonsingular (over the field $\mathbf{R}(z)$).

Two model representations (ARMA-representations and/or linear state space representations) are called equivalent if they describe the same input-output behaviour. One speaks of input-output equivalence. This is an equivalence relation on the set of *model representations*. A *model* corresponds in our set-up to an input-output equivalence class of model representations.

Starting with a general ARMA representation $\tilde{A}(z), \tilde{B}(z)$, with $\tilde{A}(z)$ nonsingular, one can always construct an equivalent representation $A(z), B(z)$ with A_0 nonsingular and $A(z), B(z)$ left coprime and furthermore, by taking suitable linear combinations (over $\mathbf{R}[z]$) of the rows of the polynomial matrix $[A(z) \mid B(z)]$ one can construct an ARMA-representation put in so-called *minimal base form*.

This means that the sum of the row degrees of $[A(z) \mid B(z)]$ is minimal over all possible ARMA representations $[C(z)A(z) \mid C(z)B(z)]$, where $C(z)$ is any square nonsingular *rational* matrix, such that $C(z)A(z)$ and $C(z)B(z)$ are *polynomial* matrices. There are algorithms to bring any pair $A(z), B(z)$ in minimal base form. There may be more than one ARMA representation in minimal base form corresponding to one and the same ARMA model, i.e. the representation of an ARMA-model in minimal base form is not unique. However the corresponding row degrees are unique, up to permutation (cf. [3]). They are the observability Kronecker indices $\kappa_i, i = 1, \ldots, m$. Their sum is $n = \sum_{i=1}^{m} \kappa_i$, the McMillan degree of the model. This is equal to the dimension of the state space in a minimal representation (F, G, H, J).

Consider the Markov matrices of a given linear dynamical model. They are parameter-independent quantities that fully determine the model. In terms of a state space representation (F, G, H, J) they are given by

$$\begin{aligned} H_0 &= J \\ H_i &= HF^{i-1}G, i = 1, 2, \ldots \end{aligned} \tag{3}$$

In terms of an ARMA representation $(A(z), B(z))$ the Markov matrices can be obtained by the following algorithm. Let

$$\begin{aligned} B^{(0)}(z) &= B(z) \\ B^{(j)}(z) &= \{B^{(j-1)}(z) - A(z)A(0)^{-1}B^{(j-1)}(0)\}/z, j = 1, 2, \ldots \end{aligned} \tag{4}$$

Then

$$\begin{aligned} H_0 &= A(0)^{-1}B(0) = A_0^{-1}B_0 \\ H_j &= A(0)^{-1}B^{(j)}(0), j = 1, 2, \ldots \end{aligned} \tag{5}$$

Now form the (block-)Hankel matrix

$$\mathcal{H} = \begin{bmatrix} H_1 & H_2 & H_3 & \ldots \\ H_2 & H_3 & H_4 & \ldots \\ H_3 & H_4 & H_5 & \ldots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{6}$$

We can now treat the four cases, numbered (i)–(iv) described in the introduction.

(i) The McMillan degree can be found from the Hankel matrix by the formula

$$rank(\mathcal{H}) = n \tag{7}$$

We will assume that the McMillan degree is finite, so $n \in \mathbb{N}$.

(ii) The observability indices can be found by the following procedure (see e.g. [4]).

Let

$$\begin{aligned} t_0 &= 0 \\ t_j &= rank \begin{bmatrix} H_1 & H_2 & \ldots \\ H_2 & H_3 & \ldots \\ \vdots & \vdots & \\ H_j & H_{j+1} & \ldots \end{bmatrix}, j = 1, 2, \ldots \end{aligned} \tag{8}$$

and

$$s_j = \Delta t_j = t_j - t_{j-1}, j = 1, 2, \ldots \tag{9}$$

Note that $\sum_{j=1}^{\infty} s_j = \sum_{j=1}^{n} s_j = \sum_{j=1}^{n} \Delta t_j = t_n = n$, so $\{s_j\}$ is a *partition* of n. Form the following socalled Young diagram of this partition:

$$\begin{array}{ccccccc}
\kappa_1 & \kappa_2 & \kappa_. & \kappa_. & \kappa_. & \kappa_m & \\
\odot & \odot & \odot & \odot & \odot & \odot & s_1 \\
 & \odot & \odot & \odot & \odot & \odot & s_2 \\
 & \odot & \odot & \odot & \odot & \odot & s_. \\
 & & & & & \odot & s_. \\
 & & & & & \odot & s_.
\end{array}$$

In this diagram the length of the i-th column is κ_i and the length of the j-th row is s_j. The total number of nonempty entries in the diagram is n. (So the Young diagram presented above corresponds to $m = 6; n = 18$; $\kappa_1 = 1, \kappa_2 = \kappa_3 = \kappa_4 = \kappa_5 = 3, \kappa_6 = 5; s_1 = 6, s_2 = s_3 = 5, s_4 = s_5 = 1, s_6 = s_7 = \ldots = 0$)

Given the partition $\{s_j\}$ of n one can read off the socalled *dual partition* of n. It consists precisely of the observability indices $\kappa_1, \kappa_2, \ldots, \kappa_m$, and clearly $\kappa_1 \leq \kappa_2 \leq \kappa_3 \ldots \leq \kappa_m$.

(iii) Next we turn to ARMA(p,q)-models. Let $arma(\bar{p}, \bar{q})$ denote the set of all ARMA-models that can be represented by a pair of polynomial matrices $(A(z), B(z))$ *with* $\deg A(z) \leq p, \deg B(z) \leq q$. Note:$(A(z), B(z))$ do *not* have to be in minimal base form here.

One has the following characterization of models with this structure.

Theorem 2.1 (cf. [4]) *A linear dynamical model is an element of $arma(\bar{p}, \bar{q})$ if and only if*

$$rk \begin{bmatrix} H_{q-p+1} & H_{q-p+2} & \cdots & H_{q+(m-1)p} \\ H_{q-p+2} & H_{q-p+3} & \cdots & H_{q+(m-1)p+1} \\ \vdots & \vdots & & \vdots \\ H_q & H_{q+1} & \cdots & H_{q+mp-1} \end{bmatrix} =$$

$$rk \begin{bmatrix} H_{q-p+1} & H_{q-p+2} & \cdots \\ H_{q-p+2} & H_{q-p+3} & \cdots \\ \vdots & \vdots & \\ \vdots & \vdots & \end{bmatrix} \leq mp \tag{10}$$

(iv) Finally we turn to the Cauchy index of a SISO (i.e. $m = 1, m' = 1$ in (1),(2)) linear dynamical model. As is well-known (cf. [1]) the Cauchy index is equal to the signature of the Hankel matrix $\mathcal{H}$. According to Sylvester's theorem this signature is equal to the number of positive eigenvalues (which we will denote by n_+) minus the number of negative eigenvalues (which we will denote by n_-) of $\Lambda\mathcal{H}\Lambda^T$, for each matrix Λ for which $\Lambda\mathcal{H}\Lambda^T$ has n well-defined non-zero eigenvalues (multiplicities included in all the eigenvalue counts). Of course $n = n_+ + n_-$. (Note that we are working here with $\infty \times \infty$ matrices and therefore we have to be careful in our formulations; for example eigenvectors and eigenvalues do not always have to exist etc. However because we are working with matrices of finite rank, many properties of these matrices are the same as for the case of standard linear algebra, if properly formulated)

3 Boundaries

In order to clarify the meaning and to indicate the practical importance of the boundaries of a set of models with a given structure, we will start with a small example. Consider the following ARMA-equations

$$y_{2,t} - \alpha y_{1,t-1} = u_t \tag{11}$$
$$y_{1,t} + \alpha^{-1} y_{2,t-1} = u_t + \alpha^{-1} u_{t-1} \tag{12}$$

This can be written in terms of polynomial matrices in the lag operator L, as follows.

$$A(L) \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = B(L) u_t, \tag{13}$$

with

$$A(L) = \begin{bmatrix} -\alpha L & 1 \\ 1 & \alpha^{-1} L \end{bmatrix}; B(L) = \begin{pmatrix} 1 \\ 1 + \alpha^{-1} L \end{pmatrix}$$

It is easy to see that these ARMA-equations are in a minimal base form and so the observability indices are $(1, 1)$. Now let us investigate what happens with the structure of the model if α converges to zero. If one takes the one period delayed version of the first equation of (11), multiplies it with α^{-1} and adds the result to the second equation of (11) one obtains the following ARMA-equations for the model:

$$\begin{aligned} y_{2,t} - \alpha y_{1,t-1} &= u_t \\ y_{1,t} + y_{1,t-2} &= u_t \end{aligned}$$

If we let α go to zero in these equations one obtains

$$\begin{aligned} y_{2,t} &= u_t \\ y_{1,t} + y_{1,t-2} &= u_t \end{aligned}$$

It is again easy to see that these ARMA-equations are in minimal base form. However now the observability indices are $(0, 2)$! So although the original ARMA-model can be represented by equations in which there is only a *one* time period delay, in the limit one has an ARMA-model that can only be represented by ARMA-equations with at least a *two* time periods delay! This shows that determining the boundaries of a set of ARMA-models is a non-trivial problem.

Let us now turn to the question of identifying the boundaries for each of the structures treated in the previous section. The choice of a topology is of course very important. Here we will work with the *topology of pointwise convergence of the Markov matrices.* This is a rather weak topology, it only requires that each of the Markov matrices converges seperately.

(i) To start with one has the following well-known result.

Theorem 3.1 (cf. [2]) *Consider, for a fixed choice of the number of inputs m' and the number of outputs m, all linear dynamical models with McMillan degree n. All limit points are linear dynamical models of McMillan degree $\leq n$. Also* all *such models are in the boundary.*

Proof. That all limit points are linear dynamical models of McMillan degree $\leq n$ is well-known and can easily be shown using (7).

That *all* models of McMillan degree $\leq n$ are in the boundary is not hard to show; one way of proving this is as follows. Take an arbitrary model with McMillan degree $n', n' < n$. Let $(F_{n'}, G_{n'}, H_{n'}, J_{n'})$ be a minimal state space representation of it. Let $f_{n'+1}, \ldots, f_n$ be $n - n'$ different real numbers, none of which is an eigenvalue of $F_{n'}$. Let $g_{n'+1}, \ldots, g_n$ be arbitrarily small positive numbers. Let

$$F = \left[\begin{array}{c|ccccc} F_{n'} & & & \bigcirc & & \\ \hline & f_{n'+1} & 0 & \cdots & \cdots & 0 \\ & 0 & f_{n'+2} & \ddots & & \vdots \\ \bigcirc & \vdots & \ddots & \ddots & \ddots & \vdots \\ & \vdots & & \ddots & \ddots & 0 \\ & 0 & \cdots & \cdots & 0 & f_n \end{array}\right],$$

$$\begin{aligned} G &= \begin{pmatrix} G_{n'} \\ \hline g_{n'+1} \\ \vdots \\ g_n \end{pmatrix}, \\ H &= (H_{n'}|1,1,\ldots,1), J = J_{n'} \end{aligned} \quad (14)$$

Then (F, G, H, J) is a minimal representation of a model with McMillan degree n which can be taken arbitrarily close to the original model by taking the positive numbers $g_{n'+1}, \ldots, g_n$ close enough to zero. □

(ii) Next consider for a fixed choice of the number of inputs m' and the number of outputs m, all linear dynamical models with a given fixed nondecreasing sequence of observability indices $\kappa_i, i = 1, 2, \ldots, m$, with $0 \leq \kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_m$ where $\kappa_i \in \{0, 1, 2, \ldots\}$. Let us now introduce the socalled *specialization order* which plays an important role.

Let $\kappa_i', i = 1, 2, \ldots, m$, with $0 \leq \kappa_1' \leq \kappa_2' \leq \ldots \leq \kappa_m'$, and $\kappa_i' \in \{0, 1, 2, \ldots\}$ be another nondecreasing sequence of observability indices, *not necessarily adding up to* $n = \sum_{i=1}^m \kappa_i$, then $\{\kappa_i'\}$ will be called "at least as special as" $\{\kappa_i\}$ if

$$\sum_{i=1}^k \kappa_i' \leq \sum_{i=1}^k \kappa_i, k = 1, 2, \ldots, m. \quad (15)$$

This will be denoted as $\{\kappa_i'\} \preceq \{\kappa_i\}$.

Some examples are: $\{1, 2, 3\} \preceq \{2, 3, 4\}; \{1, 2, 4\} \preceq \{2, 2, 3\}$.

For nonincreasing sequences of nonnegative integers one defines a partial order $\preceq'$ as follows: if $s_1, s_2, \ldots$ and $s_1', s_2', \ldots$ are two such sequences, then $\{s_j'\} \preceq' \{s_j\}$ if $\sum_{j=1}^k s_j' \leq \sum_{j=1}^k s_j, k = 1, 2, \ldots$.

It can be shown that $\preceq$ and $\preceq'$ are dual versions of each other in the following sense.

Theorem 3.2 ([4]) *If $\{\kappa_i\}_{i=1}^m$ and $\{\kappa_i'\}_{i=1}^m$ are nondecreasing sequences of nonnegative integers and if $\{s_1, s_2, \ldots\}$ and $\{s_1', s_2', \ldots\}$ are nonincreasing sequences of nonnegative integers and if furthermore $\{\kappa_i\}$ is dual to $\{s_j\}$ and $\{\kappa_i'\}$ is dual to $\{s_j'\}$ in the sense of the Young diagram of Section 2. Then*

$$\{s_j'\} \preceq' \{s_j\} \Longleftrightarrow \{\kappa_i'\} \preceq \{\kappa_i\} \quad (16)$$

It is important to note that if $\{\kappa_i\}_{i=1}^m$ is a partition of n and if $\{\kappa_i'\}_{i=1}^m$ is a partition of n', then n and n' do *not* have to be equal in this theorem.

We will make use of the following lemma.

Lemma 3.3 *Consider the set of all polynomial matrices of the form* $[A(z) \mid B(z)]$ *with maximal row degrees* $d_1, d_2, \ldots, d_m$*. The subset of all ARMA model representations in minimal base form is open and dense in the Euclidean topology of the parameters of the ARMA equations.*

Proof. A polynomial matrix of the form $[A(z) \mid B(z)]$ is an ARMA-model representation if $A(z)$ is nonsingular, which is the case on an open and dense subset of the set of all polynomial matrices in the Euclidean topology of the parameters. The main theorem of [3] states, among other things, that a polynomial matrix of the form $[A(z) \mid B(z)]$ is in minimal base form iff (a) the greatest common divisor of all the $k \times k$ minors is 1 and (b) their greatest degree is n.

The matrix whose i-th row consists of the coefficients of z^{d_i} in the i-th row of $[A(z) \mid B(z)]$ is called the high order coefficient matrix. Condition (b) means that the high order coefficient matrix has full row rank.

Standard considerations about relative prime polynomials, determinants and matrices having full rank show that the combination of (a) and (b) holds on an open and dense set in the Euclidean topology of the coefficients of the polynomial entries of $A(z), B(z)$.

It follows that the subset of all ARMA model representations in minimal base form is the intersection of two open and dense subsets of the set of all polynomial matrices of the form $[A(z) \mid B(z)]$ with maximal row degrees $d_1, d_2, \ldots, d_m$ and therefore is itself an open and dense subset. □

In the proof of the main result for the present case use will be made of the following corollary (the main result will in fact encompass this corollary).

Corollary 3.4 *If* $\kappa_i' \leq \kappa_i, i = 1, \ldots, m$*, then all models with observability indices* $\{\kappa_i'\}_{i=1}^m$ *are in the boundary of the set of all linear dynamical models with observability indices* $\{\kappa_i\}_{i=1}^m$*.*

Proof. Consider the set of all ARMA model representations $[A(z) \mid B(z)]$ with maximal row degrees $d_1, d_2, \ldots, d_m$ with $d_1 \leq d_2 \leq \ldots \leq d_m$. Any linear dynamical model with observability indices $\{\kappa_i'\}_{i=1}^m$ having the property $\kappa_i' \leq d_i, i = 1, \ldots, m$, has of course a representation in this set (not necessarily unique). Choosing $d_i = \kappa_i, i = 1, \ldots, m$ it follows from the lemma that there exists a sequence of ARMA-models in minimal base form with indices $\{\kappa_i\}_{i=1}^m$ which converges in the Euclidean topology of the coefficients to the equations for our linear dynamical model. It then follows that this sequence also converges in the pointwise topology of the Markov parameters to our linear dynamical model (because an ARMA-representation and its Markov matrices are related as described in (4),(5)). □

We can now state the following result, which in this general form appears to be new.

Theorem 3.5 *Let the number of inputs m' and the number of outputs m be fixed. The closure of the set of all linear dynamical models with observability indices $\{\kappa_i\}_{i=1}^m$ consists precisely of all linear dynamical models with observability indices $\{\kappa_i'\}_{i=1}^m$ such that $\{\kappa_i'\} \preceq \{\kappa_i\}$.*

Let $arma(\{\bar{d}_i\})$ denote the set of all linear dynamical models which can be represented by a set of ARMA-equations with the property that the i-th equation has maximal delay $\leq d_i$.

Corollary 3.6 *The closure of the set of models $arma(\{\bar{d}_i\})$ is equal to $\bigcup_{\{d_i'\}\preceq\{d_i\}} arma(\{\bar{d}_i'\})$.*

Proof of theorem 3.5 Let $\{s_j\}$ be dual to $\{\kappa_i\}$. From (8),(9) it follows that if $\{s_j'\}$ corresponds to a model in the boundary of the set of all linear dynamical models with observability indices $\{\kappa_i\}$, then $\{s_j'\} \preceq' \{s_j\}$.From theorem 3.2 it follows that $\{\kappa_i'\} \preceq \{\kappa_i\}$.

To show that *all* linear dynamical models with observability indices $\{\kappa_i'\}$ such that $\{\kappa_i'\} \preceq \{\kappa_i\}$ are in the boundary, we will adapt a similar result for controllable pairs of matrices (F, G) (with n fixed) reported in [6]. The proof will be given in a number of steps in each of which the problem is reduced to a simpler one; in the last step the remaining problem is solved.

Step 1. Choose an arbitrary linear dynamical model with observability indices $\{\kappa_i'\}_{i=1}^m$ such that $\{\kappa_i'\} \preceq \{\kappa_i\}$. We distinguish between two possibilities, namely $\sum_{i=1}^m \kappa_i' = n$ and $\sum_{i=1}^m \kappa_i' < n$. In the last case consider the observability indices $\{\kappa_i''\}_{i=1}^m$ defined by

$$\begin{aligned} \kappa_i'' &= \kappa_i', i = 1, \ldots, m-1 \\ \kappa_m'' &= n - \sum_{i=1}^{m-1} \kappa_i' \end{aligned}$$

which have the properties

$$\sum_{i=1}^{m} \kappa_i'' = n$$

$$\{\kappa_i'\} \preceq \{\kappa_i''\} \preceq \{\kappa_i\}$$

and

$$\kappa_i' \leq \kappa_i'', i = 1, \ldots, m.$$

From corollary 3.4 it follows that our linear dynamical model is in the boundary of the set of all linear dynamical models with indices $\{\kappa_i''\}_{i=1}^m$. Because a boundary of a set of boundary points is included in the closure of the original set, it now follows that it suffices to show the desired result for the case in wich the observability indices add up to n. Therefore in the remainder of the proof we will assume $\sum_{i=1}^m \kappa_i' = n$.

Step 2. It is a basic result for the specialization order on partitions of n, that if two such partitions, $\{\kappa_i'\}_{i=1}^m$ and $\{\kappa_i\}_{i=1}^m$, with $\{\kappa_i'\} \preceq \{\kappa_i\}$, have the property that $\{\kappa_i'\} \preceq \{\kappa_i''\} \preceq \{\kappa_i\}$ implies $\{\kappa_i''\} = \{\kappa_i'\}$ or $\{\kappa_i''\} = \{\kappa_i\}$, then the Young diagram of $\{\kappa_i'\}$ can be obtained from the one of $\{\kappa_i\}$ by transporting *one* atom of the diagram. This means that there exist numbers i_0 and i_1 with $i_0 < i_1$ such that

$$\begin{aligned} \kappa_i' &= \kappa_i, \text{ if } i \neq i_0, i_1 \\ \kappa_{i_0}' &= \kappa_{i_0} - 1 \\ \kappa_{i_1}' &= \kappa_{i_1} + 1. \end{aligned} \tag{17}$$

Expressing this same property in terms of the dual partitions $\{s_j'\}$ and $\{s_j\}$ (both nonincreasing sequences) one has that there exist numbers j_0 and j_1 with $j_0 < j_1$ such that

$$\begin{aligned} s_j' &= s_j, \text{ if } j \neq j_0, j_1 \\ s_{j_0}' &= s_{j_0} - 1 \\ s_{j_1}' &= s_{j_1} + 1. \end{aligned} \tag{18}$$

Because the boundary of a set of boundary points is in the closure of the original set, this implies that it suffices to consider only the case where indeed (17) holds.

Step 3. Consider a minimal state space representation (F, G, H, J) of our linear dynamical model with observability indices $\{\kappa_i'\}$. We will make use of its dual (F^T, H^T, G^T, J^T). Of course the reachable pair (F^T, H^T) has *reachability* indices $\{\kappa_i'\}$.

Given these reachability indices and therefore the Young diagram, a special reachable pair (F_0^T, H_0^T) can be defined as follows. Let $e_1, e_2, \ldots, e_n$ denote the n standard basis vectors in $\mathbf{R}^n$. Then

$$H_0^T = [e_1 \mid \ldots \mid e_m] \tag{19}$$

and F_0^T is given by an adapted version of the Young diagram, which is obtained by putting $e_1, e_2, \ldots, e_n$ in the diagram, starting with filling up the first row from left to right, then filling up the second row from left to right, etc. until the last row is filled up with the last basis vectors and so the south-east corner element of the diagram contains e_n;

F_0^T is then defined by

$$\begin{aligned} F_0^T(e_i) &= e_{i'}, \text{ if } e_{i'} \text{ occurs just below } e_i \\ F_0^T(e_i) &= 0 \quad \text{otherwise} \end{aligned} \tag{20}$$

It follows directly that

$$\begin{aligned} span[H_0^T \mid F_0^T H_0^T \mid \ldots \mid (F_0^T)^{j-1} H_0^T] &= span[e_1, e_2, \ldots, e_{t'_j}], \\ j &= 1, 2, \ldots \end{aligned} \tag{21}$$

where $\{t'_j\}$ is related to $\{s'_j\}$, which is the dual partition of $\{\kappa'_i\}$, as $\{t_j\}$ is related to $\{s_j\}$ in (8),(9). From this it follows directly that the reachability indices of the pair (F_0^T, H_0^T) are $\{\kappa'_i\}$.

A famous result of Kalman states that using the elements of the socalled feedback group, which can be represented by $(S, K, R), S \in GL_n(\mathbf{R}), K \in \mathbf{R}^{m \times n}, R \in GL_n(\mathbf{R})$ and which act on the reachable pair (F^T, H^T) to give the reachable pairs $(S^{-1}F^TS + S^{-1}H^TK, S^{-1}H^TR)$ as a result, each reachable pair can be transformed into any other reachable pair which has the *same reachability indices.*

It follows that there exists an element of the feedback group represented by (S, K, R) which transforms (F^T, H^T) to the reachable pair (F_0^T, H_0^T). The inverse of the element of the feedback group is represented by

$(S^{-1}, -R^{-1}KS^{-1}, R^{-1})$ and this transforms (F_0^T, H_0^T) back to (F^T, H^T). (Warning: note that we have *not* introduced a matrix G_0^T. In fact there may for some cases not even exist a minimal realization (given m and m') with reachable pair (F_0^T, H_0^T)!)

Step 4. Define the reachable pair (F_1^T, H_1^T) as follows.

$$H_1^T = H_0^T \tag{22}$$

and F_1^T is defined by filling up the Young diagram of $\{\kappa_i\}_{i=1}^m$ with the standard basis vectors in the following way: at each entry where this Young diagram overlaps with the one of $\{\kappa'_i\}_{i=1}^m$ take the *same* basis vector as in the diagram of F_0^T. Because of what was said in Step 2 all standard basis vectors except $e_{t_{j_1-1}+1}$ are placed by this rule. Also there is (of course) only one spot in the diagram of $\{\kappa_i\}$ which is not yet filled, namely the one at the entry which lies at the intersection of column i_0 and row j_o. There we put $e_{t_{j_1-1}+1}$ and the definition of F_1^T is complete.

It follows that

$$
\begin{aligned}
span[H_1^T \mid F_1^T H_1^T \mid \ldots \mid (F_1^T)^{j-1} H_1^T] &= span[e_1, e_2, \ldots, e_{t'_j}], \\
j &= 1, 2, \ldots, j_0 - 1 \\
span[H_1^T \mid F_1^T H_1^T \mid \ldots \mid (F_1^T)^{j-1} H_1^T] &= span[e_{t'_{j_1-1}+1}; e_1, e_2, \ldots, e_{t'_j}], \\
j &= j_0, j_0 + 1, \ldots, j_1 - 1 \\
span[H_1^T \mid F_1^T H_1^T \mid \ldots \mid (F_1^T)^{j-1} H_1^T] &= span[e_1, e_2, \ldots, e_{t'_j}], \\
j &= j_1, j_1 + 1, \ldots \qquad (23)
\end{aligned}
$$

It follows using (17), (18) from Step 2, that the reachability indices of the pair (F_1^T, H_1^T) are $\{\kappa_i\}_{i=1}^m$.

Now let for each $\tau \in [0,1]$ the pair (F_τ^T, H_τ^T) be defined by

$$
\begin{aligned}
F_\tau^T &= (1-\tau) F_0^T + \tau F_1^T \\
H_\tau^T &= H_0^T = H_1^T. \qquad (24)
\end{aligned}
$$

Then for all $\tau \neq 0$ the spans of the initial parts of the reachability matrix are the same as for the case $\tau = 1$ and are therefore given by (23) if F_1^T is replaced by F_τ^T and H_1^T by H_τ^T in that equation. So for all $\tau \neq 0$ the pair (F_τ^T, H_τ^T) has reachability indices $\{\kappa_i\}_{i=1}^m$, while of course for $\tau = 0$ the pair has reachability indices $\{\kappa_i'\}_{i=1}^m$, by construction.

So the reachable pair (F_0^T, H_0^T) with reachability indices $\{\kappa_i'\}$ is the limit (in the Euclidean topology of the entries of the pair of matrices) of a curve of reachable pairs with reachability indices $\{\kappa_i\}$.

Step 5. Now we can apply the inverse element of the feedback group, as described in Step 3, to each of the pairs $(F_\tau^T, H_\tau^T), \tau \in [0,1]$. Because the same element of the feedback group is applied to all the pairs the resulting curve of pairs converges to the pair (F^T, H^T) for τ converging to zero. Because the pair (G^T, F^T) is observable (i.e. the observability matrix has full row rank) there exists an interval $[0, \epsilon)$, for some $\epsilon > 0$, such that for each $\tau \in [0, \epsilon)$ the corresponding reachable pair from this curve, together with the matrix G^T is observable and therefore forms a minimal triple. Joining J^T and taking the duals again, one obtains a curve of linear dynamical models each with observability indices $\{\kappa_i\}$, which converges, in the Euclidean topology of state space representations and therefore also in the pointwise topology of the Markov matrices, to the linear dynamical model with observability indices $\{\kappa_i'\}$ that we started with in Step 1. □

(iii) Let us now consider the set of all linear dynamical models (with m' input components and m output components) which have an ARMA(p,q) representation. The closure of this set is the same as the closure of the set $arma(\bar{p},\bar{q})$, which we defined in Section 2. Let $arma(\{p_i\}_{i=1}^m,\{q_i\}_{i=1}^m)$ denote the set of all linear dynamical models which have what we will call an ARMA$(\{p_i\},\{q_i\})$-representation, namely an ARMA representation $(A(z),B(z))$ with the property that the degree of the i-th row of $A(z)$ is $\leq p_i$ and the degree of the i-th row of $B(z)$ is $\leq q_i$. We have the following result.

Theorem 3.7 *Let m,m',p and q be fixed. Let $d=q-p\in \mathbf{Z}$. The closure of the set $arma(\bar{p},\bar{q})$ is equal to the union of all sets $arma(\{p_i\}_{i=1}^m,\{q_i\}_{i=1}^m)$ for which*

$$\sum_{i=1}^{m}\max(p_i,q_i-d)\leq mp. \tag{25}$$

Remark. One can reformulate the theorem in a slightly less formal way by saying that the closure of the set of all linear dynamical models which have an ARMA(p,q)-representation, consists of all linear dynamical models which have an ARMA$(\{p_i\},\{q_i\})$-representation satisfying (25).

Proof. We will consider a linear dynamical model with Hankel matrix $\mathcal{H}$ and direct feedthrough matrix H_0. Let $\mathcal{H}_{d+1}$ be defined by

$$\mathcal{H}_{d+1}=\begin{bmatrix} H_{d+1} & H_{d+2} & \cdots \\ H_{d+2} & H_{d+3} & \cdots \\ \vdots & \vdots & \\ \vdots & \vdots & \end{bmatrix} . \tag{26}$$

Consider the linear dynamical model with Hankel matrix $\tilde{\mathcal{H}}$ equal to $\tilde{\mathcal{H}}=\mathcal{H}_{d+1}$ and with direct feedthrough matrix $\tilde{H}_0=H_d$. Let its McMillan degree be denoted by $\tilde{n}$, its observability indices by $\{\tilde{\kappa}_i\}_{i=1}^m$ and the corresponding dual partition by $\{\tilde{s}_j\}$. From theorem 2.1, combined with (8),(9) it follows that if the original linear dynamical model (with Hankel matrix $\mathcal{H}$) is an element of the set $arma(\bar{p},\bar{q})$ then

$$\tilde{s}_{p+1}=0 \tag{27}$$

and therefore

$$\tilde{\kappa}_i\leq p, i=1,2,\ldots,m \tag{28}$$

and

$$\tilde{n} \leq mp. \tag{29}$$

If alternatively our original linear model lies in the *closure* of $arma(\bar{p}, \bar{q})$ then (28) no longer has to hold, only (29) remains valid, which implies

$$\sum_{i=1}^{m} \tilde{\kappa}_i \leq mp, i = 1, 2, \ldots, m. \tag{30}$$

This implies that there is an ARMA representation $(\tilde{A}(z), \tilde{B}(z))$ of this model (with Hankel matrix $\tilde{\mathcal{H}}$) with the property $\sum_{i=1}^{m} \max(\tilde{p}_i, \tilde{q}_i) \leq mp$. As one would expect, $\tilde{p}_i$ denotes the degree of the i-th row of $\tilde{A}(z)$ and $\tilde{q}_i$ denotes the degree of the i-th row of $\tilde{B}(z)$.

Now we distinguish the cases $d \geq 0$ and $d < 0$.

If $d \geq 0$, let

$$\begin{aligned} B^{(d)}(z) &= \tilde{B}(z) \text{ and} \\ B^{(j-1)}(z) &= B^{(j)}(z)z + \tilde{A}(z)H_{j-1}, j = d, d-1, \ldots, 1. \end{aligned} \tag{31}$$

Taking $(A(z), B(z)) = (\tilde{A}(z), B^{(0)}(z))$ one has an ARMA representation of the original model in the closure, i. e. its Hankel matrix is $\mathcal{H}$ and its direct feedthrough matrix is H_0, and it has the property

$$\sum_{i=1}^{m} \max(p_i, q_i - d) \leq mp. \tag{32}$$

If $d < 0$ then the first d Markov matrices in $\tilde{\mathcal{H}}$ are zero. Applying the algorithm (4) to $(\tilde{A}(z), \tilde{B}(z))$ it follows from that algorithm and (5) that the coefficient matrices of $z^0, z^1, \ldots, z^{d-1}$ in $\tilde{B}(z)$ are zero and that the *polynomial* matrix pair $(A(z), B(z))$ given by $(A(z), B(z)) = (\tilde{A}(z), z^{-d}\tilde{B}(z))$ is an ARMA-representation of the original model. Clearly

$$\tilde{p}_i = p_i \text{ and } \tilde{q}_i = q_i - d, i = 1, 2, \ldots, m \tag{33}$$

and therefore for such a model in the closure one finds an ARMA representation with the property $\sum_{i=1}^{m} \max(p_i, q_i - d) \leq mp$ indeed. This shows that all models in the closure are of the form prescribed by the theorem.

It remains to show that all models of the prescribed form are in the closure indeed.

Again we distinguish the cases (i) $d \geq 0$ and (ii) $d < 0$.

(i) Suppose $d \geq 0$. As noted above, (2.1) is equivalent to the statement that a linear dynamical model belongs to $arma(\bar{p},\bar{q})$ iff the corresponding Hankel matrix $\tilde{\mathcal{H}} = \mathcal{H}_d$ satisfies the property (28). There are *no* other restrictions on $\tilde{\mathcal{H}}$ and therefore it follows from Theorem 3.1 that all linear dynamical models for which (29) holds are in the closure indeed. In exactly the same way as above one can deduce that the set so obtained is the set of all models which have an ARMA-representation for which (32) holds, as was to be shown.

(ii) Finally, suppose $d < 0$. In this case $\tilde{\mathcal{H}}$ will in general contain many zero entries by construction. Therefore Theorem 3.1 cannot be applied directly. Let $(A(z), B(z)) \in arma(\bar{p},\bar{q})$. As noted in the proof of (i), this implies that the corresponding Hankel matrix $\tilde{\mathcal{H}} = \mathcal{H}_d$ satisfies the property (28). As before let $(\tilde{A}(z), \tilde{B}(z))$ be an ARMA representation of a model with Hankel matrix $\tilde{\mathcal{H}}$. If we do not take account of the zero restrictions for the moment, Theorem 3.1 tells us that there exists a sequence of ARMA(p,p) models that converges, in the topology of pointwise convergence of the Markov matrices, to $(\tilde{A}(z), \tilde{B}(z))$. We know that the coefficient matrices of $z^0, z^1, \ldots, z^{d-1}$ in $\tilde{B}(z)$ go to zero in the limit. If in each element $(A^{(k)}(z), B^{(k)}(z)), k = 1, 2, \ldots$ of the converging sequence of ARMA(p,p) models we put the coefficient matrices of $z^0, z^1, \ldots, z^{d-1}$ in $B^{(k)}(z)$ equal to zero, the resulting sequence, the k-th element of which we denote by $(\tilde{A}^{(k)}(z), \tilde{B}^{(k)}(z))$, converges to $(\tilde{A}(z), \tilde{B}(z))$. In terms of the Markov matrices this corresponds to putting the first d Markov matrices *in* $\tilde{\mathcal{H}}$ equal to zero and therefore convergence of the resulting sequence is clear. Finally, consider the sequence $\{(\tilde{A}^{(k)}(z), z^{-d}\tilde{B}^{(k)}(z))\}_{k=1}^{\infty}$, of ARMA$(p,q)$ models. It converges to $(A(z), B(z)) = (\tilde{A}(z), z^{-d}\tilde{B}(z))$ and the theorem is proved. □

(iv) Now we turn to the Cauchy index. As we have seen in the previous section, the Cauchy index is equal to $n_+ - n_-$. Because $n = n_+ + n_-$ we can in turn determine n_+ and n_- from the Cauchy index if n is known. We can therefore work just as well with (n_+, n_-) as with the Cauchy index and we will in fact do so. The pair (n_+, n_-) will be called the Hankel inertia of the system.

We will use the obvious ordering on the set $\{(n_+, n_-) | n_+ \in \mathrm{N}, n_- \in \mathrm{N}\}$, namely the one given by:

$$(n'_+, n'_-) \leq (n_+, n_-) \Longleftrightarrow n'_+ \leq n_+ \text{ and } n'_- \leq n_- \tag{34}$$

We can now state the results for this case. First a formulation in terms of the Hankel inertia.

Theorem 3.8 *Let (n_+, n_-) be fixed. The closure of the set of SISO linear dynamical models with Hankel inertia (n_+, n_-) consists of all SISO linear dynamical models for which the Hankel inertia (n'_+, n'_-) satisfies the inequality $(n'_+, n'_-) \leq (n_+, n_-)$.*

Corollary 3.9 *The closure of the set of SISO linear dynamical models with McMillan degree n and Cauchy index $\eta, \eta \in \mathbf{Z}$ consists of all SISO linear dynamical models with McMillan degree $n' = n - d$ with $d \in \{0, 1, \ldots, n\}$ and with Cauchy index η' with $\eta' \in \{\eta - d, \eta - d + 1, \ldots, \eta + d\}$.*

Proof of theorem 3.8. Consider a SISO linear dynamical model with Hankel inertia (n'_+, n'_-) in the closure of the set of SISO linear dynamical models with Hankel inertia (n_+, n_-). From the continuity of the eigenvalues of the Hankel matrix with respect to the Markov parameters, it follows that $n'_+ \leq n_+$ and $n'_- \leq n_-$.

Consider a SISO linear dynamical model, given by a minimal representation (F, G, H, J), with Hankel inertia (n'_+, n'_-) such that $(n'_+, n'_-) \leq (n_+, n_-)$ and $(n'_+ + n'_-) < (n_+ + n_-)$. We will show in a number of steps that this linear dynamical model lies in the boundary of the set of SISO linear dynamical models with Hankel inertia (n_+, n_-).

Step 1. Because the boundary of a set of boundary points is included in the closure of the original set, it suffices to show the case in which $(n_+ + n_-) - (n_+ + n_-) = 1$. Because multiplication of the model by -1 has the effect that n_+ and n_- are interchanged, it will be clearly sufficient to show the result for the case in which $n'_+ = n_+ - 1$ and $n'_- = n_-$.

Step 2. We will now first treat the case in which the linear dynamical model is asymptotically stable, i.e. the eigenvalues of F all have modulus smaller than or equal to one. Without loss of generality it can be then assumed that the quadruple (F, G, H, J) is a balanced realization, as described in [8,9]. This means that there are singular values $\sigma_1 > \sigma_2 > \ldots > \sigma_k > 0$ with multiplicities $n_1, n_2, \ldots, n_k$ adding up to the McMillan degree $n'_+ + n'_-$; signs $s_j \in \{-1, 1\}, j = 1, 2, \ldots, k$ and sequences of positive numbers $\{g_j\}_{j=1}^k; \{\alpha_{j,1}, \ldots, \alpha_{j,n_j-1}\}, j = 1, 2, \ldots, k$, such that

$$\begin{aligned} G^T &= [g_1 e_1^T \mid g_2 e_2^T \mid \ldots \mid g_k e_k^T] \\ H &= [s_1 g_1 e_1^T \mid s_2 g_2 e_2^T \mid \ldots \mid s_k g_k e_k^T] \\ F &= [F(i,j)]_{i=1,j=1}^{i=k,j=k}, \text{ with } F(i,j) \in \mathbf{R}^{n_i \times n_j} \text{ given by} \end{aligned}$$

$$F(j,j) = \begin{bmatrix} -g_j^2/(2\sigma_j) & \alpha_{j,1} & & & & 0 \\ -\alpha_{j,1} & 0 & \alpha_{j,2} & & & \\ & -\alpha_{j,2} & 0 & \alpha_{j,3} & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & 0 & \alpha_{j,n_j-1} \\ & & & & -\alpha_{j,n_j-1} & 0 \end{bmatrix}$$

$$F(i,j) = \begin{bmatrix} -g_i g_j/(s_i s_j \sigma_i + \sigma_j) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \qquad (35)$$

It is well-known (cf. [8]) that the Cauchy index is given by $\eta = \sum_{j=1}^{k} s_j$. It can now easily be seen from this canonical form, that by choosing a small positive singular value σ_{k+1}, with multiplicity $n_{k+1} = 1$, such that $\sigma_{k+1} < \sigma_k$ and by choosing furthermore $g_{k+1} = \tau, \tau \geq 0$ and $s_{k+1} = 1$ one obtains, for each positive value of τ a system with McMillan degree $n'_+ + n'_- + 1$ and with Cauchy index $n'_+ - n'_- + 1$. So the Hankel inertia of such a system is $(n_+, n_-) = (n'_+ + 1, n'_-)$. If τ converges to zero from above the corresponding curve of matrix quadruples converges in the topology of the entries of the four matrices to the matrix quadruple

$$\left(\left[\begin{array}{c|c} F & 0 \\ \hline 0 & 0 \end{array} \right], \left(\frac{G}{0} \right), (H|0), J \right)$$

and therefore the corresponding curve of linear dynamical models converges in the topology of pointwise convergence of the Markov parameters, to the model that we started with. This proofs the case of asymptotically stable F.

Step 3. If F is not asymptotically stable, there exists a $\lambda \in (0,1)$ such that $\lambda^2 F$ is asymptotically stable. According to the results of Step 2 there exists a curve of models with Hankel inertia (n_+, n_-) converging to the model with state space representation $(\lambda^2 F, G, H, J)$. Multiplication of F with λ^2 is equivalent to pre- and post-multiplying the Hankel matrix by $\Lambda = diag(1, \lambda, \lambda^2, \ldots)$. Therefore according to Sylvester's theorem the resulting Hankel inertia will not have changed. Pre- and post-multiplying the Hankel matrices of all the linear dynamical models on the curve by Λ^{-1} one obtains a curve of linear dynamical models having Hankel inertia (n_+, n_-) which converges in terms of the matrix entries of a state space representation and therefore also in the topology of pointwise convergence of the Markov parameters, to the linear dynamical model that we started with. □

4 Conclusions

The results presented on the boundaries of several families of ARMA models (linear time-invariant dynamical models) show that these boundaries are nontrivial, especially in the multivariable case. We have shown how the boundaries can be identified using results from mathematical systems theory. Understanding these boundaries can be important in practical situations. E.g. if one uses the maximum likelihood method to identify (i.e. estimate) the model, and the maximizing model happens to lie on the boundary, in running some iterative optimization algorithm one will typically find that some of the parameters tend to infinity, while in fact convergence is taking place. To avoid this kind of problem one could use an overlapping parametrizations approach to system identification (see e.g. [4] and the references given there). An interesting open question is: do there exist alternative parametrizations for ARMA models from which convergence to another "structure" can be seen directly, just as convergence to a lower order system can be seen directly from a balanced realization by monitoring the smallest Hankel singular value, and if so how can they be constructed?

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