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MONOTONE IMPROVEMENT OF THE SOCIAL WELFARE IN AN INDUSTRIAL NETWORK

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MONOTONE IMPROVEMENT OF THE SOCIAL WELFARE IN AN INDUSTRIAL NETWORK[1)]

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Abstract

In this paper we consider the problem of achieving an optimal state in an industrial network. In such a network of production sectors each sector maximizes its profit. For given prices an network equilibrium is attained by quantity rations on the inputs of the production sectors. It is shown that there exists a price adjustment process with the property that the utility of the consumption sector at the resulting equilibrium network state increases monotonically. As a result of this property the process converges to an optimal network state.

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MONOTONE IMPROVEMENT OF THE SOCIAL WELFARE IN AN INDUSTRIAL NETWORK.

1. Introduction

In this paper we consider an industrial network, which consists of n production sectors. Each sector produces one good and uses the other goods as inputs. A network state is a collection of production vectors $y = \{y^1,...,y^n\}$, with y^i the production vector of inputs and output of sector i, $i = 1,...,n$. A network state yields a net output vector $\Sigma_i y^i$, which is consumed by an external consumption sector. A network state is optimal if it maximizes the utility of the consumption sector. Initially, the prices of the commodities are fixed. Given the fixed prices, each production sector maximizes its profit over the production set. However, the inputs are controlled by a central planning institution. This institution rations the inputs of the production sectors. The task of the central planning institution is to maximize the utility of the consumption sector by setting an efficient rationing scheme. Under the profit maximizing behaviour of the production sectors, a rationing scheme is efficient if there is no other rationing scheme, such that the resulting network state yields a higher utility to the consumption sector. In Braverman and Levin [1] the necessary conditions for an efficient rationing scheme have been discussed. In this paper we first discuss the question how to reach an efficient rationing scheme. Therefore we introduce an adjustment process, which adjusts the rationing scheme until an efficient rationing scheme has been achieved. Along this adjustment path the utility of the corresponding net output vector increases monotonically.

In the second part of the paper prices and rationings are adjusted until under the profit maximizing behaviour of the production sectors an optimal state has been reached. Again we show that along the path the utility increases monotonically. Furthermore, the price adjusting process is based on local information.

2. The model

We consider a production network consisting of n production sectors, indexed $j = 1,..,n$. There are also n commodities. Production sector j produces commodity j and uses the commodities $i = 1,...n$, $i \neq j$ as inputs. So, production sector j is characterized by a production set $Y^j \subset \{y \in R^n \mid y_i \leq 0, i \neq j\}$. Each production sector is profit maximizing given the prices of the commodities. Let $Y = \Sigma_i Y^i$ be the total production set. Furthermore R^n_+ denotes the set $\{y \in R^n \mid y_j \geq 0, j = 1,...,n\}$. We make the following assumptions on the production sets.

Assumptions.

A_1. Y^i is closed, convex and strict convex with respect to the output.

A_2. Y^i has a smooth boundary with respect to the output.

A_3. $-R^n_+ \subset Y^i$.

A_4. for all i, $0 \in Y^i$.

A_5. $Y \cap -Y = \{0\}$.

With strict convex with respect to the output we mean that if $y^i \in Y^i$ and $z^i \in Y^i$ and $y^i_i > 0$ and/or $z^i_i > 0$, then x^i is not in the boundary of Y^i with $x^i = \alpha y^i + (1-\alpha)z^i$ for any $0 < \alpha < 1$. With smooth boundary with respect to the output we mean that the boundary is smooth at any point y^i in the boundary with $y^i_i > 0$. Let $\delta Y^i = \{y^i \in Y^i | z^i \notin Y^i$ if $z^i_i > y^i_i$ and $z^i_k \leq y^i_k$, $k \neq i\}$, i.e., δY^i is the upper boundary of Y^i. So, according to A_2 and A_3, δY^i contains the zero point and all production vectors y^i on the boundary of Y^i with positive output of commodity i. Under assumption A_1 and A_2 this upper boundary δY^i of the production set can be characterized by a differentiable strictly convex transformation function f^i satisfying $f^i(y^i) = 0$ for all $y^i \in \delta Y^i$ and $f^i(y^i) < 0$ for all $y^i \in Y^i \backslash \delta Y^i$. At a point $y^i \in \delta Y^i$, let $D^i(y^i)$ be the gradient of f^i at y^i, i.e., $D^i(y^i) = (\delta f^i/\delta y^i_1, \ldots, \delta f^i/\delta y^i_n)^T$.

A network state is a collection $y = \{y^1, \ldots, y^n\}$, such that $y^i \in Y^i$, $i = 1, \ldots, n$. Let $x(y) = \Sigma_i y^i$ be the net output of network state $y = \{y^1, \ldots, y^n\}$. Assume $w > 0$ to be a vector of initial resources.

Definition 2.1.

A network state $y = \{y^1, \ldots, y^n\}$ is admissible if $x(y) + w \geq 0$.

The consumption sector is modelled by an agent having a utility function $u: R^n_+ \to R$. An admissible network state yields a utility $u(y) \equiv u(x(y)+w)$ to the consumption sector. We make the following assumptions on the utility function.

Assumptions.

B_1. The utility function u is strict quasi-concave and continuously differentiable.

B_2. The utility function u is strict monotone increasing, i.e., $u(x) > u(y)$ for all $x,y \in R^n_+$, such that with $x_j \geq y_j$, $j = 1, \ldots, n$ and for at least one j, $x_j > y_j$.

A central planning institution wants to maximize the utility of the consumption sector. In this paper we want to discuss two questions. First of all, we consider the situation that all commodities have fixed positive prices. Under these fixed prices the production sectors want to maximize their profits. However, profit maximizing does not lead to a utility maximizing network state. To improve on the situation the central planning institution sets maximum quantities on the inputs of the production, i.e., the inputs are rationed by the planning institution. Let $E^i = \{e^i \in R^n \mid e^i_k \geq 0$, $k = 1, \ldots, n$ and $e^i_i = \infty\}$ be the set of rationing vectors of sector i. So, $e^i \in E^i$ is a vector of maximum inputs for production sector i, $i = 1, \ldots, n$, with $e^i_i = \infty$. Then, for $e^i \in E^i$, let $Y^i(e^i) = \{y^i \in Y^i \mid y^i \geq -e^i\}$ be the feasible production set of sector i under the rationing vector e^i. Furthermore, let $y^i(p,e^i) = \{y^i \in Y^i(e^i) \mid p^T y^i \geq p^T z^i$ for all $z^i \in Y^i(e^i)\}$ be the set of profit maximizing production vectors of sector i, given the vector p of fixed prices and the feasible production set $Y^i(e^i)$. Under assumption A_1 we have that $y^i(p,e^i)$ is unique and that $y^i(p,.) : R^n \to Y^i$ is a continuous function. Finally, let $y(p,e)$ denote the network state $\{y^1(p,e^1), \ldots, y^n(p,e^n)\}$. A rationing scheme $e = \{e^1, \ldots, e^n\}$ is efficient if it maximizes the utility under the profit maximizing behaviour of the production sectors. So, efficiency is defined with respect to a vector p of fixed prices.

Definition 2.2.

Let $p = (p_1,...,p_n)^T$ be a vector of fixed positive prices. Then the rationing scheme $e = \{e^1,...,e^n\}$ is admissible if $y(p,e)$ is an admissible network state.

Lemma 2.3. There exists an admissible rationing scheme e.

Proof. For all i, take $e^i{}_k = 0$, $k \neq i$. From assumptions A_4 and A_5 it follows that $y^i(p,e^i) = 0$ and hence $x(y(p,e)) + w = w >> 0$.

Definition 2.4.

Let $p = (p_1,...,p_n)^T$ be a vector of fixed positive prices. Then the rationing scheme $e = \{e^1,...,e^n\}$ is efficient if it is admissible and if $u(y(p,e)) \geq u(y(p,d)$ for each admissible rationing scheme $d = \{d^1,...,d^n\}$. For an efficient rationing scheme $e = \{e^1,...,e^n\}$, $y(p,e)$ is an efficient network state.

An efficient network state maximizes the utility under the profit maximizing behaviour of the production sectors given the vector of fixed prices. In the next definition we define an optimal network state. In such an optimal network state the utility of the consumption sector is maximized, given the production technologies.

Definition 2.6.

A network state y is optimal if y is admissible and if $u(y) \geq u(z)$ for any admissible network state z.

We want to give an adjustment process which starts at an admissible rationing scheme under a vector of fixed prices. In the next section we describe the adjustment to an efficient rationing scheme given the vector of fixed prices. We show that there exists an adjustment process along which the utility of the consumption sector increases monotonically. In section 4 we show the existence of a price adjustment process such that the utility of the corresponding efficient network state increases monotonically until an optimal state has been reached.

3. Adjustment of the rationing scheme

In this section we give an adjustment process from an admissible rationing scheme to an efficient rationing scheme given a vector p of fixed positive prices. The main result is that there exists an adjustment path along which the utility of the consumption sector increases monotonically. This implies that at any admissible rationing scheme the utility can be increased by an arbitrarily small change of the rationing scheme. Hence, at any admissible rationing scheme it is possible to improve the network state locally. We first introduce some notation. The set

$$X^i(p) = \{y^i \in Y^i | \text{ there exists some } e^i \in E^i, \text{ such that } y^i = y^i(p,e^i)\}.$$

Since $y^i(p,e^i)$ maximizes the profit of the production sector given the vector p of prices and the rationing scheme e^i, we have that $y^i(p,e^i) \in \delta Y^i$. Under profit maximization of the production sector, any $y^i \in X^i(p)$ can be forced by the central planning institution through rationing of the inputs of sector i. We have the following lemma.

Lemma 3.1.

Under Assumptions A_1 and A_2, $X^i(p) = \{y^i \in \delta Y^i | D^i_k(y^i)/D^i_i(y^i) \geq p_k/p_i, k \neq i\}$.

Proof. The vector $y^i(p,e^i)$ solves the maximization problem

(3.1) max py^i subject to $y^i \in Y^i$ and $y^i_k \geq -e^i_k$, $k \neq i$.

Since $p >> 0$ it follows that $y^i(p,e^i) \in \delta Y^i$ so that $y^i(p,e^i)$ satisfies $f^i(y^i(p,e^i)) = 0$. Since py^i is a concave function of y^i and $f^i(y^i)$ is a convex function we can apply the Kuhn Tucker theorem and it follows from the extended Lagrangian

$$L = py^i - \lambda f^i(y^i) - \Sigma_{k \neq i} \mu_k(y^i_k + e^i_k)$$

that the solution $y^i(p,e^i)$ satisfies

$$p_i = \lambda D^i_i(y^i)$$

and

$$p_k = \lambda D^i_k(y^i) + \mu_k$$

with $\mu_k \geq 0$ the Kuhn Tucker multiplier of $y^i_k + e^i_k \geq 0$. From this it follows that $D^i_k(y^i)/D^i_i(y^i) \geq p_k/p_i$, $k \neq i$, for any $y^i \in X^i(p)$. On the other hand, if $y^{*i} \in \delta Y^i$ satisfies $D^i_k(y^{*i})/D^i_i(y^{*i}) \geq p_k/p_i$, then y^{*i} solves (3.1) with $y^{*i}_k = -e^i_k$, $k \neq i$.

Lemma 3.1 says that $X^i(p)$ is the set of production vectors on δY^i, such that the rate of commodity transformation between the commodities k and i is at least equal to the ratio of the prices of the commodities k and i. So, $X^i(p)$ is the set of prodution vectors at which the shadow prices of the inputs are at least equal to the real prices. Furthermore, let $P^i(p)$ be the orthogonal projection of $X^i(p)$ on $\{x \in R^n | x_i = 0\}$, i.e.,

$$P^i(p) = \{z \in R^n | z_i = 0 \text{ and there exists some } y_i \text{, such that } z_1,...,z_{i-1},y_i,z_{i+1},...,z_n)^\top \in X^i(p)\}$$

From Assumption A_3 it follows that $P^i(p) \subset Y^i$.

Assumption.

C. for each i, $P^i(p)$ is convex.

For $y \in X^i(p)$, let $y^{-i} = (y_1,...,y_{i-1},0,y_{i+1},...,y_n)^\top \in P^i(p)$ be the corresponding vector of inputs. Then, assumption C says that for each two x, $y \in X^i(p)$ and for any $0 < \alpha < 1$, there is a vector $z \in X^i(p)$, such $z^{-i} = \alpha x^{-i} + (1-\alpha)y^{-i}$, i.e., for each mixture of the inputs of x and y, there is a $z \in X^i(p)$ which inputs are exactly that mixture. So, if two production vectors can be forced by the central planning institution, then the production vector in δY^i corresponding to any mixture of the two input vectors can be forced. This is a rather strong assumption, which does not follow from assumption A_1. In appendix A we show that a Cobb-Douglas transformation function satisfies assumption C. In appendix B we give an example of a transformation function, which satisfies assumption A_1, but does not satisfy assumption C.

Finally, we define $C^i(p) = \text{Conv}\ \{X^i(p), P^i(p)\}$, where Conv {A,B} means the convex hull of the sets A and B and let $C(p) = \Sigma_i C^i(p)$.

Lemma 3.2. For all i, $C^i(p) \subset Y^i$ and $0 \in C^i(p)$. Moreover, C(p) is convex and $C(p) \subset Y$.

Proof.
Since $X^i(p) \subset Y^i$, $P^i(p) \subset Y^i$ and Y^i is convex it follows that $C^i(p) \subset Y^i$, $i = 1,...,n$. Moreover, analogously to lemma 2.3 we have that $0 \in X^i(p)$ and hence $0 \in C^i(p)$. Clearly, since the sum of convex sets is convex, C(p) is convex. Finally, from $C^i(p) \subset Y^i$ it follows that $C(p) = \Sigma_i C^i(p) \subset \Sigma_i Y^i = Y$.

From lemma 3.1 it follows that $X^i(p)$ belongs to the boundary of $C^i(p)$, while $X(p) \equiv \Sigma_i X^i(p)$ belongs to C(p). Now, let $e = \{e^1,...,e^n\}$ be an admissible rationing scheme. From lemma 2.3 we know that such a rationing scheme exists. Furthermore, let $e^*(p) = \{e^{*1}(p),...,e^{*n}(p)\}$ be an efficient rationing scheme and $x^*(p) = x(y(p,e^*(p)) = \Sigma_i y^i(p,e^{*i}(p))$ the corresponding net output vector.

Lemma 3.3. There exists an efficient rationing scheme $e^*(p) = \{e^{*1}(p),...,e^{*n}(p)\}$. Moreover, $x^*(p)$ is unique.

Proof. Clearly $x^*(p)$ follows from

$$\max u(z) \text{ such that } z \in C(p) \text{ and } z + w \geq 0.$$

From Lemma 3.1 it follows that $X^i(p)$ is closed in δY^i, so that $C^i(p)$ is a closed subset of Y^i. By definition, we have that $C^i(p)$ is convex. So, C(p) is closed and convex as the sum of closed and convex sets. From assumption A_1 it follows that Y is closed and convex, so that also $Y_w \equiv Y \cap \{ z \in R^n | z + w \geq 0\}$ is closed and convex. Moreover, from the assumptions A_3 and A_5 it follows that the set Y_w is bounded and hence Y_w is compact and convex and hence $C(p) \cap Y_w$ is compact and convex. Since $C(p) \subset Y$ we have that $C(p) \cap Y_w = C(p) \cap \{z \in R^n \mid z + w \geq 0\}$ and hence $C(p) \cap \{z \in R^n \mid z + w \geq 0\}$ is compact and convex. Since $0 \in C(p)$ it follows that there exists a feasible z with $z \in C(p)$. From assumption B_1 it follows that u(z) has a unique maximum $x^*(p)$ on $C(p) \cap \{z \in R^n \mid z + w \geq 0\}$. Since $C(p) = \Sigma_i C^i(p)$ we have that there exists elements $x^i \in C^i(p)$, $i = 1,...,n$, such that $x^*(p) = \Sigma_i x^i$. From assumption B_2 it follows that $x^i \in \delta Y^i$ and hence it follows with assumption C that $x^i \in X^i(p)$, $i = 1,...,n$. By definition this implies that there exists a rationing scheme $\{e^{*1}(p),...,e^{*n}(p)\}$ such that $x^i = y^i(p,e^{*i}(p))$, $i = 1,...,n$.

Let $x^*(p)$ be the utility maximizing element of $C(p)$ and $x^i = y^i(p,e^{*i})$ corresponding profit maximizing elements in δY^i under e^{*i}. Although $x^*(p)$ is unique, it may be sustained by various rationing schemes. For instance, suppose that e^{*i} satisfies $y^i{}_k(p,e^*) > -e^{*i}{}_k$. Then each rationing e^i with $y^i{}_k(p,e^*) \geq -e^i{}_k$ and $e^i{}_j = e^{*i}{}_j$, $j \neq k$ yields $x^i = y^i(p,e^i)$. To obtain a unique choice of the rationing scheme we define the set $E^i(p)$ of minimal rationing vectors given the price vector p, i.e.,

$$E^i(p) = \{ e^i \in R^n \mid e^i{}_i = \infty \text{ and } e^i{}_k = -y^i{}_k(p,e^i),\ k \neq i \}.$$

So, if $e^i \in E^i(p)$, we have that in the profit maximizing point each input is equal to the maximium input. Now, let $E(p) = \Pi_i E^i(p)$ and let $e^*(p) = \{e^{*1}(p),...,e^{*n}(p)\}$ be the unique rationing scheme in $E(p)$ sustaining x^i, $i = 1,...,n$, with $x^i \in X^i(p)$, $i = 1,...,n$, such that $x^*(p) = \Sigma_i x^i$.

We consider now the problem how to reach the efficient rationing scheme $e^*(p) = \{e^{*1}(p),...,e^{*n}(p)\} \in E(p)$. Let $e = \{e^1,...e^n\}$ be an arbitrarily admissible rationing scheme given a vector of fixed prices p and let $y^i(p,e^i)$ be the corresponding profit maximizing production of production sector i. Furthermore, let $d = \{d^1,...,d^n\} \in E(p)$ be the corresponding minimal rationing scheme. We now define a path of rationing schemes in $E(p)$ from d to e^* and we show that adjusting the rationing scheme from d to e^* the utility of the network state increases monotonically by using a Liapunov function. For $t \in [0,\infty)$, let $e(t)$ be defined by

(3.2) $\delta e(t)/\delta t = e^*(p) - e(t)$ with $e(0) = d$.

<u>Lemma 3.4.</u>
For any t, e(t) is a rationing scheme in E(p).

<u>Proof.</u>
The solution to (3.2) is $e(t) = (1 - e^{-t})e^*(p) + e^{-t}d$, $t \geq 0$. Hence, for each t, e(t) is a convec combination of $e^*(p)$ and d. Since $e^*(p)$ and d are both minimal rationing schemes we have for all i that

(3.3) $y^i{}_k(p,e^*(p)) = -e^{*i}{}_k(p)$ and $y^i{}_k(p,d) = -d^i{}_k$, for all $k \neq i$.

Because of assumption C there is for all t a rationing scheme $e^0(t) \in E(p)$, such that for all i

(3.4) $y^i{}_k(p,e^0(t)) = (1 - e^{-t})y^i{}_k(p,e^*(p)) + e^{-t}y^i{}_k(p,d)$, $k \neq i$,

Since $e^0(t)$ is a minimal rationing scheme we also have that

(3.5) $y^i{}_k(p,e^0(t)) = -e^{0i}{}_k(t)$, $k \neq i$.

From (3.3), (3.4) and (3.5) it follows that $e^{0i}{}_k(t) = (1-e^{-t})e^{*i}{}_k(p) + e^{-t}d^i{}_k$ and hence $e^{0i}{}_k(t) = e^i{}_k(t)$, $k \neq i$, $i = 1,...,n$, which proves that $e(t) \in E(p)$ for all t.

We now define the function V: E → R by $V(e) = u(y(p,e^*(p))) - u(y(p,e))$. Recall that $u(y(p,e)) = u(x(y(p,e))+w)$ with $x(y(p,e)) = \Sigma_i y^i(p,e)$.

Lemma 3.5. The function V(e) is a Liapunov function.

Proof.
By definition, $V(e^*(p)) = 0$ and $V(e) > 0$ for all $e \neq e^*(p)$. It remains to show that $\delta V(e(t))/\delta t < 0$ for $t \geq 0$. To do so we define for each t the function $f_t(\tau)$: $[0,1] \to E(p)$ by

$$f_t(\tau) = \tau e^*(p) + (1 - \tau)e(t).$$

We have that

(3.6) $\delta f_t(\tau)/\delta\tau|_{\tau=0} = e^*(p) - e(t) = \delta e(t)/\delta t.$

According to Lemma 3.4 we have that $e(t) \in E(p)$, so that for all $\tau \in [0,1]$, $f_t(\tau)$ is a convex combination of the two rationing schemes in E(p). Analogously to the proof of Lemma 3.4, we have that $f_t(\tau) \in E(p)$ for all τ. Hence, it follows that for all i

$$y^i{}_k(p,f_t(\tau)) = \tau y^i{}_k(p,e^*(p)) + (1-\tau)e^{-t}y^i{}_k(p,e(t)),\ k \neq i$$

and hence we have that the inputs in τ are a convex combination of the inputs at $\tau = 0$ and $\tau = 1$. Since the production sets are strict convex and $y^i(p,f_t(\tau)) \in \delta Y^i$, this implies that

$$y^i{}_1(p,f_t(\tau)) > \tau y^i{}_1(p,e^*(p)) + (1-\tau)y^i{}_1(p,e(t))$$

Since the production sets have a smooth boundary this implies that there exisits a smooth function r_t: $[0,1] \to R^n$, such that

(3.7) $x(y(p,f_t(\tau)) = \Sigma_i y^i(p,f_t(\tau)) = \tau\Sigma_i y^i(p,e^*(p)) + (1-\tau)\Sigma_i y^i(p,e(t)) + r_t(\tau),$

with $r_t(0) = r_t(1) = 0$ and for all τ, $0 < \tau < 1$, $r_{ti}(\tau) > 0$, $i = 1,\ldots,n$. From (3.7) it follows that

(3.8) $\delta[x(y(p,f_t(\tau))]/\delta\tau = \Sigma_i y^i(p,e^*(p)) - \Sigma_i y^i(p,e(t)) + \delta r_t(\tau)/\delta\tau =$

$$= x(y(p,e^*(p)) - x(y(p,e(t))) + \delta r_t(\tau)/\delta\tau.$$

On the other hand we have that

(3.9) $\delta[x(y(p,e(t))]/\delta t = \Sigma_i\Sigma_k\ [\delta x/\delta e^i{}_k][\delta e^i{}_k(t)/\delta t].$

From (3.6) and (3.9) it follows that

(3.10) $\delta[x(y(p,e(t))]/\delta t = \Sigma_i\Sigma_k\ [\delta x/\delta e^i{}_k][\delta f_t(\tau)^i{}_k/\delta\tau|_{\tau=0}] = \delta[x(y(p,f_t(\tau))]/\delta\tau|_{\tau=0}$.

Hence it follows with (3.8) and (3.10) that

$$\delta V(e(t))/\delta t = -\ \delta u(y(p,e(t)))/\delta t = -\ (Du)^T\delta x(y(p,e(t)))/\delta t =$$

$$= -(Du)^T[x(y(p,e^*(p)) - x(y(p,e(t))) + \delta r_t(\tau)/\delta\tau|_{\tau=0}]$$

with $Du = (\delta u/\delta x_1,...,\delta u/\delta x_n)^T$ the gradient vector of the utility function u in x(y(p,e(t))). Since u is strict quasi-concave we have that $(Du)^T[x(y(p,e^*(p)) - x(y(p,e(t)))] > 0$. Moreover, since $r_{ti}(\tau) > 0$, i = 1,...,n and $r_t(0) = 0$, we have that $\delta r_{ti}(\tau)/\delta\tau|_{\tau=0} > 0$, i = 1,...,n. Since u is strictly monotone increasing it follows thast $(Du)^T[\delta r_t(\tau)/\delta\tau|_{\tau=0}] > 0$. This proves that $\delta V(e(t))/\delta t < 0$ for all t.

Theorem 3.6. Under the assumptions A, B and C we have that
(i) The solution path of the system of differential equations $\delta e(t)/\delta t = e^*(p) - e(t)$ with $e(0) = d$ converges to the efficient rationing scheme $e^*(p)$ for each $d \in E(p)$.
(ii) The utility u(y(p,e(t))) increases monotonically along the solution path.

Proof.
(i) From Lemma 3.4 we have that e(t) is in the compact set E(p) for all t. Lemma 3.5 says that V(e) is a Liapunov function satisfying $V(e) > V(e^*(p)) = 0$ for all $e \neq e^*(p)$ and $\delta V(e(t)/\delta t < 0$ for all $t \geq 0$. This proves assertion (i).
(ii) This follows immediately from the fact that $\delta u(y(p,e(t)))/\delta t = -\ \delta V(e(t))/\delta t > 0$ for all $t > 0$.

To define the system of differential equations we have to know the efficient rationing scheme $e^*(p)$. This is rather unsatisfactorily, because it means that we have to know the global optimum in advance in order to be able to define the adjustment process. In fact we are looking for an adjustment process based on local information. Although such a process is not available, theorem 3.6 implies an important result. It says that for any initial rationing schem $d \in E(p)$, the utility increases along the line from d to $e^*(p)$. So, for any d we have that the rationing scheme $e = d + \lambda(e^*(p) - d)$ yields a higher utility for any $\lambda > 0$. So, the utility can be improved locally and to find a better rationing scheme a local seaich suffices. This gives the main result of this section.

Main result. For any $d \in E(p)$, $d \neq e^*(p)$, there is a rationing scheme $e \in E(p)$ arbitrarily close to d, such that the utility under e is higher than the utility under d.

4. Adjustment of the prices.

In this section we give a price adjustment process along which the utility under the efficient rationing scheme increases monotonically. It will be shown that the process converges to a system of prices at which the corresponding network state is optimal. Because of the strict convexity of the production sets and the strict quasi-concavity of the utility function there is a unique optimal network state $y = \{y^1,...,y^n\}$. It is well-known that under the assumptions A and B there exists a price vector $p^* > 0$, such that

(i) $p^* y^i \geq p^* y^i$ for all $y^i \in Y^i$, $i = 1,..,n$,
(ii) $p^* x > p^* x(y)$ for all x such that $u(x) > u(x(y))$

for the optimal network state $y = \{y^1,...,y^n\}$ (see e.g. Debreu [2]). So there is a price vector p^*, such that the optimal network state y is an equilibrium of the economy relative to the price system p^*. In the following we normalize the sum of the prices to be equal to one, so that each price vector $p \in S = \{p \in R^n \mid p_j \geq 0 \text{ and } \Sigma_j p_j = 1\}$. We also normalize the sum of the components of the gradient of u to be equal to one. Then, because of the differentiability of the utility function the price vector p^* is unique and equal to the normalized gradient of u(x) at x(y). We show that for any arbitrarily chosen initial price system p^0, there exists a price adjustment mechanism leading from p^0 to p^*, such that along the path of prices the utility of the efficient network state $y(p,e^*(p))$ increases monotonically. At p^*, we reach the optimal network state as the efficient network state $y(p^*,e^*(p^*))$. Let $u(p) \equiv u(x^*(p)+w)$ be the utility of the unique optimal element of C(p). Then we define Du(p) to be the normalized gradient of u at $x = x^*(p)+w$, i.e.

$$Du(p) = (\delta u/\delta x_1,...,\delta u/\delta x_1)^T/[\Sigma_j \delta u/\delta x_j]$$

at $x = x^*(p)+w$. Observe that assumption B_2 implies that at each $x \geq 0$, $\delta u/\delta x_j > 0$, $j = 1,...,n$. We now define for $t \geq 0$, the system of differential equations

(4.1) $\delta p(t)/\delta t = Du(p(t)) - p(t)$,

with $p(0) = p \in \text{int } S$ some arbitrarily chosen initial price system. We show that the solution of this system of differential equations converges to p^*.

Lemma 4.1.
For any t, $p(t) \in S$.

Proof. Take $h(t) = \Sigma_j p_j(t)$. Then $\delta h(t)/\delta t = \Sigma_j \delta p_j(t)/\delta t = \Sigma_j \{Du(p(t))_j - p_j(t)\} = 1 - 1 = 0$. Hence h(t) i a constant function of t and $\Sigma_j p_j(t) = \Sigma_j p_j(0) = \Sigma_j p_j = 1$ for all j. Moroover, $Du(p(t))_j > 0$ and hence $\delta p_j(t)/\delta t > 0$ if $p_j(t) = 0$. Hence, for any t, $p_j(t) \geq 0$, $j = 1,...,n$. This proves that $p(t) \in S$ for all t.

We now define the function $V: S \rightarrow R$ by $V(p) = u(x^*(p^*)+w) - u(x^*(p)+w)$.

Lemma 4.2. The function V(p) is a Liapunov function.

Proof.

By definition, $V(p^*) = 0$ and $V(p) > 0$ for all $p \neq p^*$. It remains to show that $\delta V(p(t))/\delta t < 0$ for $t \geq 0$, or equivalently that $\delta u(x^*(p(t))+w)/\delta t > 0$ for $t \geq 0$. With $u(t) \equiv u(p(t))$ we have by definition of $u(p)$ that

$$u(t) = u(x^*(p(t))+w) = \max_{e \in E(p(t))} u(y(p(t),e)).$$

Equivalently for a network state $y = (y^1,\ldots,y^n)$ we have with $u(y) = u(x(y)+w)$ that

$$u(t) = \max_y u(y) \text{ such that } y^i \in X^i(p(t)),\ i = 1,\ldots,n.$$

Let $y^{*i}(t)$, $i = 1,\ldots,n$, be the solution to this maximization problem. Then we have from the convexity of u that on $t = \tau$, $(Du(\tau))^T y^i \leq (Du(\tau))^T y^{*i}(t)$ for all $y^i \in X^i(p(t))$. Hence, on $t = \tau$, $y^{*i}(t)$ is the solution to

$$\max\ (Du(\tau))^T y^i \text{ such that } y^i \in X^i(p(t))$$

Let $\pi^i_\tau(t)$ be defined by

$$\pi^i_\tau(t) = \max\ (Du(\tau))^T y^i \text{ such that } y^i \in X^i(p(t)).$$

So, by definition we have that $\pi^i_\tau(t) = (Du(\tau))^T y^{*i}(t)$ on $t = \tau$. Hence,

$$\delta u(t)/\delta t|_{t=\tau} = \delta u(x^*(p(t))+w)/\delta t|_{t=\tau} = \{(Du(t))^T \Sigma_i\ \delta y^{*i}(t)/\delta t\}|_{t=\tau}$$
$$= (Du(\tau))^T \Sigma_i\ \delta y^{*i}(t)/\delta t\}|_{t=\tau} = \Sigma_i\ \delta\pi^i_\tau(t)/\delta t|_{t=\tau}.$$

It remains to prove that $\Sigma_i\ \delta\pi^i_\tau(t)/\delta t|_{t=\tau} > 0$ for all τ. Therefore we prove that for all i, $\delta\pi^i_\tau(t)/\delta t|_{t=\tau} > 0$ for all τ.

From Lemma 3.1, it follows that

$$X^i(p(t)) = \{y^i \in \delta Y^i \mid D^i_k(y^i)/D^i_i(y^i) \geq p_k(t)/p_i(t),\ k \neq i\},$$

where $D^i_j(y^i) = \delta f^i(y^i)/\delta y^i_j$, $j = 1,\ldots,n$, with $f^i\colon R^n \to R$ such that $f^i(y^i) = 0$ if $y^i \in \delta Y^i$. From production theory we know that f^i can be chosen such that $f^i(y^i) = y^i_i - g^i(y^{-i})$ with g^i the strict concave efficiency function yielding output y^i_i to the input vector $y^{-i} = (y^i_1,\ldots,y^i_{i-1},y^i_{i+1},\ldots,y^i_n)^T$. Taking $f^i(y^i) = y^i_i - g^i(y^{-i})$, we have that $D^i_i(y^i) = 1$ for all y^i. So, for this function f^i, $X^i(p(t))$ becomes

$$X^i(p(t)) = \{y^i \in \delta Y^i \mid D^i_k(y^i) \geq \bar{p}_k(t),\ k \neq i\},$$

with $\bar{p}_k(t) = p_k(t)/p_i(t)$. Hence $\pi^i_\tau(t) = \max\ (Du(\tau))^T y^i$ such that $f^i(y^i) = 0$ and $D^i_k(y^i) \geq \bar{p}_k(t)$, $k \neq i$. Let

$$L^i_\tau(y^i,\lambda,\mu;t) = (Du(\tau))^T y^i - \lambda f^i(y^i) - \Sigma_{k \neq i}\ \mu_k(\bar{p}_k(t) - D^i_k(y^i))$$

be the associated Lagrangian with solution $y^i(t)$, $\lambda(t) \geq 0$ and $\mu_k(t) \geq 0$ to the maximization problem and with $\mu_k(t) > 0$ if the corresponding constraint is binding. Then applying the envelope theorem we have that

$$(4.2)\quad \delta\pi^i_\tau(t)/\delta t = \delta L^i_\tau(y^i,\lambda,\mu;t)/\delta t = -\Sigma_{k\neq i}\,\mu_k(t)\delta \bar{p}_k(t)/\delta t.$$

Now observe that

$$(4.3)\quad \delta\bar{p}_k(t)/\delta t = \delta(p_k(t)/p_i(t))/\delta t = \{\delta(p_k(t)/\delta t\}/p_i(t) - p_k(t)\{\delta p_i(t)/\delta t\}/\{p_i(t)\}^2.$$

With $u(t) \equiv u(p(t))$ it follows from (4.1) and (4.3) that

$$(4.4)\quad \delta\bar{p}_k(t)/\delta t = \delta(p_k(t)/p_i(t))/\delta t =$$
$$\{Du(t))_k - p_k(t)\}/p_i(t) - p_k(t)\{Du(t))_i - p_i(t)\}/\{p_i(t)\}^2 =$$
$$\{Du(t))_i/p_i(t)\}\{Du(t))_k/Du(t))_i - \bar{p}_k(t)\}.$$

From (4.2) and (4.4) it follows that

$$(4.5)\quad \delta\pi^i_\tau(t)/\delta t|_{t=\tau} = -\Sigma_{k\neq i}\,\mu_k(\tau)\delta\bar{p}_k(t)/\delta t|_{t=\tau} =$$
$$-\Sigma_{k\neq i}\,\mu_k(\tau)\{Du(\tau))_i/p_i(\tau)\}\{Du(\tau))_k/Du(\tau))_i - \bar{p}_k(\tau)\} =$$
$$-\Sigma_{k\neq i}\,\mu^*_k(\tau)\{Du(\tau))_k/Du(\tau))_i - \bar{p}_k(\tau)\}$$

with $\mu^*_k(\tau) = \mu_k(\tau)\{Du(\tau))_i/p_i(\tau)\} \geq 0$, $k = 1,\dots,n$. From $\delta L^i_\tau(y^i,\lambda,\mu;t)/\delta y^i_j = 0$ we obtain that

$$(4.6)\quad Du(\tau)_j = \lambda(\tau)D_j(y^i) - \Sigma_{k\neq i}\,\mu_k(\tau)D^i_{kj}(y^i),$$

at $y^i = y^i(\tau)$, $j = 1,\dots,n$, where $D^i_{kj}(y^i) = \delta D^i_k(y^i)/\delta y^i_j = \delta^2 f^i(y^i)/\delta y^i_k\delta y^i_j$. Since $D^i_i(y^i) = 1$ and hence $D^i_{ki}(y^i) = D^i_{ik}(y^i) = 0$, for $j = i$ expression (4.6) becomes

$$(4.7)\quad Du(\tau)_i = \lambda(\tau)D_i(y^i) = \lambda(\tau).$$

Hence

$$(4.8)\quad \delta\pi^i_\tau(t)/\delta t|_{t=\tau} = -\Sigma_{j\neq i}\,\mu^*_j(\tau)\{Du(\tau))_j/Du(\tau))_i - \bar{p}_j(\tau)\} =$$
$$-\Sigma_{j\neq i}\,\mu^*_j(\tau)\{\lambda(\tau)D^i_j(y^i) - \Sigma_{k\neq i}\,\mu_k(\tau)D^i_{kj}(y^i)\}/Du(\tau))_i + \Sigma_{j\neq i}\,\mu^*_j(\tau)\bar{p}_j(\tau)\} =$$
$$\Sigma_{j\neq i}\,\mu^*_j(\tau)\{\bar{p}_j(\tau) - \lambda(\tau)D^i_j(y^i)/Du(\tau))_i\} +$$
$$[Du(\tau)_i]^{-1}\Sigma_{j\neq i}\Sigma_{k\neq i}\mu^*_j(\tau)\mu_k(\tau)D^i_{kj}(y^i) =$$
$$\Sigma_{j\neq i}\,\mu^*_j(\tau)\{\bar{p}_j(\tau) - D^i_j(y^i)\} + \Sigma_j\Sigma_k\mu_j(\tau)\mu_k(\tau)D^i_{kj}(y^i)/p_i(\tau).$$

Since for all $j \neq i$, $\mu^*_j(\tau) = 0$ if $\bar{p}_j(\tau) \neq D^i_j(y^i)$ and the Hessian matrix $[D^i_{kj}(y^i)]_{j,k=1,\dots,n}$ of the convex function $f^i(y^i)$ is positive definite, (4.8) becomes

$$\delta\pi^i_\tau(t)/\delta t|_{t=\tau} = \Sigma_j\Sigma_k\mu_j(\tau)\mu_k(\tau)D^i_{kj}(y^i)/p_i(\tau) > 0,\ i = 1,...,n.$$

This proves that $\delta u(t)/\delta t|_{t=\tau} = \Sigma_i\ \delta\pi^i_\tau(t)/\delta t|_{t=\tau} > 0$ and hence V(p) is indeed a Liapunov function.

Theorem 4.3. Under the assumptions A, B and C we have that

(i) The solution path of the system of differential equations $\delta p(t)/\delta t = Du(p(t)) - p(t)$ with $p(0) = p$ converges to the optimal price vector p^* for each positive price vector p^0.

(ii) The utility $u(p(t)) = u(x^*(p(t)))$ increases monotonically along the solution path.

Proof.
(i) From Lemma 4.1 we have that p(t) is in the compact set S for all t. Lemma 4.2 says that V(p) is a Liapunov function satisfying $V(p) > V(p^*) = 0$ for all $p \neq p^*$ and $\delta V(p(t)/\delta t < 0$ for all $t \geq 0$. This proves assertion (i).
(ii) This follows immediately from the fact that $\delta u(p(t))/\delta t = - \delta V(p(t))/\delta t > 0$ for all $t > 0$.

The main result of Theorem 4.3 is that the local adjustment of the prices according to the system of differential equations (4.1) yields a convergent process along with the utility increases monotonically.

References

[1] E.M. Braverman and M.I. Levin, Identification of Effective Network States of Industrial Elements I, II and III, *Automation and Remote Control*, Vol 39, nos. 6, 7 and 9 (translated form Russian), 1978.

[2] G. Debreu, *Theory of Value* (Yale University Press, New Haven, 1959).

Appendix A.

In this appendix we show that Cobb-Douglas production functions satisfy assumption C. In case sector i has a Cobb-Douglas production set, we have that

$$Y^i = \{y^i \in R^n | \ y^i_i \leq K_i \Pi_{j \neq i} (-y^i_j)^{\alpha_j},\ y^i_j \leq 0\}.$$

Let the transformation function be given by

$$f^i(y^i) = \ln y^i_i - \ln K_i - \Sigma_{j \neq i}\ \alpha_j \ln(-y^i_j)$$

Then we have that $D^i_k(y^i)/D^i_i(y^i) = \alpha_k y^i_i/(-y^i_k)$ and hence

$$X^i(p) = \{y^i \in \delta Y^i | \ \alpha_k y^i_i/(-y^i_k) \geq p_k/p_i,\ k \neq i\}.$$

With $y^i_i = K_i \Pi_{j \neq i} (-y^i_j)^{\alpha_j}$ for $y^i \in \delta Y^i$ we obtain

$$X^i(p) = \{y^i \in \delta Y^i | \ \{\alpha_k K_i p_i/p_k\} \Pi_{j \neq i,k} (-y^i_j)^{\alpha_j} \geq (-y^i_k)^{1-\alpha_k},\ k \neq i\}.$$

Now, let y^i an element of $P^i(p)$. Then by definition of $P^i(p)$, $y^i_i = 0$ and the other components of y^i satisfies the 'Cobb-Douglas' restriction

$$(-y^i_k)^{1-\alpha_k} \leq \{\alpha_k K_i p_i/p_k\} \Pi_{j \neq i,k} (-y^i_j)^{\alpha_j},\ k \neq i.$$

From this it follows that for any two elements $u^i \in P^i(p)$ and $v^i \in P^i(p)$ als the convex combination

$$w^i = tu^i + (1-t)v^i$$

is in $P^i(p)$ for any $t \in [0,1]$. Hence $P^i(p)$ is convex.

Appendix B.

In this appendix we show by a counterexample that the assumption of the convexity of Y^i does not imply the convexity of $P^i(p)$. For $n = 3$ we take for $i = 1$

$$f^1(y) = y_1 + (y_2 + 1)^2 + (y_3 + 1)^2 - (y_2)^2 y_3 - 2$$

and

$$Y^1 = \{y^1 \in R^3 | -1 \leq y^1{}_2, y^1{}_3 \leq 0 \leq y^1{}_1 \text{ and } f^1(y^1) \leq 0\}$$

This set Y^i is convex. However, for $p = (1,1,1)^T$, we obtain with for ease of notation y^1 denoted by y that

$$X^i(p) = \{y \in \delta Y^i | \; 2(y_2 + 1) - 2y_2 y_3 \geq 1 \text{ and } 2(y_3 + 1) - (y_2)^2 \geq 1\}$$

and hence

$$P^i(p) = \{y \in R^3 \mid y_1 = 0, -1 \leq y_2, y_3 \leq 0, 2(y_2 + 1) - 2y_2 y_3 \geq 1 \text{ and } 2(y_3 + 1) - (y_2)^2 \geq 1\}.$$

Hence, for $y \in P^i(p)$, $-1 \leq y_2, y_3 \leq 0$ satisfy

$$-y_2 \leq 1/(2 - 2y_3) \text{ and } (y_2)^2 \leq 1 + 2y_3.$$

This set of feasible values of y_2 and y_3 is not convex.