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On the Effect of Small Loss Probabilities in Input/Output Transmission Delay Systems

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ON THE EFFECT OF SMALL LOSS PROBABILITIES IN INPUT/OUTPUT TRANSMISSION DELAY SYSTEMS

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Abstract A transmission system is studied with a two-stage delay structure of multiple input and output buffers and packet or message transmission loss probabilities such as due time-slotting or transmission errors. A priori error bounds are derived for the accuracy of simple product form estimates based on disregarding the loss probabilities. The results support practical engineering and seem promising for extension.

Keywords Transmission system * Loss probability * product form estimates * Error bound.

1. INTRODUCTION

Motivation

Queueing network modeling has become a generally accepted tool to evaluate the performance of transmission systems in computer and communication networks ([1],[3],[6]). Unfortunately, simple closed form expressions are generally destroyed by practical phenomena such as message losses. Such losses may naturally arise form resource contentions, time-slotting, transmission errors or link failures.

In present-day digitized communications technology, however, the effect of these features is more and more reduced by special protocols (e.g. for transmission error corrections) and extremely large capacities (e.g. of optical fibres). For evaluation purposes, it thus seems appealing to ignore these losses so as to provide simple approximations. Clearly, such approximations can be expected to be reasonable only when losses will not occur too frequently, that is when the loss probabilities are small. In this light, the approximations should primarily be seen as first indicators of orders of magnitude rather than as accurate estimates. Nevertheless, the results do seem of practical interest for quick evaluation purposes at low computational expense. However, no formal justification for such approximations seems to be available.

Objective

This paper, aims to provide a first step in this direction. To this end, it studies a two-stage transmission delay structure with multiple input and output (store and forward) delay facilities (buffers) and state dependent transmission loss probabilities. Though clearly too simplistic and abstract for direct application in realistic situations, the store and forward structure studied can be seen as a typical generic component of more complicated realistic packet or circuit switching communications networks. Particularly, it involves the essential feature of state dependent loss probabilities. In this light the paper aims to illustrate how one can provide formal support for simplifying assumptions in practical communications engineering.

Results

Simple analytic error bounds will be provided for the accuracy of a product form estimate based on ignoring these state dependent losses. The error bounds are derived along the lines of a recently outlined approach using so-called bias-terms of Markov reward structures (cf. [8]). The actual verification of the conditions, however, requires special technical details (e.g. see lemmas 3.1 and 3.2), which form the main body of the paper, which have not been dealt with before.

For clarity of the essential steps involved, the proof will first be restricted to the single buffer case with only one input and output delay facility. The proof for the multi buffer-situation will then be argued as an immediate extension.

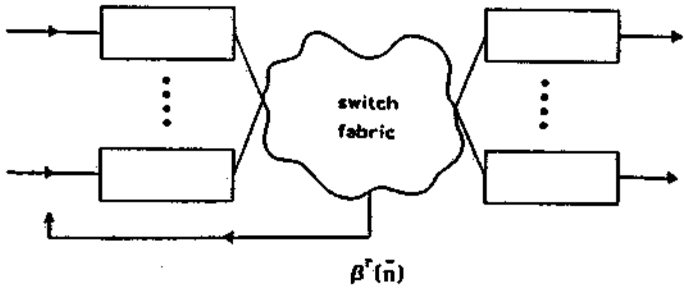
The organization is as follows. In section 2 the model is presented and motivated in some more detail. Next the product form estimates are proposed. Section 3 provides the technical proof for the single buffer case. Section 4 provides the essential extending steps for the multi-buffer case and discusses some variants and possible extensions.

2. MODEL AND ESTIMATES

First, in section 2.1, we will present the model under consideration in a somewhat abstract manner. More practical motivation of the structure and the loss probabilities involved will then be provided in section 2.2, while the simple product form estimates will be proposed in section 2.3.

2.1 Model

We consider a transmission or communication system which consists of a single switch or transmission fabric with multiple input and output delay buffers. More precisely, there are M input and associated output buffers (hereafter called queues). Packets or messages to be transmitted (hereafter called jobs) arrive at input queue r according to a Poisson process with parameter λ^r. This queue works at a first-come-first-served basis with exponential delay (or service) parameter μ_1^r. After delay at input queue r, a job (read packet message) is transmitted and enters output queue r. This output queue also works at a first-come-first-served basis with exponential delay (or service) parameter μ_2^r. The actual switching time at the switch or transmission fabric is hereby neglected.



However, at switching or transmission from an input to an output queue a job may experience a loss. More precisely, let $\bar{n} = (\bar{n}^1,\ldots,\bar{n}^M)$ with $\bar{n}^r=(n_1^r,n_2^r)$ denote the number of jobs n_1^r and n_2^r at the r-th input and output queue for all r. When the system is in state $\bar{n}$ a job from input queue r will then experience a loss at transmission with probability

$$\beta^r(\bar{n})$$

That is, the transmission will fail and the job (read packet or message) is said to be lost so that it has to be retransmitted. To this end, various retransmission protocols could be in order. For simplicity and also referring to remark 4.2.1, we assume that the lost job is instantaneously recirculated to the end of input queue r. The loss probabilities, though, should be thought of as rather small. Say, for some β

$$(2.1) \qquad \beta^r(\bar{n}) \leq \beta \qquad \text{(for all } r \text{ and } \bar{n}\text{)},$$

where, realistically, β is a rather small number, for example of order .1%. Further, referring to remark 4.2.3, for simplicity we also assume that the system in total and the queues individually have unlimited storage capacity.

2.2 Practical motivation

2.2.1 Two (multiple) delay structure

First let us give some global motivation for a two-stage delay structure. A two-stage transmission structure is a typical representative component of so-called store and forward switching (e.g. Verma, [10]). Herein, a communication between endpoints is allowed to be substantiated in intermediate stages with possible random delays at each stage. This type of switching is used, most notably, in packet switching for data communication in computer networks, where messages are broken up in individual parts or frames that are independently transmitted on a store and forward basis.

But the store and forward switching structure also applies to various more classical telecommunications or circuit switching situations, in which rather direct connections between end-to-end points, typically at large distance, along some virtual path are to be set up. For example, while in classical voice communication a channel from each intermediate trunkgroup was occupied during the total communication all at the same time, in present day asynchronous transfer modes message transmissions may take place via intermediate stages or nodes.

In particular, a two-stage delay structure directly applies to circuit switch models in which the output queues can be seen as the actual transmission devices, and thus the delays at an output queue as actual transmission time, while the input queues are used to buffer or control the input via an intermediate single switch or common input device for all transmission devices. This switch, for instance, may work in a time-slotted manner where only one packet can be processed per time-slot. In this respect we particularly refer to remark 4.2.2, in which the results that will be derived are rather directly extended to output queues with multi-channel or infinite server disciplines as would be more natural in that case.

In fact, as per remark 4.2.1, the assumption of first-come-first-served single server buffer delays is made as it seems natural in many applications, while it takes into account the complication of delays in the strongest manner which complicates the technical details of the analysis.

2.2.2 Packet or message loss probabilities

Roughly speaking, packet or message losses can be naturally caused by features like resource contentions, time-slotting transmission errors, link failures, message interactions, randomized channel allocation or external disturbances. Generally, losses are likely to be more frequent the more loaded a system becomes, though this might not even be expressable purely in a state dependent loss probability.

Let us briefly present some examples to motivate the analysis. These examples are artificial simplifications but they are meant merely to illustrate how state dependence can be involved and how it destroys analytical tractibility.

Example 1: (Time-slotted switch; input dependence)

Various switch devices work in a time-slotted manner in which only one packet (job) can be processed per time-slot. As multiple input queues may place a transmission request during one and the same time slot, this may lead to collisions and lost transmission requests. More precisely, let Δ represent the duration of one time-slot and assume that a transmission request, say of queue r, during a time-slot is succesful only if none of the other queues has requested during that same time-slot. Assuming no further causes for losses, the loss probability will then take the form:

$$\beta^r(\bar{n}) = 1 - \exp[-\Delta \Sigma_{k\neq r} 1_{(n_1^k>0)}\mu_1^k]$$

An exact (product form) solution for the steady state distribution $\pi(\bar{n})$ now applies if (and only if): $\mu_1^1 = \mu_1^2 = \ldots = \mu_1^N$ (see [9]), while without this condition no exact expression appears to be available.

Example 2: (Memory module; output dependence)

In computer network communications the actual processing of jobs may from time to time require brief communication with some memory device such as to retrieve or store data. But also to start processing of a job this memory device can be needed such as for labeling or addressing the job. As memory devices can usually handle only one job at a time this may give conflicts which lead to losses. As a somewhat artificial simplification, assume that a busy output queue k uses the memory device during a small fraction w^k of its time and that an actual transmission from an input to an output queue is possible only if none of the output queues is currently using the memory device. In state $\bar{n}$ and assuming no other causes for failures, a transmission of an arbitrary input queue r will then fail with loss probability

$$\beta^r(\bar{n}) = 1 - \prod_k (1 - w^k)\ 1_{(n_2^k>0)} .$$

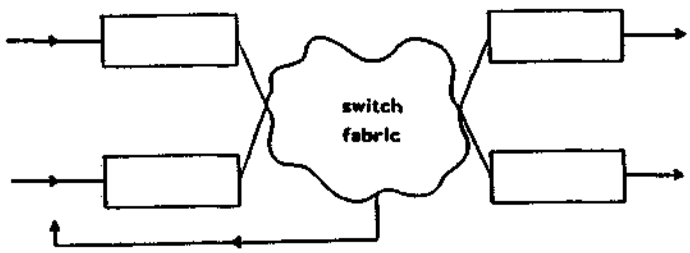
An exact (product form) solution for the steady state distribution $\pi(\bar{n})$ now applies if (and only if): $w^1 = w^2 = \ldots = w^N$, (see [9]), while without this conditions no exact expression appears to be obtainable.

Example 3: (Transmission errors)

The switching or transmission of a single packet may be subject to an error, say even with state independent error probability p. However, the detection of this error and its delay consequence after receiving a negative acknowledgement will typically depend on the loads at the queues. Also, the quality of error correction mechanisms usually depend on actual loads. For example, in the "backward error correction mechanism" (see [10], chapter 5), a fixed number of overhead slots is reserved for correcting errors. With higher loads, more consumption of these overhead slots takes place so that incorrected errors remain incorrected more frequently.

2.2.3 No product form expressions

Simple product form expressions, as per the statements in examples 1 and 2 above, seem to be restricted to very special situations. These statements were based on reversibility arguments.Let us consider a more extreme situation to provide intuitive insight in why state dependence will generally destroy product form expressions. As per [2], as well as many other related references, such expressions are directly related to notions of partial balance, in this case: "balance per station or queue", requiring that in any state the flow out of a queue has to be equal to the flow into that queue. Now consider the simple structure



with M = 2 and assume that

$$\beta^2(\bar{n}) = \begin{cases} 0 & \text{if } n_1^1 = 0 \\ 1 & \text{otherwise.} \end{cases}$$

That is, transmissions from input queue 1 are given strict priority over queue 2. Then, in a state with $n_1^1 > 0$, at input queue 2 we would have:

the flow out of input queue 2 = 0
the flow into that input queue > 0 (Poisson arrival rate λ_2).

The station or queue balance principle thus fails at input queue 2, so that, as per literature, the system cannot have an explicit product form expression. If rather than 1 in $\beta^2(\bar{n})$ one would read any small $\beta>0$, the same balance inconsistency would essentially remain, thus prohibiting a product form.

2.3 Simple product form estimates

We wish to evaluate the mean response times, that is the mean time of delay that jobs experience in the system for the various input/output connections, denoted by W^r for the r-th connection. By virtue of Little's law and the fact that jobs always get eventually through, we can compute W^r by

$$L^r = \lambda^r W^r$$

where L^r is the mean number of jobs in the r-th input/output connection. As the steady state distribution has no general simple expresion from which L^r can be computed when losses are involved, we first consider the system without losses.

Case 1 (No losses; $\beta=0$) When losses do not actually occur, that is when $\beta=0$ can be substituted in (2.1), the steady state distribution exhibits the product form, with $\rho_1^r = \lambda/\mu_1^r$ and $\rho_2^r = \lambda/\mu_2^r$:

$$(2.2) \qquad \pi(\bar{n}) = \prod_{r=1}^{M} [1-\rho_1^r]^{-1} [1-\rho_2^r]^{-1} [\rho_1^r]^{n_1^r} [\rho_2^r]^{n_2^r}$$

reflecting that both the various input/output connections and the input/output queue per r-th connection can be regarded as statistically independent with each queue as a single server system. As a particular consequence, the mean total number in the r-th connection is readily concluded as:

$$(2.3) \qquad L^r = \Sigma_{\bar{n}} \, \pi(\bar{n}) \, [n_1^r + n_2^r] = \rho_1^r/(1-\rho_1^r) + \rho_2^r/(1-\rho_2^r).$$

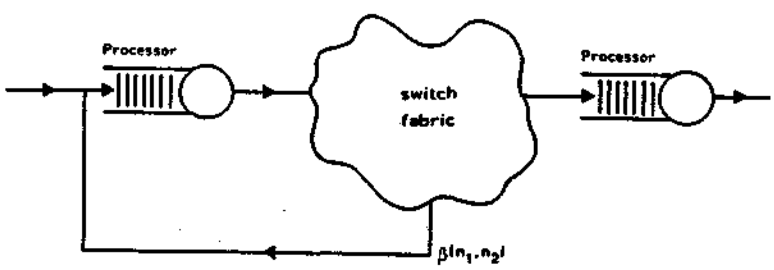
Case 2 (Losses) When state dependent losses can arise, as per the illustration and insight in section 2.2, a simple explicit expression for the steady state distribution and consequently the mean number of jobs at the queues is not generally available. But, as the loss probabilities can be small in realistic situations, say of order .1% or less, it thus seems appealing to use the above expressions for the case without losses as approximate values. More precisely, let $\bar{L}^r$ denote the mean number of jobs in the r-th input/output connection for the system with losses. To justify this practical approximation, the objective of this paper is then to provide (a priori) error bounds on:

$$|L^r - \bar{L}^r|.$$

3. Error bound: single buffer case

This section restricts to the single buffer case, that is with a single input and single output queue. The essential steps are hereby better highlighted while essential complications are covered which can directly be extended to the multiple buffer case as will be presented in the next section.

We will essentially apply a general approximation or perturbation theorem from [8] but the presentation will be kept self-contained as most of the technical details concern the verification of the conditions which are new and which involve special technicalities that have not been dealt with before



As only one input/output connection is considered, simplify the notation to $\bar{n} = (n_1,n_2)$ denoting the number of jobs n_1 at the input and n_2 at the output queue and suppress the superscript r in all notation, e.g. write L for L^r. Further, an expression for the system with losses (case 2) is denoted with an upper bar symbol, such as $\bar{\pi}$ and $\bar{P}$, while no symbol is used for the system without losses (case 1), such as π and P. The symbol "(-)" is used when an expression is to be read for both cases.

In order to compare the underlying continuous-time Markov chains in a convenient recursive manner, we will first apply the wellknown step of the uniformization (e.g. [5] or [7]). To this end, let $h = (\lambda+\mu_1+\mu_2)^{-1}$ and define discrete-time Markov chains, in correspondence with the continuous-time descriptions, with transition probabilities $\overset{(-)}{p}([n_1,n_2],[n_1',n_2'])$ for a transition from $[n_1,n_2]$ into $[n_1',n_2']$ by:

$$
(3.1) \qquad \begin{cases}
\overset{(-)}{p}([n_1,n_2],[n_1+1,n_2]) = h\lambda \\
\overset{(-)}{p}([n_1,n_2],[n_1,n_2-1]) = h\mu_2 1_{(n_2>0)} \\
p([n_1,n_2],[n_1-1,n_2+1]) = h\mu_1 1_{(n_1>0)} \\
\bar{p}([n_1,n_2],[n_1-1,n_2+1]) = h\mu_1 1_{(n_1>0)}\ [1-\beta(n_1,n_2)] \\
\overset{(-)}{p}([n_1,n_2],[n_1,n_2]) = 1 - \sum_{(n_1,n_2)\neq(n_1',n_2')} \overset{(-)}{p}([n_1,n_2],[n_1',n_2'])
\end{cases}
$$

where $1_{(A)} = 1$ if event A is satisfied and 0 otherwise. Further, also define

$$
(3.2) \qquad \overset{(-)}{V}_{t+1}(n_1,n_2) = [n_1+n_2] + \sum_{[m_1,m_2]} \overset{(-)}{p}([n_1,n_2],[m_1,m_2])\ \overset{(-)}{V}_t(m_1,m_2)
$$

for all (n_1,n_2) and $t \geq 0$ while $\overset{(-)}{V}_0(n_1,n_2) = 0$ for all (n_1,n_2).
Then by virtue of this uniformization technique (e.g. [7], p.110) we can conclude:

$$
(3.3) \qquad \overset{(-)}{L} = \lim_{N\to\infty} \frac{1}{N}\ \overset{(-)}{V}_N(\ell_1,\ell_2)
$$

for arbitrary initial state (ℓ_1,ℓ_2). This latter relation enables us to prove the following main result based on the technical lemmas 1 and 2 that will be presented below.

Theorem 3.1

with

$$
\begin{aligned}
&| L - \bar{L} | \le \beta\, C\, L_\beta \\
&C = [\mu_1+\mu_2-\lambda][\mu_1-\lambda]^{-1}\,[\mu_2-\lambda]^{-1} \\
&L_\beta = \bar{\rho}_1[1-\bar{\rho}_1]^{-1} + \rho_2[1-\rho_2]^{-1}; \quad (\bar{\rho}_1=\lambda/[\mu_1(1-\beta)];\rho_2=\lambda/\mu_2)
\end{aligned}
$$

Proof First we define operators $\overset{(-)}{T}_k$ and $\overset{(-)}{T}$ for arbitrary functions f by:

$$
(3.4)\qquad
\begin{cases}
\overset{(-)}{T}_0 f = f \\
\overset{(-)}{T}_{k+1} f = \overset{(-)}{T}\;\overset{(-)}{T}_k f \quad (k = 0,1,2,\dots) \\
\overset{(-)}{T} f(n_1,n_2) = \Sigma_{[m_1,m_2]}\, \overset{(-)}{p}([n_1,n_2],[m_1,m_2])\; f(m_1,m_2)
\end{cases}
$$

Then from (3.2):

$$
\begin{aligned}
(3.5)\qquad & (\bar{V}_N-V_N)\;(\ell_1,\ell_2) = \\
& (\bar{T}\;\bar{V}_{N-1}-T\;V_{N-1})\;(\ell_1,\ell_2) = \\
& (\bar{T}-T)V_{N-1}\;(\ell_1,\ell_2) + \bar{T}(\bar{V}_{N-1}-V_{N-1})\;(\ell_1,\ell_2) = \\
& \textstyle\sum_{k=0}^{N-1} \bar{T}_k\;(\bar{T}-T)V_{N-k-1}\;(\ell_1,\ell_2) + \bar{T}_N(\bar{V}_0-V_0)\;(\ell_1,\ell_2).
\end{aligned}
$$

where the latter expression follows by iteration. Now first note that the last term in the right hand side is equal to 0 as $\bar{V}_0(.) = V_0(.) = 0$. Further, from (3.1) and (3.4) we conclude for arbitrary t and (n_1,n_2):

$$
\begin{aligned}
(3.6)\qquad & (\bar{T}-T)V_t(n_1,n_2) = \\
& \Big\{h\lambda\; V_t(n_1+1,n_2) +
\end{aligned}
$$

$$h\,\mu_1 1_{(n_1>0)}[1-\beta(n_1,n_2)]\;V_t(n_1-1,n_2+1) + h\,\mu_2 1_{(n_2>0)}V_t(n_1,n_2-1$$

$$\left[1-h\lambda-h\mu_1 1_{(n_1>0)}[1-\beta(n_1,n_2)] - h\,\mu_2 1_{(n_2>0)}\right]V_t(n_1,n_2)\Big\}$$

$$-$$

$$\Big\{h\,\lambda\,V_t(n_1+1,n_2) + h\,\mu_2 1_{(n_1<0)}V_t(n_1-1,n_2+1) +$$

$$h\,\mu_2 1_{(n_2>0)}V_2(n_1,n_2) + \left[1-h\lambda-h\mu_1 1_{(n_1>0)}-h\mu_2 1_{(n_2>0)}\right]V_t(n_1,n_2)\Big\}$$

$$=$$

$$h\,\mu_1 1_{(n_1>0)}\beta(n_1,n_2)\left[V_t(n_1,n_2) - V_t(n_1-1,n_2+1)\right].$$

By also noting that the operators $\bar{T}_k$ are monotone operators (i.e $\bar{T}_k f \le \bar{T}_k g$ provided $f \le g$ componentwise) from (2.1) and lemma 3.1 below with $C = C_1$, we conclude that uniformly in k and for any initial state (ℓ_1,ℓ_2):

(3.7) $\quad |\bar{T}_k\,(\bar{T}-T)V_{N-k-1}\,(\ell_1,\ell_2)| \le \beta\,C\,\bar{T}_k\,\Phi\,(\ell_1,\ell_2)$

where Φ is defined by

(3.8) $\quad \Phi(n_1,n_2) = [n_1+n_2]$

Further by lemma 3.1 below with L_β given by Theorem 3.1,

(3.9) $\quad \bar{T}_k\,\Phi\,(0,0) \le L_\beta \quad (k \ge 0)$

Combining (3.5), (3.7) and (3.9) yields:

(3.10) $\quad |(\bar{V}_N-V_N)(0,0)| \le \beta\,C\sum_{k=0}^{N-1}\bar{T}_k\,\Phi\,(0,0) \le \beta\,N\,C\,L_\beta$

As this inequality holds for arbitrary N, relation (3.3) with $(\ell_1,\ell_2) = (0,0)$ completes the proof of the theorem. □

Lemma 3.1 With

$$h = [\lambda+\mu_1+\mu_2]^{-1}$$

$$C_2 = [h(\mu_2-\lambda)]^{-1}$$

$$C_1 = [h(\mu_1+\mu_2-\lambda)]\,[h(\mu_1-\lambda)]^{-1}\,[h(\mu_2-\lambda)]^{-1}$$

for all (n_1,n_2) and $t\geq 0$ we have:

$$(3.11)\qquad |\Delta_3 V_t(n_1,n_2) = V_t(n_1-1,n_2+1) - V_t(n_1,n_2)| \leq (n_1+n_2)C_1$$

$$(3.12)\qquad 0 \leq \Delta_2\, V_t(n_1,n_2) = V_t(n_1,n_2+1) - V_t(n_1,n_2) \leq (1+n_1+n_2)C_2$$

$$(3.13)\qquad 0 \leq \Delta_1\, V_t(n_1,n_2) = V_t(n_1+1,n_2) - V_t(n_1,n_2) \leq (1+n_1)C_1 + n_2C_2$$

Proof (3.11) follows from (3.12) and (3.13) by

$$(3.14)\qquad V_t(n_1-1,n_2+1) - V_t(n_1,n_2) =$$

$$[V_t(n_1-1,n_2+1)-V_t(n_1-1,n_2)] - [V_t(n_1,n_2)-V_t(n_1-1,n_2)]$$

and noting that $C_2\leq C_1$.

To prove (3.12) and (3.13) we will apply induction on t. As $V_0(.,.) = 0$, (3.12) and (3.13) trivially hold for $t=0$. Assume that (3.12) and (3.13) hold for $t \leq m$. Then by virtue of (3.2) we can write:

$$(3.15)\quad \Delta_2\, V_{m+1}(n_1,n_2)$$

$$=$$

$$\Big\{ [n_1+n_2+1] + h\lambda V_m(n_1+1,n_2+1) +$$

$$h\mu_1 1_{(n_1>0)}\, V_m(n_1-1,n_2+2) + h\,\mu_2 1_{(n_2>0)}\, V_m(n_1,n_2) +$$

$$h\mu_2 1_{(n_2=0)}\, V_m(n_1,0) + \Big[1-h\lambda-\, h\mu_1 1_{(n_1>0)} - h\mu_2\Big]\, V_m(n_1,n_2+1)\Big\}$$

$$-$$

$$\Big\{ [n_1+n_2] + h\lambda V_m(n_1+1,n_2) +$$

$$h\mu_1 1_{(n_1>0)}\, V_m(n_1-1,n_2+1) + h\,\mu_2 1_{(n_2>0)}\, V_m(n_1,n_2-1) +$$

$$h\mu_2 1_{(n_2=0)}\, V_m(n_1,0) + \Big[1-h\lambda-\, h\mu_1 1_{(n_1>0)} - h\mu_2\Big]\, V_m(n_1,n_2)\Big\}$$

$$=$$

$$1 + h\,\lambda\,\Delta_2\, V_m(n_1+1,n_2) +$$

$$h\mu_1 1_{(n_1>0)}\,\Delta_2\, V_m(n_1-1,n_2+1) + h\mu_2 1_{(n_2>0)}\,\Delta_2\, V_m(n_1,n_2-1)$$

$$h\mu_2 1_{(n_2=0)}\Big[V_m(n_1,0)-V_m(n_1,0)\Big] + \Big[1-h\lambda-h\mu_1 1_{(n_1>0)}-h\mu_2\Big]\,\Delta_2 V_m(n_1,n_2)$$

where the last fifth term is indeed equal to 0 but kept in for clarity. By substituting the lower estimate 0 from (3.12) as per induction hypothesis for $t=m$ we immediately verify the lower estimate 0 in (3.12) also for $t=m+1$. In order to conclude the upper estimate in (3.12) for $t=m+1$, by substituting the upper estimate from (3.12) for $t=m$ in (3.15), after cancelling terms we require:

$$(3.16)\qquad 1 + h\lambda C_2 - h\mu_2 C_2 \le 0$$

which in turn is satisfied provided $C_2 \ge [h(\mu_2-\lambda)]^{-1}$. Similarly, by (3.2) again:

(3.17) $\Delta_1 V_{m+1}(n_1,n_2)$

$$=$$

$$\Big\{ [n_1+n_2+1] +$$

$$h\lambda\ V_m(n_1+2,n_2) + h\mu_1 1_{(n_1>0)}\ V_m(n_1,n_2+1) + h\mu_1 1_{(n_1=0)}\ V_m(0,n_2+1) +$$

$$h\mu_2 1_{(n_2>0)}\ V_m(n_1+1,n_2-1) + \Big[1 - h\lambda - h\mu_1 - h\mu_2 1_{(n_2>0)}\Big]\ V_m(n_1,n_2+1)\Big\}$$

$$-$$

$$\Big\{ [n_1+n_2] +$$

$$h\lambda\ V_m(n_1+1,n_2) + h\mu_1 1_{(n_1>0)}\ V_m(n_1-1,n_2+1) + h\mu_1 1_{(n_1=0)}\ V_m(0,n_2) +$$

$$h\mu_2 1_{(n_2>0)}\ V_m(n_1,n_2-1) + \Big[1 - h\lambda - h\mu_1 - h\mu_2 1_{(n_2>0)}\Big]\ V_m(n_1,n_2)\Big\}$$

$$=$$

$$1 + h\,\lambda\,\Delta_1\ V_m(n_1+1,n_2) +$$

$$h\mu_1 1_{(n_1>0)}\ \Delta_1\ V_m(n_1-1,n_2+1) + h\mu_1 1_{(n_1=0)}\ \Delta_2\ V_m(0,n_2) +$$

$$h\mu_2 1_{(n_2>0)}\ \Delta_1\ V_m(n_1,n_2-1) + \Big[1 - h\lambda - h\mu_1 - h\mu_2 1_{(n_2>0)}\Big]\Delta_1\ V_m(n_1,n_2)$$

Here, note that the fourth term involves a Δ_2 rather than Δ_1-term. However, by substituting the lower estimates 0 from (3.12) and (3.13) for t=m, as per hypothesis, we immediately verify the lower estimate 0 in (3.13) also for t=m+1. By substituting the upper estimates from (3.12) and (3.13) for t=m, the upper estimate from (3.13) is concluded also for t=m+1 if

(3.18)
$$1 + h\lambda\Big[1+n_1+1+n_2\Big]C_1 + h\mu_1 1_{(n_1>0)}\Big[n_1C_1 + (1+n_2)C_2\Big] +$$

$$h\mu_1 1_{(n_1=0)}(1+n_2)\ C_2 + h\mu_2 1_{(n_2>0)}\Big[(1+n_1)C_1 + (n_2-1)C_2\Big] +$$

$$\Big[1-h\lambda-h\mu_1-h\mu_2 1_{(n_2>0)}\Big]\Big[(1+n_1)C_1 + n_2C_2\Big] \le \Big[(1+n_1)C_1 + n_2C_2\Big].$$

This holds when

$$(3.19)\qquad 1 + h\lambda\ C_1 + h\mu_1 1_{(n_1>0)}[C_2 - C_1] + h\mu_1 1_{(n_1=0)}[C_2 - C_1] \le 0$$

which in turn is satisfied by: $1 + h\mu_1 C_2 \le (h\mu_1 - h\lambda)\ C_1$ as guaranteed with = sign by $C_1 = (\mu_1 + \mu_2 - \lambda)[(\mu_1 - \lambda)(\mu_2 - \lambda)h]^{-1}$. The proof of the lemma is completed by induction. □

Lemma 3.2 For all k:

$$\bar{T}_k\ \Phi\ (0,0) \le L_\beta$$

Proof Consider the process where $\beta(n_1,n_2) = \beta$ for all (n_1,n_2) and let S and S_k be the corresponding one-step and k-step expectation operators as defined by (3.1) and (3.4) for $\bar{T}$ and $\bar{T}_k$ with $\beta(.,.) = \beta$ for all (n_1,n_2). The proof will involve three steps.

<u>Step 1</u> First we will prove that for all k:

$$(3.20)\qquad \bar{T}_k\ f(0,0) \le S_k\ f(0,0)$$

for any function $f(n_1,n_2)$ such that

$$\text{(a)}\quad \Delta_1\ f(n_1,n_2) = f(n_1+1,n_2) - f(n_1,n_2) \ge 0$$

$$(3.21)\quad \text{(b)}\quad \Delta_2\ f(n_1,n_2) = f(n_2,n_2+1) - f(n_1,n_2) \ge 0$$

$$\text{(c)}\quad \Delta_3\ f(n_1,n_2) = f(n_1-1,n_2+1) - f(n_1,n_2) \le 0.$$

To this end, from (3.4) we obtain as in (3.5):

$$(3.22)\qquad (\bar{T}_k - S_k)\ f(0,0) = \textstyle\sum_{t=0}^{k-1} \bar{T}_t(\bar{T} - S)S_{k-t-1}\ f(0,0).$$

However, by comparing (3.1) for the original process (for $\bar{T}$) and the modified process with $\beta(.,.) = \beta$ (for S) where $\beta \ge \beta(n_1,n_2)$ for all (n_1,n_2) we also derive for any function g and state (n_1,n_2):

(3.23) $(\bar{T}-S)g(n_1,n_2)$

$$=$$

$$h\ \mu_1 1_{(n_1>0)}\left\{\left[1-\beta(n_1,n_2)\right]g(n_1-1,n_2+1)+\beta(n_1,n_2)g(n_1,n_2)\right\}-$$

$$h\ \mu_1 1_{(n_1>0)}\left\{\left[1-\beta\right]g(n_1-1,n_2+1)+\beta g(n_1,n_2)\right\}$$

$$=$$

$$h\ \mu_1 1_{(n_1>0)}\left[\beta-\beta(n_1,n_2)\right]\ \left[g(n_1-1,n_2+1)-g(n_1,n_2)\right].$$

Recalling that $\bar{T}_t$ is a monotone operator, (3.20) is now concluded from (3.22) and (3.23) provided (3.21) also holds with f replaced by $(S_t f)$, where f itself satisfies (3.21), for all $t \geq 0$.

(Proof of (3.21) *for* $S_t f$*)*

This will follow by induction on t. As $S_0 f=f$ by assumption it applies for t=0. Suppose (3.21) holds for t=m with f_t replaced by S f, As

(3.24) $\Delta_1\ g(n_1,n_2) = \Delta_2\ g(n_1,n_2) - \Delta_3\ g(n_1+1,n_2)$

for arbitrary g, (3.21a) with f replaced by $(S_{m+1}f)$ is shown by verifying (3.21b) and (3.21c) with f replaced by $(S_{m+1}f)$. To this end, we derive similarly to (3.15):

(3.25) $\Delta_2(S_{m+1}f)(n_1,n_2)$

$$= S(S_m f)(n_1,n_2+1) - S(S_m f)(n_1,n_2)$$

$$= h\ \lambda\ \Delta_2(S_m f)(n_1+1,n_2) + h\mu_1 1_{(n_1>0)}[1-\beta]\ \Delta_2(S_m f)(n_1-1,n_2+1) +$$

$$h\mu_2 1_{(n_2>0)}\ \Delta_2(S_m f)(n_1,n_2-1)+\left[1-h\lambda-h\mu_1 1_{(n_1>0)}[1-\beta]\ -\mu_2 1_{(n_2>0)}\right]$$

$$\Delta_2(S_m f)(n_1,n_2).$$

The induction hypothesis $\Delta_2(S_m f) \geq 0$ thus implies $\Delta_2(S_{m+1}f) \geq 0$, that is

(3.21b) for f replaced by $(S_{m+1}f)$. Similarly,

$$(3.26) \qquad \Delta_3(S_{m+1}f)(n_1,n_2)$$

$$= S(S_m f)(n_1-1,n_2+1) - S(S_m f)(n_1,n_2)$$

$$= h\ \lambda\ \Delta_3\ S_m f\ (n_1+1,n_2) + h\ \mu_1 1_{(n_1>1)} [1-\beta]\ \Delta_3(S_m f)(n_1-1,n_2+1) +$$

$$h\ \mu_1 1_{(n_1=1)}\ [1-\beta] \left[S_m f(0,n_2+1) - S_m f(0,n_2+1)\right] + h\,\mu_2 1_{(n_2>0)} +$$

$$\Delta_3(S_m f)(n_1,n_2-1) + h\mu_2 1_{(n_2=0)} \left[-\Delta_1(S_m f)(n_1-1,)\right] +$$

$$\left[1-h\ \lambda-h\ \mu_1 1_{(n_1>0)}[1-\beta]-h\ \mu_2\right] \Delta_3(S_m f)(n_1,n_2)$$

where we note that the third term is indeed equal to 0 while the fifth involves a $-\Delta_1$ term. The induction hypothesis $\Delta_3(S_m f) \le 0$ and $\Delta_1(S_m f) \ge 0$ thus imply also $\Delta_3(S_{m+1}f) \le 0$, that is (3.21c) with f replaced by $(S_{m+1}f)$. By induction we can thus conclude that (3.21) holds for f replaced by $S_t f$ for all t and f satifying (3.21). As argued, this in turn completes the proof of (3.20) for all k and f satisfying (3.21).

<u>Step 2</u> Next, we will inductively show that for any f satisfying (3.21):

$$(3.27) \qquad S_k f(0,0) \le S_{k+1} f(0,0) \quad (k \ge 0).$$

For k = 0, (3.27)is satisfied by the identity:

$$(3.28) \qquad (S\ f)(0,0) = \lambda h\ f(1,0) + [1-\lambda h]\ f(0,0) \ge f(0,0).$$

Suppose that (3.27) holds for $k \le m$ and all f satisfying (3.21). Then by recalling, as proven above, that (3.21) also holds with f replaced by $(Sf)=S_1 f$ provided f satisfies (3.21), (3.27) is inductively proven by:

$$(3.29) \qquad (S_{m+2}f-S_{m+1}f)(0,0) = (S_{m+1}-S_m)(S\ f)(0,0).$$

<u>Step 3</u> To finalize the proof of the lemma, now note that $f = \Phi$, with $\Phi(n_1,n_2) = [n_1+n_2]$ as defined in theorem 3.1, satisfies (3.21). Consequently, combining (3.20) with (3.27) yields for all k:

$$\bar{T}_k\ \Phi(0,0) \le S_k\ \Phi(0,0) \le S_{k+1}\ \Phi(0,0) \le \lim_{k\to\infty} S_k\ \Phi(0,0) = L_\beta \tag{3.30}$$

where L_β is the mean total number of jobs in queues 1 and 2 together, for the system with constant loss probability $\beta(n_1,n_2) = \beta$. However, as this system exhibits the product form expression (2.2) with ρ_1 replaced by $\bar{\rho}_1 = \lambda[\mu_1(1-\beta)]^{-1}$, L_β is given accordingly by (2.3), which completes the proof of lemma 3.2. □

As a special application of Lemma 3.2 also a relative error bound of the unknown quantity $\bar{L}$ can be concluded.

Theorem 3.2 With C as in theorem 3.1

$$\boxed{\frac{|\bar{L} - L|}{\bar{L}} \le \frac{\beta}{(1-\beta)}\ C} \tag{3.31}$$

Proof Almost identically to (3.20)-(3.26) one can also show

$$T_k f(0,0) \le \bar{T}_k f(0,0) \tag{3.32}$$

As only difference (3.23) is to be replaced by (also see (3.6)):

$$(T - \bar{T})\ g(n_1,n_2) = \tag{3.33}$$

$$h\ \mu_1\ 1_{(n_1>0)}\ \beta(n_1,n_2)\Big[g(n_1-1,n_2+1) - g(n_1,n_2)\Big].$$

By combination of (3.20) and (3.23) we then observe that

$$L = \lim_{k\to\infty} T_k\Phi(0,0) \le \bar{L} = \lim_{k\to\infty} \bar{T}_k\Phi(0,0) \le L_\beta = \lim_{k\to\infty} S_k\Phi(0,0). \tag{3.34}$$

And thus

$$\frac{|\bar{L} - L|}{\bar{L}} \le \frac{|\bar{L} - L|}{L} \le \beta\ C\ \frac{L_\beta}{L} \le \frac{\beta}{(1-\beta)}\ C.$$

Here the last inequality follows from comparing L_β and L as given in theorem 3.1 and noting that for arbitrary a,b>0: $a/(1-\beta)+b = (1-\beta)^{-1}[a+b(1-\beta)] \le (1-\beta)[a+b]$. □

4. MULTI BUFFER CASE AND OTHER EXTENSIONS

4.1 Error bound: mutlti-buffer case.

Now let us return to the multi-buffer case as described in sections 2.1 and 2.3 and investigate whether error bounds similar to the single-buffer case of section 3 can also be concluded. Indeed this will appear to be possible along exactly the same lines. As, as all steps are essentially identical, let us only present the key-steps.

To this end, let e_1^r and e_2^r denote a vector with 0-values allover except for a value 1 for the r-th input respectively output queue. For example, the state $\bar{n}-e_1^r+e_2^r$ then indicates the state identical to $\bar{n}$ with one job moved from the r-th input to the r-th output queue.

Similarly to (3.1), and now with $h = [\Sigma_i(\lambda_1^r+\mu_1^r+\mu_2^r)]^{-1}$, we then define the uniformized discrete-time Markov chains with transition probabilities given

$$
(4.1)\qquad
\begin{cases}
\overset{(-)}{p}(\bar{n},\bar{n}+e_1^r) = h\,\lambda^r \\[2mm]
\overset{(-)}{p}(\bar{n},\bar{n}-e_2^r) = h\,\mu_2^r\,1_{(n_2^r>0)} \\[2mm]
p(\bar{n},\bar{n}-e_1^r+e_2^r) = h\,\mu_1^r\,1_{(n_1^r>0)} \\[2mm]
\bar{p}(\bar{n},\bar{n}-e_1^r+e_2^r) = h\,\mu_1^r\,1_{(n_1^r>0)}[1-\beta^r(\bar{n})] \\[2mm]
\overset{(-)}{p}(\bar{n},\bar{n}) = 1-\Sigma_{\bar{n}'\neq\bar{n}}\,\overset{(-)}{p}(\bar{n},\bar{n}')
\end{cases}
$$

Then, to obtain an error bound on $|L^r-\bar{L}^r|$ we also define functions $\overset{(-)}{V}{}_t^r$ for $t\geq0$ as in (3.2) by:

$$
(4.2)\qquad \overset{(-)}{V}{}_{t+1}^r(\bar{n}) = [n_1^r + n_2^r] + \Sigma_{\bar{n}'}\,\overset{(-)}{p}(\bar{n},\bar{n}')\,\overset{(-)}{V}{}_t^r(\bar{n}')
$$

and conclude by uniformization, for initial state $\bar{0}$ (the zero vector)

$$
(4.3)\qquad \overset{(-)}{L}{}^r = \lim_{N\to\infty}\frac{1}{N}\overset{(-)}{V}{}_N^r(\bar{0}).
$$

The following theorem is then almost identical to theorem 3.1.

Theorem 4.1

$$|L^r - \bar{L}^r| \le \beta\ C\ L^r_\beta$$

$$C = [\mu^r_1 + \mu^r_2 - \lambda^r]\ [\mu^r_1 - \lambda^1]^{-1}\ [\mu^r_2 - \lambda^r]^{-1}$$

$$L^r_\beta = \bar{\rho}^r_1\ [1-\bar{\rho}^r_1]^{-1} + \rho^r_2[1-\rho^r_2]^{-1}\ (\bar{\rho}^r_1 = \lambda/[\mu^r_1(1-\beta)];\ \rho^r_2 = \lambda^r/\mu^r_2)$$

Proof

Similarly to (3.4) defining operators T, $\bar{T}$ and $\bar{T}_k$ with

(4.4) $\overset{(-)}{T} f(\bar{n}) = \Sigma_{\bar{n}'} \overset{(-)}{p}(\bar{n},\bar{n}')\ f(\bar{n}'),$

we derive as in (3.5) and noting that $\overset{(-)}{V}_0(.) = 0$

(4.5) $(\overset{(-)}{V}_N - V_N)(\bar{0}) = \Sigma^{N-1}_{k=0}\ \overset{(-)}{T}_k\ (\bar{T}-T)\ \overset{(-)}{V}_{N-k-1}(0).$

Further, now by substituting (4.1) and (4.4) we obtain as in (3.6):

(4.6) $(\bar{T} - T)V_t(\bar{n}) = \Sigma_r\ h\ \mu^r_1\ 1_{(n^r_1>0)}\ \beta^r(\bar{n})\left[V_t(\bar{n}) - V_t(\bar{n}-e^r_1+e^r_2)\right]$

Essentially the comparison of the two models with and without losses is hereby transformed in terms $V_t(.)$ for only the system without losses. As a consequence, by using lemmas 4.1 and 4.2 below, we conclude as in (3.7)-(3.10):

(4.7) $|(\bar{V}_N-V_N)(\bar{0})| \le \beta\ C\ \Sigma^{N-1}_{k=0}\ \bar{T}_k\ \Phi^r(\bar{0}) \le \beta\ N\ C\ L^r_\beta$

so that applying (4.3) completes the proof. □

Lemma 4.1 With

$$h = \Sigma_r[\lambda^r + \mu^r_1 + \mu^r_2]$$

$$C^r_2 = [\mu^r_2 - \lambda^r]^{-1}h^{-1}$$

$$C^r_1 = [\mu^r_1 + \mu^r_2 - \lambda^r]\ [\mu^r_1 - \lambda^r]^{-1}\ [\mu^r_2 - \lambda^r]^{-1}h^{-1}$$

we have for all $\bar{n}$ and $t \geq 0$:

(4.8) $$|\Delta_3^r V_t(\bar{n}) = V_t(\bar{n} - e_1^r + e_2^r) - V_t(\bar{n})| \leq \left(1 + n_1^r + n_2^r\right) C_1^r$$

(4.9) $$0 \leq \Delta_2^r V_t(\bar{n}) = V_t(\bar{n} + e_2^r) - V_t(\bar{n}) \leq \left(1 + n_1^r + n_2^r\right) C_2^r$$

(4.10) $$0 \leq \Delta_1^s V_t(\bar{n}) = V_t(\bar{n} + e_1^r) - V_t(\bar{n}) \leq \left(1 + n_1^r\right) C_1^r + n_1^r C_2^r$$

Proof

Note that we are now dealing with the system without losses so that interdependencies of the various input/output connections is no longer involved. The proof for the s-th connection will therefore be almost identical to that of lemma 3.1 for the single connection case. To illustrate this and to also argue the actual bounds, similarly to (3.15) we can conclude:

(4.11) $$\Delta_2^r V_{m+1}(\bar{n}) =$$

$$1 + h\, \Sigma_s\, \lambda^s\, \Delta_2^r V_m(\bar{n} + e_1^s) +$$

$$h\, \Sigma_s\, \mu_1^s\, 1_{(n_1^s > 0)}\, \Delta_2^r V_m(\bar{n} - e_1^s + e_2^s) +$$

$$h\, \Sigma_s\, \mu_2^s\, 1_{(n_2^s > 0)}\, \Delta_2^r V_m(\bar{n} - e_2^s) +$$

$$h\, \mu_2^r\, 1_{(n_2^r = 0)} \left[V_m(\bar{n} + e_2^r - e_2^r) - V_m(\bar{n}) \right] +$$

$$\left[1 - h\, \Sigma_s\, \lambda^s - h\, \Sigma_s\, \mu_1^s\, 1_{(n_2^s > 0)} - h\, \Sigma_{s \neq r}\, \mu_1^s\, 1_{(n_2^s > 0)} - h\, \mu_2^r\right] \Delta_2^r\, V_m(\bar{n}).$$

Here the fifth term in the right hand side is indeed equal to 0 but kept in for clarity. Assuming (4.9) for t=m thus directly yields: $\Delta_2^1 V_{m+1}(\bar{n}) \geq 0$, while $\Delta_2^1 V_{m+1}(\bar{n}) \leq (1 + n_1^1 + n_2^1)\, C_2^1$ provided

$$1 + h\, \lambda^r\, C_2^r - h\, \mu_2^r\, C_2^r \leq 0$$

as satisfied by $C_2^r \geq [h(\mu_2^r - \lambda^r)]^{-1}$. As $\Delta_2 V_0^r(.) = 0$, induction proves (4.9).

In the same manner, as in (3.17), we obtain

(4.12) $\Delta_1^r V_{m+1}(\bar{n}) =$

$$1 + h\, \Sigma_s \lambda^s\, \Delta_1^r V_m(\bar{n} + e_1^s) +$$

$$h\, \Sigma_s\, \mu_1^s\, 1_{(n_1^s>0)}\, \Delta_1^r V_m(\bar{n} - e_1^s + e_2^s) +$$

$$h\, \Sigma_s\, \mu_2^s\, 1_{(n_2^s>0)}\, \Delta_1^r V_m(\bar{n} - e_2^s) + h\, \mu_1^r\, 1_{(n_1^r=0)}\, \Delta_2^r V_m(\bar{n})$$

$$\left[1 - h\, \Sigma_s \lambda^s - h\, \Sigma_{s\neq r} \mu_1^s\, 1_{(n_1^s>0)} - h\, \mu_r - h\, \Sigma_s \mu_2^s\, 1_{(n_2^s>0)}\right] \Delta_1 V_m^r(\bar{n}).$$

Here we note that indeed the fifth term in the right hand side has a Δ_2^r -rather than Δ_1^r-term. By assuming $\Delta_1^r V_m \geq 0$ and $\Delta_2^r V_m \geq 0$ we would directly conclude $\Delta_1^r V_{m+1} \geq 0$. By assuming the upper estimate from (4.9) and (4.10) for t=m, as in (3.18) and (3.19) we would also conclude

$$\Delta_1^r V_{m+1}(\bar{n}) \leq (1+n_1+n_2)C_1^r$$

provided

$$1 + h\, \lambda^r C_1^r + h\, \mu_1^r\, 1_{(n_1^r>0)}\, [C_2^r - C_1^r] + h\, \mu_1^r\, 1_{(n_1^r=0)}\, [C_2^r - C_1^r] \leq 0.$$

This is satisfied with = sign by: $C_1^r = (\mu_1^r + \mu_2^r - \lambda^r)[(\mu_1^r - \lambda^r)(\mu_2^r - \lambda^r)h]^{-1}$ and $C_2^1 = [h(\mu_2^r - \lambda^r)]^{-1}$. As $\Delta_1^r V_0(.)=0$ and $\Delta_2^r V_0(.)=0$, induction proves (4.10). Inequality (4.8) finally, follows from (4.9) and (4.10) similarly to (3.14). □

Lemma 4.2 With

$$\Phi^r(\bar{n}) = n_1^r + n_2^r$$

$$\rho_1^r = \lambda_1^r/[\mu_1^r(1-\beta)],\ \rho_2^r = \lambda_2^r/\mu_2^r$$

$$L_\beta^r = \bar{\rho}_1^r/(1 - \bar{\rho}_1^r) + \rho_2^r/(1 - \rho_2^r)$$

we have for all k:

(4.13) $\bar{T}_k \Phi^r(\bar{0}) \leq L_\beta^r.$

Proof Consider the process with $\beta^r(\bar{n}) = \beta$ for all r and $\bar{n}$ and let the corresponding one-step and k-step transition operators as defined by (3.4), (4.1) and (4.4) be denoted by S and S_k.
Also, define H as the class of functions f such that for all i:

$$\text{(4.14)} \quad \begin{aligned} &\text{(a)} \quad \Delta_1^r f(\bar{n}) = f(\bar{n} + e_1^r) - f(\bar{n}) \geq 0 \\ &\text{(b)} \quad \Delta_2^r f(\bar{n}) = f(\bar{n} + e_2^r) - f(\bar{n}) \geq 0 \\ &\text{(c)} \quad \Delta_3^r f(\bar{n}) = f(\bar{n} - e_1^r + e_2^r) - f(\bar{n}) \leq 0 \end{aligned}$$

Then as in the proof of lemma 3.2 the proof will follow in two steps showing that for any f∈H:

$$\text{(4.15)} \qquad \bar{T}_k f(\bar{0}) \leq S_k f(\bar{0}) \qquad (k\geq 0)$$

$$\text{(4.16)} \qquad S_k f(\bar{0}) \leq S_{k+1} f(\bar{0}) \qquad (k\geq 0)$$

The proof is then completed by noting

$$\text{(4.17)} \qquad \begin{cases} \lim_{k\to\infty} S_k f(\bar{0}) = L_\beta^r \\ \\ \Phi^r(\bar{n}) = n_1^r + n_2^r \in H \end{cases}$$

Step 1 (Proof of (4.15)) As in (3.22)

$$\text{(4.18)} \qquad (\bar{T}_k - S_k)\, f(\bar{0}) = \Sigma_{t=0}^{k-1} \bar{T}_t (\bar{T}-S)\, S_{t-k-1}\, f(\bar{0}).$$

By comparing the transition probabilities (4.1) with state dependent loss probabilities $\beta(.,.)$ and with state independent loss probabilities $\beta(.,.) = \beta$, we obtain as in (3.23):

$$\text{(4.19)} \qquad (\bar{T}-S)\, f(\bar{n}) = \Sigma_i\, h\, \mu_1^r\, 1_{(n_1^r>0)} \left[\beta - \beta_1(\bar{n})\right] \left[f(\bar{n}-e_1^r-e_2^r) - f(\bar{n})\right].$$

Noting that $\bar{T}$ is a transition and thus monotone operator, inequality (4.15) would thus be proven by (4.18) and (4.19) provided for all t:

(4.20) $\quad S_t f \in H \quad$ for any $f \in H$.

To prove (4.20), let $f \in H$ and assume that (4.19) holds for $t = m$, that is (4.14)(a),(b),(c) with f replaced by $S_m f$. Then by proving (4.14)(b), (c) with f replaced by $(S_{m+1} f)$ we have also shown (4.14)(a) as by (3.24).

To prove (4.14)(b) with f replaced by $(S_{m+1} f)$, we conclude as in (4.11) and (3.25):

(4.21)
$$\Delta_2^r(S_{m+1} f)(\bar{n}) =$$

$$h\,\Sigma_s\ \lambda^s \Delta_2^r(S_m f)(\bar{n}+e_1^s)\ +$$

$$h\,\Sigma_s\ \mu_1^s\ 1_{(n_1^s>0)}\ \Delta_2^r\ (S_m f)(\bar{n}-e_1^s+e_2^s)\ +$$

$$h\,\Sigma_s\ \mu_2^s\ 1_{(n_2^s>0)}\ \Delta_2^r\ (S_m f)(\bar{n}-e_2^s)\ +$$

$$\left[1 - h\,\Sigma_s\,\lambda^s - h\,\Sigma_s\,\mu_1^s\ 1_{(n_1^s>0)} - h\,\Sigma_{s\neq r}\,\mu_2^s\ 1_{(n_2^s>0)} - h\,\mu_2^r\right]\Delta_2^r(S_m f)(\bar{n})$$

so that substitution of $\Delta_2^r(S_m f)\geq 0$ directly yields $\Delta_2^r(S_{m+1} f)\geq 0$. Similarly, to prove (4.14)(c) with f replaced by $(S_{m+1} f)$, we conclude as in (4.11) and (3.26):

$$(4.22)\quad \Delta_3^r(S_{m+1}f)(\bar{n}) =$$

$$h\,\Sigma_s\ \lambda^s\Delta_3^r\ (S_m f)(\bar{n}+e_1^r) +$$

$$h\,\Sigma_s\ \mu_1^s\ 1_{(n_1^s>1)}\Delta_3^r\ (S_m f)\ (\bar{n} - e_1^s + e_2^s) +$$

$$h\,\Sigma_s\ \mu_2^s\ 1_{(n_2^s>0)}\Delta_3^r\ (S_m f)(\bar{n} - e_2^s) +$$

$$h\,\Sigma_s\ \mu_2^s\ 1_{(n_2^s=0)}\Big[(S_m f)(\bar{n}-e_1^s) - (S_m f)(\bar{n})\Big] +$$

$$h\,\Sigma_s\ \mu_1^s\ 1_{(n_1^s=1)}\Big[(S_m f)(\bar{n}-e_1^s+e_2^s) - (S_m f)(\bar{n}-e_1^s+e_2^s)\Big] +$$

$$[1 - h\,\Sigma_s\lambda^s - h\,\Sigma_s\ \mu_1^s\ 1_{(n_1^s>0)} - h\,\Sigma_s\ \mu_2^s]\ \Delta_3^r\ (S_m f)(\bar{n}).$$

Here the fifth inequality is equal to

$$h\,\Sigma_s\ \mu_2^s\ 1_{(n_2^s=0)}\ \Big[-\Delta_1^s\ (S_m f)(\bar{n} - e_1^s)\Big].$$

By substituting $\Delta_3^r(S_m f)\le 0$ and $\Delta_1^s(S_m f)\ge 0$ for all s as per induction assumption we have thus shown $\Delta_3^r(S_{m+1}f)\le 0$. By induction we have thus proven (4.20) for all t, which as argued implies (4.15).

Step 2 (Proof of (4.16)) For k=0 and $f\in H$ we have

$$(4.23)\quad (Sf)(\bar{o}) = h\,\Sigma_r\ \lambda^r\ f(\bar{o}+e_1^r) + [1 - h\,\Sigma_r\ \lambda^r]\ f(\bar{o})$$

$$= h\,\Sigma_r\ \lambda^r\ [f(\bar{o}+e_1^r) - f(\bar{o})] + f(\bar{o}) \ge f(\bar{o}).$$

Now assume that (4.16) holds for $k\le m$. Then by recalling as in (3.29)

$$(S_{m+2}f - S_{m+1}f)(\bar{o}) = (S_{m+1} - S_m)(Sf)(\bar{o})$$

and noting that $(Sf) = (S_1 f)\in H$ for any $f\in H$, as per the above proof for (4.19), induction completes the proof of (4.16) for all $f\in H$, and thus by (4.16) the proof of lemma 4.2. □

Similarly to theorem 3.2 relative error bounds can also be provided as based on theorem 3.1 and the above steps (4.15) and (4.16):

Theorem 4.2 With L^r and C as in theorem 4.1 we have:

$$(4.22) \qquad \boxed{\frac{|\bar{L}^r - L^r|}{\bar{L}^r} \le \frac{\beta}{(1-\beta)} C}$$

4.2 Further extensions

1 Retransmission protocol The retransmission protocol of recycling lost packets or messages to the end of the input queue is just one possible realistic protocol. As exponential assumptions are made, the same error bounds apply if losses are recycled into the input queue at the front or in any other randomized manner. As another extreme case, losses or errors may lead to a "real loss" or system departure. In that case, all derivations are almost identical and would give the same error bounds. As only difference in (3.6) and (4.6) we would then obtain Δ_2 rather than Δ_3-terms and lemmas 3.1 and 4.1 could be restricted accordingly. As another possibility, retransmission can take place at another speed than μ_1. Similar results can then still be derived but these would require a more detailed state description.

2 Other multi-buffer modelling/finite source input A multiple buffer or source input could also have been modeled by assuming an infinite server type input queue. The same steps could then be performed with related error bounds. The FCFS-case as studied herein though, is more complicated (e.g. in the infinite server case lemma 3.1 would apply with merely the constants C_1 and C_2 in the right hand sides of (3.11)-(3.13) and is more sensitive to the effect of losses.

3 Finite capacity constraints As for the proofs and the error bounds, also capacity limitations for the input and output queues can be imposed. As difference, though, no simple product form estimates would then apply by ignoring losses.

4 Different input/output connections Rather than M fixed input/output connections, one may also think of M_1 input queues (or buffers) and M_2 output queues, while interaction of different transmission types could then by involved. Again, results along the lines of this paper can then be expected provided some natural conditions on the interactions are satisfied.

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