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### An Improved Error Bound Theorem for Approximate Markov Chains

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**AN IMPROVED ERROR BOUND THEOREM  
FOR APPROXIMATE MARKOV CHAINS**

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**Abstract**

Recently an error bound theorem was reported to conclude analytic error bounds for approximate Markov chains. The theorem required a uniform bound for marginal expectations of the approximate model.

This note will relax this bound to steady state rather than marginal expectations as of practical interest:

- (i) to simplify verification and/or
- (ii) to obtain a more accurate error bound.

An ALOHA-type communications example is studied to support the results.



## 1 Introduction

Recently, in [8], the author proposed a general approximation or perturbation theorem to conclude a priori error bounds for the accuracy of approximate Markov chains such as due to:

- perturbations (e.g. reflecting parameter interval estimates)
- state space truncations (e.g. for reducing numerical computations), or
- system protocol modifications (e.g. for proposing simple estimates).

The theorem required to establish:

- (i) A bound for so-called bias-terms of an appropriate reward structure.
- (ii) A uniform bound for marginal expectations of the approximate chain.

The first step can usually be performed analytically by induction, based on the recursive Markov reward (or dynamic) programming equation. This has been applied successfully for various non-product queueing network applications (see [8] and references therein). The second step, in contrast, relies more specifically on insight, monotonicity assumptions and technicalities for the system in order that do not generally apply.

This note presents a technically minor but practically important extension, which relaxes the second step to merely a steady state estimate of the approximate model. This relaxation is practically appealing as:

- The steady state distribution for the approximate model is often available or rather easily computed.
- The accuracy of the error bound can hereby be improved.

Though the extension is technically minor, given its simplicity and impact, the proof is kept self-contained. To illustrate the effect of the relaxation, an ALOHA-type communications example is studied. This example motivated the improvement as the proposed approximation for this example, as adopted from [10], did not fit in well with the theorem from [8]. The present improvement avoids technically complicated verifications and leads to a more accurate error bound. Further application of the improved error bound theorem seems promising.

## 2 Model and result

As most applications will concern a continuous-time setting, the presentation will directly be restricted to the continuous-time case. The discrete-time case is hereby implicitly included (see remark 2.4).

Further, a more detailed motivation and discussion of the type of error bound theorem under investigation, its practical relevance, the important role of bias-terms, and an extensive list of related references to motivate the theorem can be found in [8]. Throughout, a vector  $V$  is assumed to be a column vector, while as a row vector it is denoted by  $V^T$ .

## 2.1 Model

Let  $(S, Q, r)$  denote a continuous-time Markov reward chain with state space  $S$ , one-step reward rate  $r(i)$  when the system is in state  $i$  and transition rate  $q(i, j)$  for a transition from state  $i$  to  $j$ . We assume the Markov chain to be irreducible at  $S$  with unique steady state distribution:

$$\{\pi(i)\}_{i \in S}$$

as well as to be uniformizable, that is, for some  $M < \infty$ :

$$(2.1) \quad \sum_j q(i, j) \leq M \quad (i \in S)$$

so that the uniformized one-step transition matrix  $P$  can be defined by:

$$(2.2) \quad p(i, j) = \begin{cases} q(i, j) M^{-1} & (j \neq i) \\ 1 - \sum_{j \neq i} q(i, j) M^{-1} & (j = i). \end{cases}$$

Let  $P^k$  denote the  $k$ -th power of  $P$  and for  $t=0, 1, 2, \dots$  define functions  $V_t$  by:

$$(2.3) \quad V_t(i) = M^{-1} \sum_{k=0}^{t-1} P^k r(i) = M^{-1} \sum_{k=0}^{t-1} \left[ \sum_j P^k(i, j) r(j) \right]$$

representing the expected cumulative reward over  $t$ -steps of length  $M^{-1}$  each when starting in state  $i$  at time 0. Now let  $\beta$  be an arbitrary probability distribution at  $S$  at time 0. Then, by the irreducibility assumption, standard Markov reward arguments and the method of uniformization (cf. [7], p.110) the value

$$(2.4) \quad g = \lim_{N \rightarrow \infty} \frac{M}{N} \left[ \beta^T V_N \right] = \lim_{N \rightarrow \infty} \frac{M}{N} \sum_{i \in S} \beta(i) V_N(i) = \sum_{i \in S} \pi(i) r(i)$$

is well-defined as a single scalar, independent of  $\beta$ , which represents the average expected reward per unit of time of the continuous-time Markov reward chain. It is noted here that the factor  $M$  could have been excluded in (2.3) and (2.4). However, it is included as condition (2.5) below then becomes more natural with both the reward and transition rates per unit of time while also the verification of this condition, most notably the estimation of bias-terms, becomes more convenient (e.g. [10]).

## 2.2 Error bound theorem

Now suppose that we are faced with:

- { An original Markov reward chain (S,Q,r)
- { An approximate Markov reward chain ( $\bar{S}, \bar{Q}, \bar{r}$ ),

as described above, where also the approximate chain is assumed to be uniformizable with the same constant M, i.e. (2.1) applies with q replaced by  $\bar{q}$ , and where we impose the condition:  $\bar{S} \subset S$ .

With the notation of section 2.1 adopted for the approximate model with an upper bar "-", the following theorem will then enable one to conclude an error bound on the difference  $|g-\bar{g}|$ .

**Theorem (Error bound)** Suppose that for some function  $\delta(\cdot)$  at  $\bar{S}$ , all  $i \in S$  and  $t \geq 0$ :

$$(2.5) \quad \left| (\bar{r}-r)(i) + \sum_j [\bar{q}(i,j)-q(i,j)] [V_t(j)-V_t(i)] \right| \leq \delta(i)$$

Then

$$(2.6) \quad | \bar{g}-g | \leq \bar{\pi}^T \delta$$

**Proof** Let the initial distribution  $\beta$  and  $\bar{\beta}$  be the same and given by  $\beta = \bar{\beta} = \bar{\pi}$ . Then by virtue of (2.3) we can write:

$$(2.7) \quad \begin{aligned} & (\bar{\pi}^T \bar{V}_t - \bar{\pi}^T V_t) = \\ & \bar{\pi}^T [\bar{r} M^{-1} + \bar{P} \bar{V}_{t-1}] - \bar{\pi}^T [r M^{-1} + P V_{t-1}] = \\ & \bar{\pi}^T \left[ (\bar{r}-r)M^{-1} + (P-\bar{P}) V_{t-1} + \bar{P}(\bar{V}_{t-1} - V_{t-1}) \right] = \\ & \bar{\pi}^T \sum_{k=0}^{t-1} \bar{P}^k \left[ (\bar{r}-r)M^{-1} + (\bar{P}-P) V_{N-k-1} \right] + \bar{\pi}^T \bar{P}^t (\bar{V}_0 - V_0) \end{aligned}$$

where the last equality follows by iteration. First, as  $\bar{V}_0(\cdot) = V_0(\cdot) = 0$ , the last term directly cancels. Secondly, as both  $\bar{P}$  and P are stochastic matrices with row-sums equal to 1, for any s and in any state  $i \in \bar{S}$  we can write:

$$(2.8) \quad \begin{aligned} (\bar{P}-P) V_s(i) &= \sum_j [\bar{p}(i,j)-p(i,j)] V_s(j) \\ &= \sum_j [\bar{p}(i,j)-p(i,j)] [V_s(j)-V_s(i)] \\ &= \sum_{j \neq i} \frac{[\bar{q}(i,j)-q(i,j)]}{M} [V_s(j)-V_s(i)] \quad (i \in S \cap \bar{S}). \end{aligned}$$

Last but not least, as  $\bar{\pi}$  represents the steady state distribution of the approximate Markov chain with one-step transition matrix  $\bar{P}$ , we have for all  $k$ :

$$(2.9) \quad \bar{\pi}^T \bar{P}^k = \bar{\pi}^T \bar{P} (\bar{P}^{k-1}) = \bar{\pi}^T \bar{P}^{k-1} = \dots = \bar{\pi}^T$$

which remains restricted to  $\bar{S}cS$ . As a consequence, for arbitrary  $N$ :

$$(2.10) \quad \left| \bar{\pi}^T \bar{V}_N - \bar{\pi}^T V_N \right| = \left| \sum_{k=0}^{N-1} \sum_1 \bar{\pi}^T(i) \left\{ (\bar{r}-r)(i) + \sum_{j=1} [\bar{q}(i,j)-q(i,j)] \left[ V_{N-k-1}(j) - V_{N-k-1}(i) \right] \right\} M^{-1} \right| \leq \frac{N}{M} [\bar{\pi}^T \delta].$$

Application of (2.4) with  $\beta = \bar{\beta} = \bar{\pi}$  now directly yields inequality (2.6). □

#### Remark 2.1 (Bias-terms)

Essentially the theorem requires one to find bounds on the so-called bias-terms  $V_t(j) - V_t(i)$ , where  $i$  and  $j$  can be restricted to "adjacent" states, uniformly in all  $t \geq 0$ . Theoretically, such bounds are known to exist based on mean first passage time results. Practically, such bounds, or usually much more accurate bounds than by mean first passage time results, can be established analytically by inductively using the Markov reward relation:

$$V_{t+1} = \frac{r}{M} + P V_t.$$

This estimation of bias-terms is discussed in more detail and illustrated for various multi-dimensional non-product form queueing network applications in [8] and references to the author therein.

#### Remark 2.2 (Advantage)

The theoretically minor but practically major advantage of the above version over the earlier one in [8] is that it only requires to estimate

$$(2.11) \quad \bar{\pi}^T \delta$$

rather than

$$(2.12) \quad \sup_k \bar{\beta}^T \bar{P}^k \delta$$

for some appropriate initial distribution  $\bar{\beta}$ , usually concentrated at one appropriate initial state. Indeed, (2.12) can always be estimated from above by  $\|\delta\|_\infty$ . However, in various applications this will not yield a reasonable error bound. For example, if the approximate model is a truncated version of the original, the difference  $\delta(i)$  will generally be 0 for states  $i < L$  where  $L$

represents the state where truncation takes place, but bounded away from 0 in states  $i=L$  when truncation takes place. However, the likelihood of being in such states must be thought of as being very small, say  $\bar{\pi}_k(L) \approx \bar{\pi}(L) \approx 2\%$  or less, so that a small estimate for (2.12) can be expected. Small estimates of (2.2) have so been found by analytically showing

$$(2.13) \quad \bar{\beta}^T \bar{P}^k \delta \uparrow \bar{\pi}^T \delta.$$

However, monotonicity conditions on  $\delta$  and system input characteristics, such as of state dependent service rates in queueing networks, technical exploitation of the specific approximate transition structure  $\bar{q}$  and an appropriate choice of  $\bar{\beta}$ , are hereby usually required (see for example references [32] and [34] in [8]).

In the present version, in contrast, a step like (2.13) and its underlying conditions are no longer required. For example, in the application of the next section, the function  $\delta(\cdot)$  will not be monotone and even more importantly, the actual structure of  $\bar{q}$  of the proposed approximation  $\bar{\pi}$  has no simple physical probabilistic interpretation as it stems from a purely analytical truncation of a (so-called) Möbius series.

In various applications the approximate model will just have been devised to guarantee easily computable approximate values  $\bar{\pi}$ , so that an estimate for (2.11) can be obtained rather directly once  $\delta(\cdot)$  is known.

#### Remark 2.3 (Further related literature)

Next to the related reference [8] with the time-dependent condition (2.12) rather than the steady state condition (2.11), as other related literature, a result similar to (2..) has already been reported in [2], p.42 and 43, and implicitly been obtained in [3] and [6]. As essential difference though, in these references the time-dependent bias-terms in (2.5) are replaced by time-independent relative value (or gain) terms (or relatedly coefficients of the fundamental matrix). This time-dependent form is crucial as it allows us to prove bounds for the bias-terms by induction using the Markov reward relation as per remark 2.1.

#### Remark 2.4 (Fundamental matrices)

In fact, the bias-terms  $V_t(j) - V_t(i)$  for  $t \rightarrow \infty$  correspond to the  $i, j$ -th coefficient of the fundamental matrix. The time-dependent formulation which enables one to inductively prove bounds on these bias-terms uniformly in  $t$  thus implicitly also provides a means to evaluate fundamental matrices. This in turn can be of interest for estimating mean first passage times or for employing the average policy improvement technique.

#### Remark 2.5 (Discrete-time case)

The theorem directly applies to discrete-time Markov reward chains  $(S, P, r)$  and  $(\bar{S}, \bar{P}, \bar{r})$ , where  $P$  and  $\bar{P}$  denote one-step transition matrices if we substi-



tute:

$$\begin{cases} M = 1 \\ q(i,j) = p(i,j) \\ \bar{q}(i,j) = \bar{p}(i,j) \end{cases}$$

**Summary:** To conclude an error bound  $|g-\bar{g}|$  it suffices to establish:

1. Difference estimates  $|q(i,j)-\bar{q}(i,j)|$
2. Bounds for bias-terms  $|V_t(j)-V_t(i)|$
3. The approximate steady state distribution  $\bar{\pi}$

### 3 An application: An ALOHA-model

In this section an illustration of the error bound theorem will be provided which will lead to accurate analytic error bounds for a specific elegant analytic approximation technique, based on truncating Möbius expansions, as recently developed and applied in [10], section 2.4.1. The actual application concerns an ALOHA-type communication (or computer) model with different source characteristics. This application actually motivated the research of this paper.

#### 3.1 Model

Consider an ALOHA-system with  $M$  sources (transmitters or processors) numbered  $1, \dots, M$ , which are alternatively idle and busy (transmitting) as follows. When  $H = \{h_1, \dots, h_n\}$  denotes that sources  $h_1, \dots, h_n$  are currently busy, an idle source  $h$  will request to become busy (to start a transmission) at an exponential rate  $\gamma_h$ . This request is accepted with probability:

$$\prod_{s \in H} (1-w_s),$$

while otherwise it is rejected and lost. Conversely, a busy source  $h$  will become idle again (complete its transmission) at an exponential rate  $\mu_h$ .

### Example 3.1 (Memory module)

As an example,  $w_s$  may represent the fraction of time that a busy source  $s$  communicates (retrieves or stores data) at some memory module  $M$ , but where a transmission can be started only if this memory module is free (such as for addressing).

### Example 3.2 (Slotted-ALOHA)

As another example, when transmissions are time-slotted in time-slots of length  $\Delta$ , as in slotted-ALOHA, while a transmission can be started only if the actual transmission switch is free, this is modeled by:

$$(1-w_s) = e^{-\Delta\mu_s}$$

#### No simple solution

Despite the simple formulation, as per section 3.1 in [10], the present system has a simple product form solution  $\pi(H)$  only if all values  $w_s$  are equal:

$w_1 = \dots = w_M$  while otherwise no simple expression exists.

### 3.2 Approximation

To evaluate the above system with unequal  $w_s$ -values, in [10] approximations  $\{\bar{\pi}(H)\}$  were computed by analytically truncating the Möbius-expansion (related to [4]):

$$(3.1) \quad \begin{cases} \pi(H) = \exp W(H) \\ W(H) = \sum_{B \subseteq H, |B| \leq K} W(B) \sum_{j=0}^{K-|B|} (-1)^j \binom{|H|-|B|}{j} \end{cases}$$

with  $|B|$  the cardinality of a set  $B$ , at truncation level  $K=2$ .

Roughly speaking, this approximation comes down to neglecting interactions between states that differ in more than two sources, as corresponding to particle-interaction approximations in statistical mechanics.)

### 3.3 Error bound

To determine an error bound on the accuracy of this approximate steady state distribution  $\bar{\pi}$ , or rather associated performance measures  $\bar{g}$  such as the system throughput, let  $q(H, H+h)$  and  $q(H, H-h)$  denote the transition rates of the original model from state  $H$  into  $H+h$  respectively  $H-h$ . As the approximate model has no direct probabilistic interpretation or Markovian transition structure, define artificial transition rates  $\bar{q}$  by:

$$(3.2) \quad \bar{q}(H, H') = \frac{1}{2} \left[ q(H, H') + q(H', H) \frac{\bar{\pi}(H')}{\bar{\pi}(H)} \right]$$

for  $H' = H+h$  or  $H-h$  while  $\bar{q}(H, H')$  is defined to be 0 otherwise. One then directly verifies for all  $H$  and  $H'$ :

$$(3.3) \quad \bar{\pi}(H) \bar{q}(H, H') = \bar{\pi}(H') \bar{q}(H', H).$$

This relation, also known as reversibility (cf. [2]), directly guarantees that  $\bar{\pi}$  is the steady state distribution of this artificially defined chain with transition rates  $\bar{q}$ . To compare  $\pi$  and  $\bar{\pi}$  or, relatedly, associated measures  $g$  and  $\bar{g}$ , we can thus apply theorem 1. To this end, in the first place, note that for all  $H, H' = H \pm h$ :

$$(3.4) \quad |q(H, H') - \bar{q}(H, H')| = \frac{1}{2} \left| q(H, H') - q(H', H) \frac{\bar{\pi}(H')}{\bar{\pi}(H)} \right|.$$

Secondly, as proven more generally in [9], where we need to substitute  $\beta(h|H) = \prod_{s \in H} (1-w_s)$  for the present application,

$$(3.5) \quad |V_t(H) - V_t(H')| \leq 1 \quad (H' = H \pm h) \quad (t \geq 0)$$

for various monotone reward rate functions, in particular, for

$$(3.6) \quad r(H) = \sum_{h \in H} \gamma_h \prod_{s \in H} (1-w_s),$$

in which case  $g$ , as per (2.4), represents the throughput of the system. By choosing  $\bar{r} = r$  so that

$$(3.7) \quad \boxed{\bar{g} = \sum_H \bar{\pi}(H) r(H)}$$

is an approximate value of the system throughput, and substituting (3.4) and (3.5) in (2.5), the error bound theorem then provides the error bound:

(3.8)

$$|\bar{g} - g| \leq \frac{1}{2} \sum_H \bar{\pi}(H) \left\{ \sum_{h \in H} \left[ q(H, H+h) - q(H+h, H) \frac{\bar{\pi}(H+h)}{\bar{\pi}(H)} \right] + \sum_{h \in H} \left[ q(H, H-h) - q(H-h, H) \frac{\bar{\pi}(H-h)}{\bar{\pi}(H)} \right] \right\}$$

### 3.4 Numerical illustration

Numerical results are given for  $M=4$  so that also the exact values  $\pi(\cdot)$  could be computed numerically for the purpose of comparison. However, in [10] the approximation technique as sketched in section 3.2 has also shown to be accurate for larger  $M$ -values.

As performance measure the number of busy sources or equivalently the throughput is evaluated. Two error bounds are compared. The first one that could be provided as based in [8] and showing (2.13) which, however, requires the term within braces in the right hand side of (3.8) to be bounded by a monotone function. The second is the error bound as per (3.8) which is an improvement in avoiding a technical proof for (2.13) and, most notably, in accuracy as illustrated. (Between parenthesis the relative error bound with respect to  $g$  is presented). For a measure as the number of idle sources the accuracy in the given examples is over a factor 3 better.

#### Remark 3.1 (Nearly reversible?)

As  $\bar{\pi}$  turns out to be an accurate approximation at first impression, from (3.4) and (3.8) one might also conclude that  $\pi$  is more or less reversible. However, this is true only if all the probabilities  $w_s$  are more or less equal, while in the examples they are not.

#### Remark 3.2 (Advantage of relaxation)

As mentioned earlier in remark 2.2, the advantage of (2.11) over (2.12), is particularly reflected in the above application. In the first place, the actual transition structure  $\bar{q}$  as defined by (3.2) is not of simple probabilistic form. Secondly, the differences (3.4) are not monotone in  $H$ . A statement like (2.13) would thus become more complicated to prove.

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## Numerical examples

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$$

Error bound 1 by (2.13)

Error bound 2 by (3.8)

Performance measure  $g$ : Expected number of busy sources.

## Example 1

$\gamma_1 = .4$	$w_1 = .45$	$\text{Max}_1  \pi(i) - \bar{\pi}(i) $	: $17 \times 10^{-7}$
$\gamma_2 = .3$	$w_2 = .40$	$g$	: .6463969
$\gamma_3 = .2$	$w_3 = .45$	$\bar{g}$	: .6463961
$\gamma_4 = .1$	$w_4 = .40$	$ g - \bar{g} $	: $8 \times 10^{-7}$

Error bound 1: .0789 (12.2%)

Error bound 2: .0096 (1.5%)

## Example 2

$\gamma_1 = .40$	$w_1 = .20$	$\text{Max}_1  \pi(i) - \bar{\pi}(i) $	: $34 \times 10^{-7}$
$\gamma_2 = .35$	$w_2 = .15$	$g$	: .887883
$\gamma_3 = .30$	$w_3 = .20$	$\bar{g}$	: .887876
$\gamma_4 = .25$	$w_4 = .15$	$ g - \bar{g} $	: $7 \times 10^{-6}$

Error bound 1: .049 (5.5%)

Error bound 2: .012 (1.5%)

## Example 3

$\gamma_1 = .40$	$w_1 = .25$	$\text{Max}_1  \pi(i) - \bar{\pi}(i) $	: $9.9 \times 10^{-6}$
$\gamma_2 = .30$	$w_2 = .20$	$g$	: .700818
$\gamma_3 = .20$	$w_3 = .30$	$\bar{g}$	: .700801
$\gamma_4 = .10$	$w_4 = .15$	$ g - \bar{g} $	: $1.7 \times 10^{-5}$

Error bound 1: .138 (19.8%)

Error bound 2: .016 (2.3%)

## Example 4

$\gamma_1 = .10$	$w_1 = .40$	$\text{Max}_1  \pi(i) - \bar{\pi}(i) $	: $1.6 \times 10^{-5}$
$\gamma_2 = .20$	$w_2 = .30$	$g$	: .407325
$\gamma_3 = .05$	$w_3 = .20$	$\bar{g}$	: .407297
$\gamma_4 = .15$	$w_4 = .10$	$ g - \bar{g} $	: $2.8 \times 10^{-5}$

Error bound 1: .3265 (80.2%)

Error bound 2: .0159 (3.9%)

**Example 5**

$\gamma_1 = .05$	$w_1 = .40$	$\text{Max}_i  \pi(i) - \bar{\pi}(i) $	:	$2.8 \times 10^{-7}$
$\gamma_2 = .20$	$w_2 = .45$	$g$	:	.4485327
$\gamma_3 = .15$	$w_3 = .40$	$\bar{g}$	:	.4485324
$\gamma_4 = .20$	$w_4 = .45$	$ g - \bar{g} $	:	$2.9 \times 10^{-7}$

Error bound 1: .0817 (18.2%)  
 Error bound 2: .0044 (1.0%)

**Example 6**

$\gamma_1 = .01$	$w_1 = .20$	$\text{Max}_i  \pi(i) - \bar{\pi}(i) $	:	$3.0 \times 10^{-8}$
$\gamma_2 = .02$	$w_2 = .30$	$g$	:	.09546422
$\gamma_3 = .03$	$w_3 = .20$	$\bar{g}$	:	.09546414
$\gamma_4 = .04$	$w_4 = .30$	$ g - \bar{g} $	:	$8.0 \times 10^{-8}$

Error bound 1: .133 (140%)  
 Error bound 2: .0004 (.5%)

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