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ON ERROR BOUND ANALYSIS FOR
TRANSIENT CONTINUOUS-TIME MARKOV REWARD STRUCTURES

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Abstract

Continuous-time Markov reward structures over a finite time interval are studied. Conditions are provided to conclude error bounds or comparison results when studying systems under modified data assumptions such as for sensitivity, computational or bounding purposes. A reliability network is studied as an application. An explicit sensitivity error bound on the effect of breakdown and repair rates is obtained.

Keywords Continuous-time Markov reward process * Transient analysis * Error bounds * Sensitivity analysis * Reliability Network.

1. Introduction

Motivation

Continuous-time Markov chains have gained a widespread popularity over the last decade for modeling and evaluation purposes in computer performance evaluation, telecommunications and reliability. Particularly, transient analysis has hereby become more and more important. Most notably, for example, system availability or reliability during finite time interval currently receives considerable attention (e.g. [1], [2], [3], [4], [5], [6], [7], [9], [10], [13]).

In contrast with steady state analysis explicit expressions are usually not available for transient characteristics and numerical computation is most commonly involved (cf. [3], [10]). As exact computations rapidly become costly, approximate procedures and truncations are frequently employed. Error bounds on the accuracy of such approximations are usually not available.

More generally, systems may have to be studied under different parameter data or protocols. To give some examples, one may wish to evaluate the effect of parameter imprecisions such as due to statistical interval estimation (perturbation or sensitivity analysis). Or, for computational purposes, large or infinite state spaces can have been truncated by transition modifications (truncations). Similarly, a simple performance estimate can have been suggested by modifying some system assumptions (simple estimates). Or, relatedly, a system might be compared under two protocols or policies for optimizing or bounding purposes.

Roughly speaking, error bound or comparison analysis to compare modified versions of some given Markovian reward structure are thus of practical interest. Particularly, transient structures may hereby be thought of for present-day applications such as performability analysis.

Literature

For the steady state case, a variety of perturbation results for numerical computations have been reported (cf. [11], [12], [14], [15] and references therein). For the transient continuous-time case such similar results are much less common. In [6] an elegant expression is derived for the derivative of marginal probability distributions with respect to a single system parameter as based on the well-known randomization method. Related parameter sensitivity results for reward structures can also be concluded (cf. [4], [6]). These results, though, still involve a large or in fact infinite recursive numerical scheme and do not concern other type perturbations such as truncations or system modifications.

Results

This note aims to show that by combining randomization with results from [14], one can provide conditions to conclude an error bound or monotonicity result when evaluating reward structures for a continuous-time Markov chain over a finite time interval under different system data. The verification of this condition is essentially based on estimating so-called bias-terms. For concrete applications this can be achieved in an analytic manner. To illustrate the conditions and this estimation of bias-terms, a reliability model will be studied. An explicit a priori error bound will be provided for the sensitivity in breakdown and repair rates.

2 General model and result

2.1 Model and discrete-time transformation

Consider a continuous-time Markov reward chain with state space $S = \{0,1,2...\}$, transition rates $q(i,j)$ for a transition from a state i in a state j and reward rate $r(i)$ whenever the system is in state i. Assume that for some constant Q

$$(2.1) \qquad \Sigma_j q(i,j) \le Q < \infty \qquad (i \in S)$$

and define the uniformization transition probability matrix P by

$$(2.2) \qquad P(i,j) = \begin{cases} q(i,j)/Q & \text{for } j \neq i \\ 1-\Sigma_{j \neq i} q(i,j)/Q & \text{for } j=i \end{cases}$$

Let $p_t(i,j)$ denote the transition probabilities over time t. Then by the standard uniformization technique (e.g. [4], [5])

$$(2.3) \qquad P_t(i,j) = \Sigma_{k=0}^{\infty} e^{-tQ} \frac{(tQ)^k}{k!} P^k(i,j) \qquad (i,j \in S)$$

where P^k is the k-th power of the uniformization transition matrix P. Further, define the expectation operators T, T^k and T_t on real-valued functions $f: S \to R$ by:

$$(2.4) \qquad \begin{aligned} Tf(i) &= \Sigma_j P(i,j)f(j) \\ T^k f(i) &= \Sigma_j P^k(i,j)f(j) \\ T_t f(i) &= \Sigma_j P_t(i,j)f(j) \end{aligned}$$

The function V_t as defined by

$$(2.5) \qquad V_t(i) = \int_0^t T_s r(i)ds \qquad (i \in S)$$

then represents the expected total reward over time t conditional to the initial state at time 0. Now similarly define the uniformized expected total reward functions V^n by:

$$(2.6) \qquad V^n(i) = Q^{-1} \Sigma_{k=0}^{n-1} T^k r(i) \qquad (i \in S)$$

Lemma 1 (Reward uniformization) For all $t \ge 0$:

$$(2.7) \qquad V_t = \Sigma_{k=0}^{\infty} e^{-tQ} \frac{(tQ)^k}{k!} V^k$$

Proof By using the Gamma-Poisson relation

$$\int_0^t \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds = \Sigma_{k=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

we obtain by substituting (3), $p(k,\nu) = e^{-\nu} \nu^k/k!$ and using (4)

$$V_t = \int_0^t \Sigma_{k=0}^{\infty} p(k,sQ) T^k r \, ds$$

$$= \Sigma_{k=0}^{\infty} (\Sigma_{\ell=k+1}^{\infty} p(\ell,tQ)) Q^{-1} T^k r$$

$$= \Sigma_{\ell=0}^{\infty} p(\ell,tQ) (\Sigma_{k=0}^{\ell-1} T^k r) Q^{-1} = \Sigma_{\ell=0}^{\infty} p(\ell,tQ) V^{\ell}$$ □

2.2 Error bounds

Now suppose that we consider a similar Markov reward process with state space $\bar{S}$, transition rates $\bar{q}(i,j)$ and reward rates $\bar{r}(i)$ where we make the assumptions

(i) $\bar{S} \subset S$

(ii) $\Sigma_j \bar{q}(i,j) \le Q \quad (i \in S)$

The notation from section 2.1 is adopted for this Markov reward process with an upper bar "-" symbol, e.g. $\bar{P}$ as by (2.2) and $\bar{V}_t$ as by (2.5) with $\bar{T}_s$ as by (2.4). A symbol "(-)" is used when both the original and modified process are meant.

The following key-theorem strongly resembles theorem 2.1 in [14]. It differs though in that it concerns finite horizon (thus transient) reward structures as opposed to average rewards for continuous-time Markov chains. Basically, it shows that error bounds can be concluded by comparing only differences in transition rates and estimating so-called bias-terms. Though similar in nature to the proof in the above reference, because of its simplicity and some differences in view of the transient continuous-time case, we prefer to give a complete self-contained proof. The essential new step though is no more than Lemma 1 above.

Theorem 1 (Error bound) Suppose that for some nonnegative function $\Phi(.)$, some initial state $\ell \in \bar{S}$, some constants ε, δ, $\beta > 0$, all $i \in \bar{S}$ and all $k \ge 0$:

(2.8) $|\Sigma_j [\bar{q}(i,j) - q(i,j)][V^k(j) - V^k(i)]| \le \varepsilon \, \Phi(i)$

(2.9) $|(\bar{r} - r)(i)| \le \delta \Phi(i)$

(2.10) $\bar{T}^k \Phi(\ell) \le \beta$

Then

(2.11) $\quad |\bar{V}_Z - V_Z| \le [\varepsilon+\delta]\beta\, Z$

Proof By virtue of (2.6), we have

(2.12) $\quad {}^{(\bar{V})}{}^{k+1}(i) = {}^{(\bar{r})}(i)Q^{-1} + {}^{(\bar{T})}V^k(i)$

As the transition probabilities $\bar{P}(.,.)$ remain restricted to $\bar{S}\subset S$, for arbitrary $\ell\in\bar{S}$ we can write:

$$(2.13) \qquad (\bar{V}^k - V^k)(\ell) = (\bar{r}-r)(\ell) + (\bar{T}\bar{V}^{k-1} - TV^{k-1})(\ell)$$

$$= (\bar{r}-r)(\ell) + (\bar{T}-T)V^{k-1}(\ell) + \bar{T}(\bar{V}^{k-1} - V^{k-1})(\ell)$$

$$= \Sigma_{s=0}^{k-1}\ \bar{T}^s([\bar{r}-r] + [(\bar{T}-T)V^{k-s-1}])(\ell) + \bar{T}^k(\bar{V}^0 - V^0)(\ell)$$

where the last step follows by iteration. First note that the last term in the latter right hand side is equal to 0 as $V^0(.)=V^0(.)=0$. Further, by (2) and (4) we can also write

$$(2.14) \qquad (\bar{T}-T)V^s(i) = \Sigma_{j\ne i}[\bar{q}(i,j) - q(i,j)]Q^{-1}V^s(j)$$

$$+ \Sigma_{j\ne i}[\bar{q}(i,j) - q(i,j)]Q^{-1}V^s(i)$$

$$= \Sigma_{j\ne i}[\bar{q}(i,j) - q(i,j)][V^s(j) - V^s(i)]Q^{-1}$$

By substituting (2.14) in (2.13), taking absolute values and noting that $\bar{T}^s$ is a monotone operator for all s (i.e. $\bar{T}^s f \le \bar{T}^s g$ if $f\le g$ componentwise), we obtain from (2.8)-(2.10) for any $h\ge 0$:

$$(2.15) \qquad |(\bar{V}^k - V^k)(\ell)| \le [\delta+\varepsilon]\ \Sigma_{s=0}^{k-1}\ \bar{T}^s\Phi(\ell) \le [\delta+\varepsilon]\beta k Q^{-1}$$

By lemma 1 and (2.15) we thus conclude:

$$(2.16) \qquad |(\bar{V}_Z - V_Z)(\ell)| \le [\delta+\varepsilon]\beta\ \Sigma_{k=0}^{\infty}\ kQ^{-1}e^{-ZQ}\ \frac{(ZQ)^k}{k!} = [\varepsilon+\delta]\beta Z \qquad \square$$

As a less restricted version, similar to theorem 2.2 in [14] we can also conclude a monotonicity result as follows:

Theorem 2.2 (Monotonicity result) Suppose that for all $i\in\bar{S}$ and $k\ge 0$:

$$(2.17) \qquad [\bar{r}-r](i) + \Sigma_j[\bar{q}(i,j) - q(i,j)][V^k(j) - V^k(i)] \ge (\le)\ 0$$

Then for all Z and $\ell \in \bar{S}$, we have

(2.18) $\quad \bar{V}_Z(\ell) \geq (\leq)\ V_Z(\ell)$

Proof This follows directly by substituting (2.14) in (2.13), noting that $\bar{V}^0(.)=V^0(.)=0$ and recalling that the operators $\bar{T}^s$ are monotone. □

Remark (Bias-terms) The essential step for verifying the conditions is to find upper bounds on the so-called bias terms $V^k(j)-V^k(i)$ of the form

$$|V^k(j)-V^k(i)| \leq B_{i,j}$$

Bias-terms are generally known to be bounded of the form (e.g. [15])

$$|V^k(j)-V^k(i)| \leq 2\ R \min[R_{ij}, R_{ji}]$$

where R is some upper bound on the reward rate and where R_{ij} denotes the mean number of steps (first passage time) to reach state j out of state i. This estimate, however, can be very rough (e.g. [15]). More importantly, except for simple random walk type models bounds on mean first passage times are extremely hard to obtain such as most notably for queueing or communication network (and thus multi-dimensional) applications.

In the next section, therefore, we will illustrate how for a concrete communication network application estimates $B_{i,j} \leq B$ can also be obtained analytically by inductively using the Markov reward equation (2.12). This technique has already proven succesful in a number of non-product form queueing networks (cf. [14]). Roughly speaking, by theorem 1 we can thus conclude error bounds for continuous-time and transient behaviour basically by inductive verification (particularly estimation of bias terms) of the conditions.

Remark (Φ-function) For a detailed discussion of the role of the bounding function Φ we refer to section 2.2 of [14]. It enables for instance state space truncations, say at state L, by choosing Φ to be the indicator of state L (i.e. $\Phi(i)=1_{\{i=L\}}$). (see section 3.4 of [14]).

As another application, by choosing $\Phi(\bar{n})=1+n$, where $\bar{n}$ is the population vector of jobs in a queueing network and n the total number of jobs, a finite source input with M sources with exponential parameter γ could be compared with a Poisson input rate $\lambda=\gamma M$ (see section 3.5 of [14]).

Though in the remainder of this paper we will only use $\Phi(.)=1$, we have included the Φ-function in the theorem for completeness.

3 Application: An availability model.

3.1 Model.

Consider an availability model which consists of N components, numbered 1,...,N. Each component is alternatively up and down as follows. When $H = (h_1,...,h_n)$ is the vector of components $h_1,...,h_n$ that are down, a component $h \notin H$ will go down at an exponential rate

$$\beta(h|H)$$

Here we assume that the breakdown rate of a particular component will become less if more components are down, that is

(3.1) $\beta(h|H) \geq \beta(h|H+s)$,

which may reflect for instance that a common maintenance device is shared over the remaining up components or that the breakdown rate of a component depends on the system speed which in turn slows down when more components are down. When a component h is down it is repaired at an exponential rate ρ_h independently of the status of the other components.

Here, as natural, we also assume the overal breakdown rate to be relatively small as opposed to the repair rates of individual components, that is for any state H and component h:

(3.2) $\Sigma_{h' \notin H}\ \beta(h'|H) \leq \rho_h$

As performance measure of interest we wish to investigate the *total number of breakdowns* $B_Z(H)$ *during a fixed time interval of length Z* when starting in a particular down state H. More precisely, we wish to provide an error bound on the effect of imprecisions or changes in the breakdown and repair rates on this value $B_Z(H)$. That is, we will employ sensitivity analysis for all parameters synchronously.

3.2 Sensitivity error bounds

Consider the original system with parameters $\beta(h|H)$ and ρ_h as well as a perturbed system with parameters $\bar{\beta}(h|H)$ and $\bar{\rho}_h$ where we assume that for some Δ>0 and all h,H

(3.3) $\Sigma_{h\in H}|\bar{\rho}_h - \rho_h| + \Sigma_{h\notin H}|\bar{\beta}(h|H) - \beta(h|H)| \leq \Delta$

We adopt all notation from section 2 without upper bar "-" symbol for the original and with upper bar symbol "-" for the perturbed system. With

$$\bar{S} = S = \{H|H=(h_1,\ldots,h_n),\ h_i\in\{1,\ldots,M\}\}$$

and choosing

$$Q \geq \Sigma_s \rho_s + \Sigma_{s\notin H}\beta(h|H)$$

the corresponding transition probabilities become:

(3.4)
$$^{(\bar{\ })}p(H,H+h) = {}^{(\bar{\ })}\beta(h|H)Q^{-1}$$
$$^{(\bar{\ })}p(H,H-h) = {}^{(\bar{\ })}\rho_h Q^{-1}$$

Further, by setting a reward rate

(3.5) $^{(\bar{\ })}r(H) = \Sigma_{h\notin H}{}^{(\bar{\ })}\beta(h|H)$

the measure $^{(\bar{\ })}B_Z(H)$ is given by

(3.6) $^{(\bar{\ })}V_Z(H) = {}^{(\bar{\ })}B_Z(H)$

To apply theorem 2.1, the following lemma is essential.

Lemma 3.1 For all h,H:

$$(3.7)\qquad 0 \le V^k(H) - V^k(H+h) \le 1$$

Proof We will apply induction to t. (3.7) holds for k=0 by $V^0(.)=0$. Suppose that (3.7) holds for k≤m. The following relation is then obtained by comparing the one-step reward relation (2.12) in state H and state H+h where (3.4) and (3.5) are substituted. Hereby we note in advance that some terms are artificially split (e.g. $\beta(h|H)$ in $\beta(h|H+h) + [\beta(h|H)-\beta(h|H+h)]$) or added and subtracted (e.g. $\rho_h Q^{-1} V_m(H)$), in order to compare transitions with equal coefficients pairwise.

$$(3.8)\qquad V^{m+1}(H) - V^{m+1}(H+h)$$

$$=$$

$$\Big\{\Sigma_{s\notin H+h}\ \beta(s|H)Q^{-1} + \beta(h|H)Q^{-1} +$$
$$\Sigma_{s\in H}\rho_s\ Q^{-1}V^m(H-s) + \rho_h Q^{-1}V^m(H) +$$
$$\Sigma_{s\notin H+h}\ \beta(s|H+h)Q^{-1}V^m(H+s) +$$
$$\Sigma_{s\notin H+h}\ [\beta(s|H)-\beta(s|H+h)]Q^{-1}V^m(H+s)+\beta(h|H)Q^{-1}V^m(H+h) +$$
$$[1 - \Sigma_{s\in H}\rho_s\ Q^{-1} - \Sigma_{s\notin H+h}\beta(s|H)Q^{-1}-\rho_h Q^{-1}-\beta(h|H)Q^{-1}]V^m(H)\Big\}$$

$$-$$

$$\Big\{\Sigma_{s\notin H+h}\beta(s|H+h)Q^{-1} +$$
$$\Sigma_{s\in H}\rho_s\ Q^{-1}V^m(H+h-s)+\rho_h Q^{-1}V^m(H) +$$
$$\Sigma_{s\notin H+h}\ \beta(s|H+h)Q^{-1}V^m(H+h+s) +$$
$$\Sigma_{s\notin H+h}\ [\beta(s|H)-\beta(s|H+h)]Q^{-1}V^m(H+h)+\beta(h|H)Q^{-1}V^m(H+h) +$$
$$[1 - \Sigma_{s\in H}\rho_s\ Q^{-1}-\rho_h Q^{-1}-\Sigma_{s\notin H+h}\beta(s|H)Q^{-1}-\beta(h|H)Q^{-1}]V^m(H+h)\Big\}$$

$$=$$

$$\beta(h|H)Q^{-1} +$$
$$\Sigma_{s\notin H+h}\ [\beta(s|H)-\beta(s|H+h)]Q^{-1} +$$
$$\Sigma_{s\in H}\rho_s\ Q^{-1}[V^m(H-s)-V^m(H-s+h)] +$$
$$\rho_h Q^{-1}[V^m(H)-V^m(H)]+\beta(h|H)Q^{-1}[V^m(H+h)-V^m(H+h)] +$$
$$\Sigma_{s\notin H+h}\ \beta(s|H+h)Q^{-1}[V^m(H+s)-V^m(H+s+h)] +$$
$$\Sigma_{s\notin H+h}\ [\beta(s|H)-\beta(s|H+h)]Q^{-1}[V^m(H+s)-V^m(H+h)] +$$
$$[1 - \Sigma_{s\in H}\ \rho_s Q^{-1}-\rho_h Q^{-1}-\Sigma_{s\notin H}\ \beta(s|H)Q^{-1}][V^m(H)-V^m(H+h)]$$

Here, first of all, note that the fourth and fifth erm are indeed equal to 0, but kept in for clarity and arguments below. Further, note that

$$(3.9) \qquad V^m(H+s) - V^m(H+h) = [V^m(H+s) - V^m(H)] + [V^m(H) - V^m(H+h)]$$

where the first term in the right hand side is nonpositive but estimated from below by $V_m(H+s)-V_m(H)\geq-1$, as per induction hypothesis. As a consequence, by combining the one but last term of the right hand side of (3.8), with (3.9) substituted, with the second term

$$\Sigma_{s\notin H+h}[\beta(s|H)-\beta(s|H+h)]Q^{-1},$$

substituting the induction hypothesis $V_m(H)-V_m(H+s)\geq0$ for all H and s in the other terms and noting that all coefficients sum up to 1, we have proven $V^{m+1}(H)-V^{m+1}(H+h) \geq 0$.

To prove the upper estimate 1 of (3.8) with k=m+1, now recall that the fifth item is equal to 0 which compensates for the first additional term $\beta(h|H)Q^{-1}$, while also the fourth term is 0 which compensates for the second additional term by virtue of the estimation

$$\Sigma_{s\notin H+h}[\beta(s|H)-\beta(s|H+h)]Q^{-1} \leq \Sigma_{s\notin H}\beta(s|H)Q^{-1} \leq \rho_h Q^{-1}$$

as guaranteed by assumption (3.2). Further, by (3.9) and the induction hypothesis (3.8) for t=m, we also have

$$V^m(H+s) - V^m(H+h) \leq V^m(H) - V^m(H+h)$$

With these arguments, substitution of the induction hypothesis $V^m(H)-V^m(H+s)\leq1$ for all H and s, and using again that all coefficients sum up to 1, we also conclude $V^{m+1}(H)-V^{m+1}(H+h)\leq1$. □

Theorem 3.1 For all finite time intervals [0,Z] and initial states H at time 0:

$$(3.10) \qquad |\bar{B}_Z(H)-B_Z(H)| \leq \Delta[1+Z].$$

Proof We will apply theorem 2.1. Condition (2.9) hodls with $\delta=\Delta$ by virtue of (3.3) and (3.5) and when choosing $\Phi(.)=1$. Condition (2.10) is satisfied with $\beta=1$. And by lemma 3.1 above, condition (2.8) is verified with $\varepsilon=\Delta$ by

$$(3.11) \qquad \Sigma_{H'}[\bar{q}(H,H')-q(H,H')][V^k(H')-V^k(H)] =$$

$$\Sigma_{s\in H}[\bar{\rho}_s-\rho_s][V^k(H-s)-V^k(H)] +$$

$$\Sigma_{s\notin H}[\bar{\beta}(s|H)-\beta(s|H)][[V^k(H+s)-V^k(H)]$$ □

As a special case, we can also obtain monotonicity results by:

Theorem 3.2 Suppose that for all h and H:

$$(3.12)\qquad \begin{cases} \tilde{\rho}_h \geq (\leq) \rho_h \\ \tilde{\beta}(h|H) \geq (\leq) \beta(h|H) \end{cases}$$

Then for all Z and H

$$(3.13)\qquad \tilde{B}_Z(H) \geq (\leq)\ B_Z(H)$$

Proof We need to verify condition (2.17) of theorem 2.2. By (3.5):

$$(3.14)\qquad \tilde{r}(H) - r(H) = \Sigma_{s \notin H} [\tilde{\beta}(s|H) - \beta(s|H)]$$

while by lemma 3.1 for all H and s:

$$(3.15)\qquad \begin{array}{l} V^k(H-s) - V^k(H) \geq 0 \\ V^k(H+s) - V^k(H) \geq -1. \end{array}$$

Combining (3.11) with (3.14) and substituting (3.15) proves (2.17). Theorem 2.2 completes the proof. □

Evaluation Transient analysis of continuous-time Markov chain reward structures is frequently demanded in practice such as to evaluate the number of breakdowns or availability level of a reliability network in a fixed time interval. As explicit parametric expressions are most rarely available while input data are often subject to imprecisions, sensitivity or error bound analysis with respect to data perturbations are of practical interest. This paper addresses this issue by providing conditions from which a priori error bounds can be concluded. In concrete situations verification of these conditions can be performed in an inductive analytic manner, as is illustrated by a reliability network. Further application in the area of performability analysis seems promising.

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