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PRODUCT FORMS FOR AVAILABILITY

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Product Forms for Availability

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Abstract

This paper characterizes a class of availability models that exhibit a product form steady state solution. A condition for the models to have a product form solution is stated explicitely. The resulting product form is presented together with a proof. A number of examples is discussed that show the usefulness of our product form results for the analysis of computer system availability.

1 Introduction

Over the last decade computer systems have penetrated in almost every part of society. Moreover, computer systems become more and more involved in highly responsible activities. This often implies that the malfunctioning of a computer system, i.e. the failure of crucial components, results in great losses. These arguments justify the need to access the probability of malfunctioning or conversely, the probability of well functioning.

The most commonly used measures to quantify the well functioning of a system are reliability and availability. The reliability function is typically of interest for systems that have to remain operational during their whole (limited) mission length, such as air or spacecraft systems. If components can be repaired and if some downtime can be tolerated then availability is a more appropriate concept. There are several types of availability measures. [12] provides an overview on numerical methods for calculating transient measures such as point- and interval availability. In this paper we focus on the steady state availability of repairable computer systems.

Basically we consider a system consisting of N components that are alternatively up and down. We denote by $\mathcal{H}$ the set of all possible states of the system and partition $\mathcal{H}$ into $\mathcal{G}$ and $\mathcal{B}$, the set of states in which the system is up (the "good" states) and the set of states in which the system is down (the "bad" states) respectivily. Let the function $\alpha: \mathcal{H} \to \{0,1\}$ be such that $\alpha(H) = 1$ if $H \in \mathcal{G}$ and $\alpha(H) = 0$ if $H \in \mathcal{B}$. Then the steady state availability, A, is given by

$$A = \sum_{H \in \mathcal{H}} \pi(H)\alpha(H), \tag{1.1}$$

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where $\pi(H)$ denotes the steady state probability that the system is in state H. Note that steady state availability can be easily generalised to long term average reward by allowing the fuction $\alpha(\cdot)$ to be any real valued function on the state space $\mathcal{H}$. From (1.1) it is clear that for our purpose it suffices to obtain the steady state distribution $\pi(\cdot)$.

If all components have statistically independent life times and are seperately repaired $\pi(\cdot)$ has a very simple form. Let a state $H = \{h_1, \ldots, h_n\}$ denote that components $h_1, \ldots, h_n$ are down while the other components are up and let p_h denote the steady state probability that component h is up. If the mean life time and mean repair time for component h are given by μ_h and ν_h respectively then p_h is given by

$$p_h = \frac{\mu_h}{\mu_h + \nu_h} \tag{1.2}$$

and the steady state distribution $\pi(\cdot)$ is given by

$$\pi(H) = \prod_{h \notin H} p_h \prod_{h \in H} (1 - p_h). \tag{1.3}$$

This model is often used in availability studies of (large) communication networks. A typical performance measure in this context is the probability that two nodes can communicate via a network of links that are subject to failure (cf. [1], [2], [3], [8]). Note that in this case $\pi(\cdot)$ depends on the life time distributions and repair time distributions only through their means. This is refered to as insensitivity. For proofs on the existence of the steady state distribution and other asymptotic quantities we refer to [13].

For modelling fault tolerant computer systems the assumption that components have statistically independent life times is to restrictive. Therefore the failure- and repair times of the components are mostly assumed to be exponentially distributed. Under this assumption the steady state distribution $\pi(\cdot)$ of the corresponding Markov chain can be obtained by solving a set of linear equations numerically. If the cardinality of the state space is large then special techniques must be used, such as sparse matrix storage and successive overrelaxation (cf. [6]). For highly reliable systems a bounding technique based on state space aggregation is presented in [10].

We do not assume that failure- and repair times are exponentially distributed, but instead characterize a class of models that exhibit a insensitive product form steady state solution like (1.3) without assuming that the life times are statistically independent nor that the components are seperately maintained. The class of product form models is characterized by putting conditions on the way in which the failure-and repair processes of individual components are allowed to depend on the state of the system. The two main conditions are that the repair of a component starts immediately upon failure and that the system behaviour in a state H does not depend on the way that state was entered. In section 2 these conditions are stated explicitely.

From a mathematical point of view this product form result is not new as based on the notion of reversibility (cf. [4], [7]). The primary goal of this paper, however is to show that apart from queueing theory product form results are also useful in availability modelling.

This paper is organized as follows. In section 2 we introduce and illustrate the model and give sufficient conditions for a product form solution. In section 3 we state and proof this product form result. In section 4 we give a number of examples that show how various systems can be modelled and how the product form conditions can be verified. Section 5 concludes the paper and indicates further research.

2 Model and product form conditions

In this section we first give an abstract formulation of the model. The main feature of the model is that components go down and are repaired with state dependent speeds. We illustrate this with an example and give an equivalent queueing network interpretation. Then we state sufficient conditions on these speeds that ensure a product form for the steady state probabilities. With some examples we indicate how these conditions can be checked in practice.

2.1 The model

Consider a system consisting of N components, numbered $1, \ldots, N$. Each component is alternatively up (being able to work) and down (not able to work and needing repair). When component h goes down, it requires a random amount of repair R_h. When its repair is completed the component immediately returns to the up mode and performs a random amount of work, W_h, before going down again.

Let a state $H = \{h_1, \ldots, h_n\}$ denote that components $h_1, \ldots, h_n$ are down while the other components are up. For notational convenience we assume that $h_1, \ldots, h_n$ are given in increasing order. The state in which all components are up (none is down) is denoted by $\emptyset$. The speeds at which the components in the up mode work and the speeds at which the components that are down are repaired are allowed to be state dependent. If the system is in state $H = \{h_1, \ldots, h_n\}$ then the speed at which a component h that is up (i.e. $h \notin H$) works is denoted by

$$\beta(h|h_1, \ldots, h_n),\ h \notin \{h_1, \ldots, h_n\}. \tag{2.1}$$

and the speed at which a component h that is down (i.e. $h \in H$) is repaired is denoted by

$$\delta(h|h_1, \ldots, h_n),\ h \in \{h_1, \ldots, h_n\}. \tag{2.2}$$

If there is no danger of ambiguity we abbreviate $\beta(h|h_1, \ldots, h_n)$ and $\delta(h|h_1, \ldots, h_n)$ by $\beta(h|H)$ and $\delta(h|H)$ respectively. Further, we write $H_1 + H_2$ for the union of two sets H_1 and H_2, $H_1 - H_2$ for their symmetric difference and h for the singleton $\{h\}$.

Interpretation of the speeds The actual *time* that component h remains up depends on the random amount of work, W_h, that it performs before going down and on the varying speeds $\beta(h|\cdot)$ with which it works, i.e. on the states that the system visits during the up time of component h. This is illustrated in the example below. In the same way the time that a component h remains down depends on the amount of repair, R_h, that it needs and on the speed(s) $\delta(h|\cdot)$ with which it is repaired.

Example Consider a computer system with two disc units, numbered 1 and 2. If both discs are operational then they perform 1 instruction per unit time. However, if one disc is down then the other one takes over all its work and thus has to work twice as fast, resulting in 2 instructions per unit time. Thus $\beta(1|\emptyset) = \beta(2|\emptyset) = 1$ and $\beta(1|2) = \beta(2|1) = 2$. Suppose that at time $t = 0$ disc 2 goes down and that disc 1 can still perform 3600 instructions before going down. If the repair of disc 2 takes more than 1800 time units then disc 1 has performed all its 3600 instructions and will go down after $3600/2 = 1800$ time units. But if the repair of disc 2 requires only 1000 time units then disc drive 1 is still able to perform $3600 - 2 * 1000 = 1600$ instructions when disc 2 goes up again. This means that disc 1 will remain up another 1600 time units, provided that disc drive 2 remains up also. Thus, in

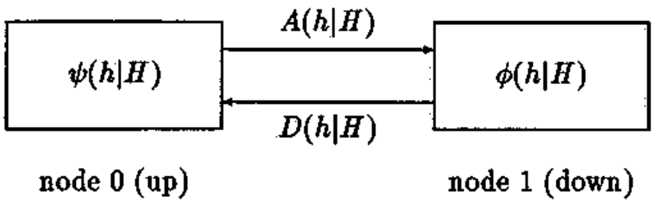

Figure 1. Queueing network model

this case disc 1 goes down at $t = 2600$.

Below we give some more examples of how the functions β and δ can be used as delay or acceleration factors to model failure- and repair dependencies between the components. In the examples H_1 and H_2 denote two disjoint sets of components not containing component h.

- $\beta(h|H_1) = 0$ reflects that component h, although it is up, does not work if the components in the set H_1 are down. Thus, component h can not go down if the system is in state H_1. We also refer to this by saying that the failure process of component h has stopped.
- $\beta(h|H_1 + H_2) < \beta(h|H_1)$ indicates that component h works at a lower speed if, apart from the components in H_1, an additional set of components, H_2, has failed. Another interpretation is that the failure process is slowed down.
- $\beta(h|H_1+H_2) > \beta(h|H_1)$ indicates that component h works faster if the additional components H_2 have failed. Another interpretation is that that the failure process is accelerated.
- $\delta(h|H_1 + H_2 + h) < \delta(h|H_1 + h)$ indicates that the repair of component h is slowed down if an additional set of components H_2 has failed.
- $\delta(h_i|h_i, h_j) > \delta(h_j|h_i, h_j) = 0$ models the (preemptive) priority of the repair of component h_i over that of component h_j.

Queueing network interpretation (cf. [6]) Our model is equivalent to the closed two–node queueing network model with N different jobs circulating between node 0 and node 1 given in figure 1. Each job corresponds to a component which is up if it is at node 0 and down if it is at node 1. A job h in node 0 (corresponding component up) will request to make a transition to node 1 (corresponding component down) after having received a random amount of service W_h. If at that moment the collection of jobs, H, is already present at node 1, then this request is granted with probability $A(h|H)$. With probability $1 - A(h|H)$ job h must receive another random amount of service W_h before submitting its next request. Analoguously, a request from a job h in node 1 to make a transition to node 0 after having received a random amount of service R_h is granted with probability $D(h|H)$ if the collection of jobs, H, is present at node 1.

Further, a job h in node 0 is served at a state dependent speed $\psi(h|H)$ and a job h in node 1 is served at a state dependent speed $\phi(h|H)$, where H is the collection of jobs at node 1. In [4] it is shown that our model is equivalent to this queueing network model. The equivalence is established by choosing $\beta(h|H) = A(h|H)\psi(h|H)$ and $\delta(h|H) =$

$D(h|H)\phi(h|H)$.

2.2 Product form condition

Denote by $\mathcal{H} \subset 2^N$ the set of states, including $\emptyset$ (no components down) that can be reached from state $\emptyset$ by components going up and down. (2^N denotes the power set of $\{1,\ldots,N\}$). Our first assumption is that in any state $H \in \mathcal{H}$, $H \neq \emptyset$, there is a component $h \in H$ such that $\delta(h|H) > 0$. This means that in any state in which one or more components are down, there is at least one component that is being repaired. This conforms condition (2.3) below. If all random variables W_h and R_h were exponential then we would have an irreducible Markov chain on $\mathcal{H}$. This follows since each state $H \in \mathcal{H}$ can be reached from $\emptyset$ by definition and $\emptyset$ can be reached from any state $H \in \mathcal{H}$ by subsequent repairs. But, as we do not restrict ourselves to exponential up- and down times and thus can not speak in terms of Markov chains, we call the set $\mathcal{H}$ an irreducible set. Thus the irreducible set $\mathcal{H}$ is the set of states, containing $\emptyset$, such that out of any state $H \in \mathcal{H}$ any other state in $\mathcal{H}$ and no state outside $\mathcal{H}$ can be reached.

We now state sufficient conditions on the functions β and δ in order for the steady state probabilities to be of product form.

Product form conditions For any state $H = \{h_1,\ldots,h_n\} \in \mathcal{H}$, $H \neq \emptyset$

$$\delta(h|H) > 0 \text{ for some } h \in H. \tag{2.3}$$

$$\delta(h|H) = 0 \Longleftrightarrow \beta(h|H-h) = 0 \text{ for all } h \in H. \tag{2.4}$$

Further, for all permutations $(h_{i_1},\ldots,h_{i_n})$ of $\{h_1,\ldots,h_n\}$ for which the denominator in (2.5) below is positive

$$\prod_{k=1}^{n} \frac{\beta(h_{i_k}|h_{i_1},\ldots,h_{i_{k-1}})}{\delta(h_{i_k}|h_{i_1},\ldots,h_{i_k})} = K(H), \tag{2.5}$$

where $K(H)$ is some positive value depending on H only. $K(\emptyset) = 1$ by definition.

Discussion of conditions Condition (2.3) guarantees that the product in (2.5) has a positive denominator for at least one permutation, while (2.4) guarantees that if the denominator of this product is zero then also the numerator is equal to zero, so that the product can be chosen equal to $K(H)$. Thus, effectively only permutations with non–zero denominators need to be considered.

The interpretation of condition (2.4) is that if component h can not be repaired when the system is in state H then component h can not go down when the system is in state $H - h$ and vice versa.

The key–condition (2.5) is related to the well–known Kolmogorov criterion for a Markov chain to be reversible (cf. [7]). For our model with state dependent speeds it can be shown that the Kolmogorov criterion is equivalent to condition (2.5). Roughly speaking, (2.5) requires that it should not matter in which order components go down, with β/δ as additional factor per component that is down. Note that if the functions β and δ take the values zero or one only then (2.5) follows directly from (2.4).

It is not difficult to see that condition (2.5) is equivalent to

$$\frac{K(H+h)}{K(H)} = \frac{\beta(h|H)}{\delta(h|H+h)} \tag{2.6}$$

for all states H and all components h such that $\delta(h|H+h) > 0$. The speeds β and δ often suggest a form for the function $K(\cdot)$ so that condition (2.5) can be easily checked by verifying (2.6). In the two examples below we will illustrate the usefulness of the equivalent condition (2.6).

Example 1 (coordinate convex) Suppose that the maximum number of components that can be down at the same time is C. I.e. if C components are down then the other components stop working (their failure process is temporarily stopped) until one of the C components is repaired. Thus $\mathcal{H} = \{H : |H| \le C\}$, where $|H|$ is the number of components that is down. Assuming for simplicity that this is the only dependency in the model, the speeds are given by

$$\beta(h|H) = \begin{cases} b_h, \text{ if } |H| < C \\ 0, \text{ if } |H| = C \end{cases} \tag{2.7}$$

$$\delta(h|H) = d_h, \text{ for all } H \in \mathcal{H}. \tag{2.8}$$

Substituting these speeds in condition (2.6) yields

$$\frac{K(H+h)}{K(H)} = \begin{cases} \frac{b_h}{d_h}, \text{ if } |H| < C \\ 0, \text{ if } |H| = C, \end{cases} \tag{2.9}$$

which immediately suggest the following form for the function $K(\cdot)$

$$K(H) = \begin{cases} \prod_{h \in H} \frac{b_h}{d_h}, \text{ if } |H| < C \\ 0, \text{ if } |H| = C. \end{cases} \tag{2.10}$$

Example 2 (neighbourhood structure) This is a slightly more complicated example involving a neighbourhood strucure, inspired by the rude CSMA protocol in communication theory (cf. [11]). As before, denote by H the set of components that is down. If the system is in state H then $N_0^h(H)$ denotes the number of neighbours of component h that is up and $N_1^h(H)$ denotes the number of neighbours of h that is down. Components that are down are always repaired at a speed $\delta = 1$. The working speed for a component h that is up depends on the number of its neighbours that are up and down in the following way

$$\beta(h|H) = p^{N_0^h(H)} q^{N_1^h(H)}, \tag{2.11}$$

where p and q are arbitrary, non negative constants. The case $q = 0$, for example, corresponds to the case where a component h only works if all its neighbours are up. Or, alternatively, the failure process of component h is stopped if one or more of its neighbours is down. Substituting these speeds into condition (2.6) yields

$$\frac{K(H+h)}{K(H)} = p^{N_0^h(H)} q^{N_1^h(H)}. \tag{2.12}$$

From (2.12) it follows that the number of extra terms p due to an additional component h that is down is exactly the number of pairs (h, h') such that h and h' are neighbours and h' is up in state H. The same holds for the number of extra terms q, with h' is up replaced by h' is down. This suggests the following form for the function $K(\cdot)$, which can be easily checked to satisfy (2.6).

$$K(H) = p^{B_0(H)} q^{B_1(H)}, \tag{2.13}$$

where $B_0(H)$ denotes the number of pairs of neighbours that are up and $B_1(H)$ denotes the number of pairs of neighbours that are down when the system is in state H.

3 Product form result

In this section we obtain a product form solution for the steady state probabilities $\pi(H)$. For this result we need to expand our state description. First, in order to indicate for each component h if it is up or down we define the N–dimensional macro–state vector $s = (s_1, \dots, s_N)$ by

$$s_h = \begin{cases} 0, \text{ if component } h \text{ is up} \\ 1, \text{ if component } h \text{ is down} \end{cases} \tag{3.1}$$

However, not all macro–state vectors $s \in \{0,1\}^N$ may correspond to a state H in the earlier defined irreducible set of states $\mathcal{H}$. In order to have that $\mathcal{H} \overset{1-1}{\longleftrightarrow} S$ for the space of macro-state vectors S, we define the function $\phi: \{0,1\}^N \to 2^N$ by

$$\phi(s) = \{h \in \{1,\dots,N\}: s_h = 0\}. \tag{3.2}$$

Thus $\phi(s)$ is the set of components that are down if the system is in macro–state s. Now, defining the macro–state space $S = \left\{s \in \{0,1\}^N: \phi(s) \in \mathcal{H}\right\}$ and defining the N–dimensional vector $t = (t_1, \dots, t_N) \in T = \Re_+^N$ by

$$t_h = \begin{cases} \text{residual amount of work for component } h, \text{ if } s_h = 0 \\ \text{residual amount of repair for component } h, \text{ if } s_h = 1, \end{cases} \tag{3.3}$$

the state of the system is completely determined by a pair $(s,t) \in S \times T$, the micro–state space.

Refering to remark 3.1 for the general case, we assume that the distribution functions F_h and G_h of the random variables W_h and R_h have continuous density functions f_h and g_h respectively. Let $p(s,t)$ denote the steady state density for the micro–states $(s,t) \in S \times T$ and let $\pi(H)$ be the steady state distribution for the states $H \in \mathcal{H}$. First we will proof the (technical) key–theorem, yielding a product form for the steady state density $p(s,t)$. The second, more practical theorem is a direct consequence of the first and provides a simple product form expression for the steady state distribution $\pi(H)$, which is insensitive (i.e. depends only on the distributions F_h and G_h through their means).

Theorem 3.1 *Under conditions (2.3)–(2.5) we have for all $(s,t) \in S \times T$ with c a normalizing constant, $H = \phi(s)$ and $K(H)$ given by (2.5) that*

$$p(s,t) = cK(H) \prod_{h:s_h=0} [1 - F_h(t_h)] \prod_{h:s_h=1} [1 - G_h(t_h)]. \tag{3.4}$$

Proof All we have to do is to verify the global balance or forward Kolmogorov equation (3.9) below, assuming (without loss of generality) that it has a unique solution. But before doing so we will first illustrate how the forward Kolmogorov equation, which requires that in any state (s,t) the "total rate of change" is equal to 0, is derived.

Consider a fixed state (s,t) with $H = \phi(s)$ representing the components that are down and let e_h denote the h–th N–dimensional unit vector. The change in state (s,t) due to the completion of an infinitesimal amount of residual work, Δ, performed by a component h that is up ($s_h = 0$) is given by

$$\{p(s,t) - p(s,t-\Delta e_h)\}\, \beta(h|H) + o(\Delta), \tag{3.5}$$

where $o(\Delta)/\Delta \to 0$ as $\Delta \to 0$. The system makes a transition to state (s,t) from a state $(s+e_h, t+[\tau - t_h]e_h), 0 < \tau < \Delta$, by the completion of the infinitesimal amount of residual repair τ of component h if, after its repair, component h can perform an amount of work t_h. This results in a change

$$\int_0^\Delta p(s+e_h, t+[\tau - t_h]e_h)\delta(h|H+h)f_h(t_h)d\tau + o(\Delta). \tag{3.6}$$

Summing (3.5) and (3.6), deviding by Δ and letting $\Delta \downarrow 0$ yields the rate of change due to a component h that is up ($s_h = 0$) given by

$$\frac{\partial}{\partial t_h} p(s,t)\beta(h|H) + \lim_{\Delta\downarrow 0} p(s+e_h, t+[\Delta - t_h]e_h)\delta(h|H+h)f_h(t_h). \tag{3.7}$$

Analogously, the rate of change due to a component h that is down ($s_h = 1$) is given by

$$\frac{\partial}{\partial t_h} p(s,t)\delta(h|H) + \lim_{\Delta\downarrow 0} p(s-e_h, t+[\Delta - t_h]e_h))\beta(h|H-h)g_h(t_h) \tag{3.8}$$

Summing all these rates of change for components that are up and down and equating this sum to zero yields the forward Kolmogorov equation given by

$$\sum_{h:s_h=0} \left\{ \frac{\partial}{\partial t_h} p(s,t)\beta(h|H) + \lim_{\Delta\downarrow 0} p(s+e_h, t+[\Delta - t_h]e_h)\delta(h|H+h)f_h(t_h) \right\} +$$

$$\sum_{h:s_h=1} \left\{ \frac{\partial}{\partial t_h} p(s,t)\delta(h|H) + \lim_{\Delta\downarrow 0} p(s-e_h, t+[\Delta - t_h]e_h))\beta(h|H-h)g_h(t_h) \right\}$$

$$= 0. \tag{3.9}$$

We will proof that the product form (3.4) satisfies the forward Kolmogorov equation (3.9) by showing that all seperate terms (3.7) and (3.8) are equal to zero. First consider the terms (3.7) for components h that are up. From (2.6) it follows directly that we can restrict ourselves to the case $\beta(h|H) > 0$ and $\delta(h|h+H) > 0$. From the postulated product form solution (3.4), the assumption that the distributions F_h and G_h have continuous densities f_h and g_h and by noting that $\lim_{\Delta\downarrow 0}[1 - F(\Delta)] = 1$ it follows that

$$\frac{\partial}{\partial t_h} p(s,t) = \frac{-f_h(t_h)}{1 - F_h(t_h)} p(s,t) = -f_h(t_h) \lim_{\Delta\downarrow 0} p(s, t+[\Delta - t_h]e_h) \tag{3.10}$$

Using the equivalent (2.6) of the permutation invariant expression (2.5) it follows that

$$\lim_{\Delta\downarrow 0} p(s, t+[\Delta - t_h]e_h)\beta(h|H) = \lim_{\Delta\downarrow 0} p(s+e_h, t+[\Delta - t_h]e_h)\delta(h|H+h). \tag{3.11}$$

Equation (3.10) and (3.11) together imply that the terms (3.7) are equal to 0. In the same way it can be shown that the terms (3.8) are equal to 0, which completes the proof.

Theorem 3.2 *Under conditions (2.3)–(2.5) we have for all $H \in \mathcal{H}$ that*

$$\pi(H) = \bar{c}K(H) \prod_{h\in H} \frac{E[R_h]}{E[W_h]}, \tag{3.12}$$

where $\bar{c}$ is a normalizing constant.

Proof The result follows directly from theorem 4.1 by integrating over all possible residual quantities t_h of work (if $s_h = 0$) or repair (if $s_h = 1$) for all components $h = 1, \ldots, N$, where s is such that $\phi(s) = H$. The change of order of integration and taking products below is justified by the fact that all integration variables occur in exactly one seperate term in the product form (3.4). Thus,

$$\begin{aligned} \pi(H) &= \int_T p(s,t)dt \\ &= cK(H) \prod_{h:s_h=0} \int_0^\infty [1-F_h(t_h)]dt_h \prod_{h:s_h=1} \int_0^\infty [1-G_h(t_h)]dt_h \\ &= cK(H) \prod_{h:s_h=0} E[W_h] \prod_{h:s_h=1} E[R_h] = \bar{c}K(H) \prod_{h\in H} \frac{E[R_h]}{E[W_h]}, \end{aligned} \tag{3.13}$$

where $\bar{c} = c\prod_{h=1}^N E[W_h]$.

Remark 3.1 To avoid technicalities, we gave the proof of theorem 3.1 under the assumption that F_h and G_h have continuous densities f_h and g_h respectively. However, this assumption may be relaxed and in fact the theorem holds for arbitrary distributions F_h and G_h, provided that multiple transitions at the same time are avoided.

4 Examples

In this section we give a number of examples to illustrate how the product form result can be used and how the product form conditions can be easily verified. For each example we show that the speeds β and δ satisfy the product form conditions (2.3) and (2.4) and that there exists a function $K(H)$ such that (2.5), or equivalently (2.6), is satisfied. Once we have this function $K(H)$ the stationary distribution $\pi(H)$ follows directly from (3.12).

4.1 Simple series–parallel configuration

Consider a system consisting of one critical device, C, and M secundary devices, $S_1, \ldots, S_M$ (see figure 2). The system is operational if the critical device and at least k out of the M secundary devices are up. We assume that all failure processes stop if the system is not operational and that there is a single repair unit giving preemptive priority to the critical device. Thus

$$\beta(S_i|H-S_i) = \delta(S_i|H) = 0, \text{ for all states } H \text{ with } C \in H. \tag{4.1}$$

Further we assume that the secundary devices are repaired in a processor sharing manner, i.e.

$$\delta(S_i|H) = \frac{1}{|H|}, \text{ for all states } H \text{ with } C \notin H, \tag{4.2}$$

where $|H|$ denotes the number of secundary devices that is down and that all other speeds are equal to 1. Observe that if $C \in H = \{h_1, \ldots, h_n\}$ then the only permutations $(h_{i_1}, \ldots, h_{i_n})$ of H that yield a non–zero denominator in (2.5) are those with $h_{i_n} = C$, i.e. the critical component is the last one that goes down. From this observation it follows directly that the values $K(H)$ are given by

$$K(H) = \begin{cases} |H|!, \text{ if } C \notin H \\ (|H|-1)!, \text{ if } C \in H \end{cases} \tag{4.3}$$

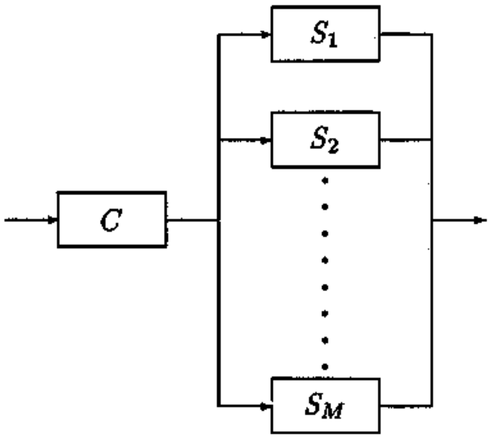

Figure 2. Series–parallel configuration

4.2 Nested structure

Consider a database system consisting of a front–end, two processing subsystems and a database (see figure 3). The system is operational if the front–end and the database are up and at least one of the processing subsystems is operational. Each subsystem consists of a switch, a memory and two processors and is operational if the switch, the memory and at least one of its processors are up. We assume that all failure processes stop if the system is not operational and that the failure processes within a subsystem stop if the subsystem is not operational. Each subsystem has its own repair unit, giving preemptive priority to the memory and the switch. (Note that, within a subsystem, the memory and the switch can not be down at the same time since the failure processes stop if one of these components is down). If the database goes down then *both* repairman are assigned to repair the database immediately. The same holds for the front–end. Thus, the database and the front–end have preemptive priority on system level. The repair units resume the work they were doing as soon as they finish the repair of a component with higher priority, i.e., the priority disciplines are work conserving.

For notational convenience we introduce the function $\mathcal{L}: \{1, \dots, N\} \longrightarrow 2^N$ defined by

$$\mathcal{L}(h) = \{H | \text{the failure process of component } h \text{ is stopped in state } H\}, \tag{4.4}$$

where we use the convention that $h \notin \mathcal{L}(h)$. I.e., $\mathcal{L}(h)$ is the collection of states in which component h does not work due to other failed components. From this definition it follows that

$$\beta(h|H) = 0 \iff H \in \mathcal{L}(h) \tag{4.5}$$

and it is easy to check that the priority rules for repair imply that

$$\delta(h|H + h) = 0 \iff H \in \mathcal{L}(h). \tag{4.6}$$

Hence condition (2.4) is satisfied. Now we have to check the permutation invariance condition (2.5). If the speeds β and δ take the values 0 or 1 only, then this follows directly from the observation that for each permutation for which the denominator in (2.5) is positive, both denominator and numerator are equal to 1. Thus $K(H) = 1$ for all states H.

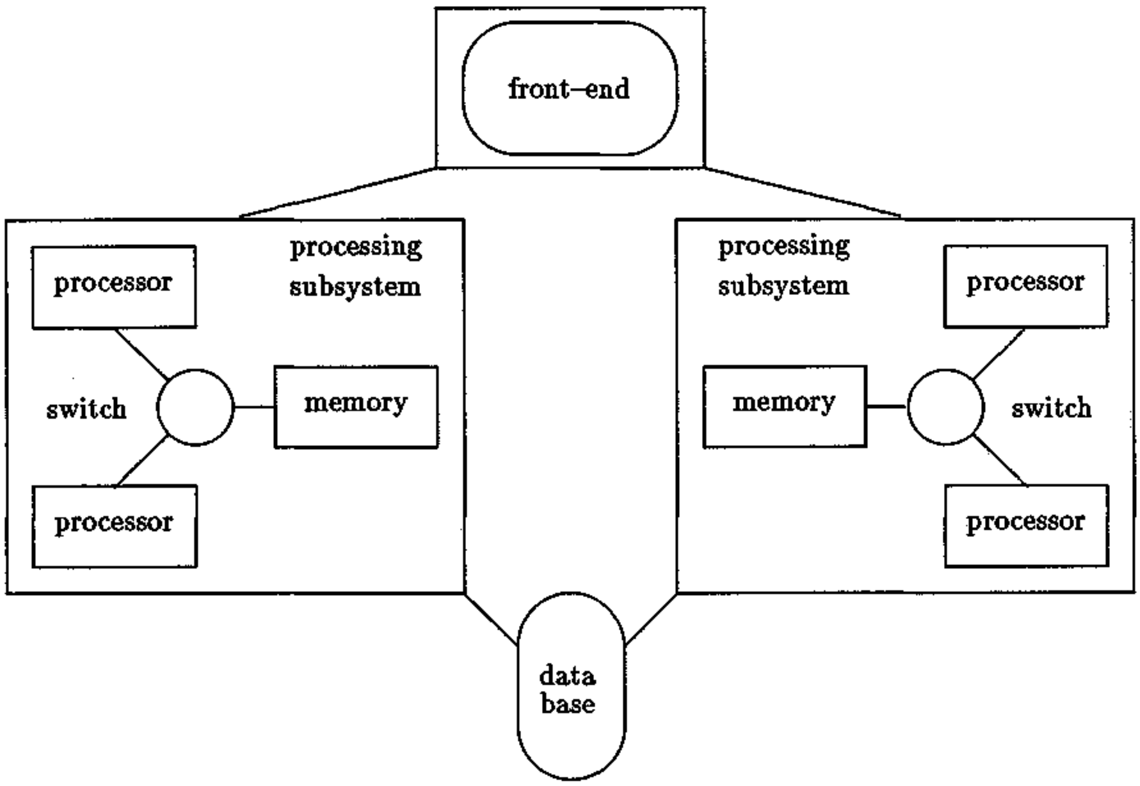

Figure 3. Fault–tolerant database system

Now suppose that the repair speed for a processor is equal to 1 if it is the only processor that is down in its subsystem but that, if the other one goes down too, this speed drops to q for both processors. Further, if in a subsystem both processors are up then they both work at speed 1 and if only one if them is up then it works at speed p. The repair (working) speeds for the processors are of course equal to 0 if the repair of some other component has priority (the failure process has stopped). The speeds of all other components take the values 0 or 1 only. Consider a state $H = \{h_1, \ldots, h_n\}$ of failed components. The only permutations $(h_{i_1}, \ldots, h_{i_n})$ that yield a non–zero denominator in (2.5) are those for which

$$\{h_{i_1}, \ldots, h_{i_{k-1}}\} \notin \mathcal{L}(h_{i_k}),\ k = 1, \ldots, n. \tag{4.7}$$

The only terms in the product in (2.5) that matter for such a permutation are those for which h_{i_k} denotes a processor, all the other terms β/δ being equal to 1. Thus we obtain

$$K(H) = \begin{cases} 1, \text{ if in both subsystems at most one processor is down} \\ \frac{p}{q}, \text{ if in one subsystem both processors are down} \\ (\frac{p}{q})^2, \text{ if all four processors are down,} \end{cases} \tag{4.8}$$

which implies that condition (2.5) is satisfied.

4.3 Multi–stage interconnection network

Consider a multi–stage interconnection network (MIN) connecting K inputs to K outputs. We assume that K is a power of two. When using 2×2 internal switches to build up the

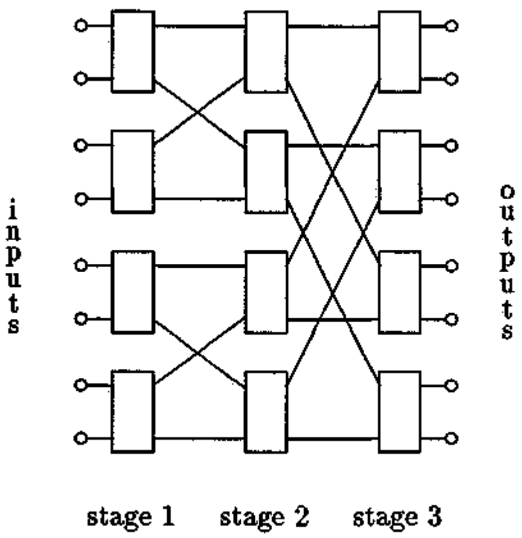

Figure 4. Multi–stage interconnection network

MIN, there are $M = {}^2\log K$ stages required. The number of switches per stage is $N = K/2$. In figure 4 an 8×8 MIN is depicted.

Assume that under normal operation all switches work at speed $\beta = 1$. Switches that are down are repaired in a processor sharing manner. However, there is a critical number, B_i, associated with each stage. As soon as B_i switches of stage i are down the network abandons normal operation. Stage i is now critical and all switches that are up stop working. The repair of all switches that are not in the critical stage is also stopped in favour of the B_i switches that are down in the critical stage i. These are repaired in a processor sharing manner. As soon as one of the switches in the critical stage is repaired the system resumes normal operation again.

Let S_i denote the set of switches in stage i and let $n_i(H) = |H \cap S_i|$ denote the number of switches that is down in stage i if the system is in state H, $i = 1, \ldots, M$. Then for all states H and all switches $h \notin H$ the speeds β are given by

$$\beta(h|H) = \begin{cases} 1, \text{ if } n_i(H) < B_i,\ i = 1, \ldots, M \\ 0, \text{ else.} \end{cases} \tag{4.9}$$

For all states H and all nodes $h \in H$ the repair speeds δ are given by

$$\delta(h|H) = \begin{cases} \frac{1}{|H|}, \text{ if } n_i(H) < B_i,\ i = 1, \ldots, M \\ \frac{1}{B_j}, \text{ if } n_j(H) = B_j \text{ and } h \in S_j \\ 0, \text{ if } n_j(H) = B_j \text{ and } h \notin S_j. \end{cases} \tag{4.10}$$

Thus the speeds satisfy the product form conditions (2.3) and (2.4. From condition (2.6) it follows that the function $K(H)$ is given by

$$K(H) = \begin{cases} |H|!, \text{ if } n_i(H) < B_i,\ i = 1, \ldots, M \\ (|H| - 1)!B_j, \text{ if } n_j(H) = B_j. \end{cases} \tag{4.11}$$

5 Conclusion

In this paper we have characterized a class of availability models that exhibit a product form steady state solution. The conditions that have to be fulfilled for a model to fall in this class (the product form conditions) have been stated and examples are given that show how these conditions can be verified.

Further research is conducted on using the product form results to obtain bounds for non product form models. This can be done by making proper adjustments in the speed functions β and δ. Another topic for research is to make the existing algorithms for computing network reliability applicable to the product form class with dependencies. Thus with computations based on (3.12) rather than on (1.3).

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