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THE AREA ENCLOSED BY THE (ORIENTED)
NYQUIST DIAGRAM AND THE
HILBERT-SCHMIDT-HANKEL NORM
OF A LINEAR SYSTEM

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THE AREA ENCLOSED BY THE (ORIENTED) NYQUIST DIAGRAM AND THE HILBERT-SCHMIDT-HANKEL NORM OF A LINEAR SYSTEM.

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Abstract In this note it is shown that the Hilbert-Schmidt-Hankel norm (HSH-norm) of a transfer function of a stable system is equal, up to a constant factor, to the square root of the area enclosed by the oriented Nyquist diagram of the transfer function (multiplicities included). A generalization is presented for the case of systems which have no poles on the stability boundary, but otherwise have no restrictions on the pole locations.

Keywords Nyquist diagram * linear systems * Hilbert-Schmidt-Hankel norm * families of linear systems.

1 INTRODUCTION

In problem areas like model reduction, robust control, system identification and system parametrization, where families of linear systems play an important role, several norms are in use, which make it possible to tell whether two systems are close or far apart. For stable finite dimensional systems we would like to mention the H^{∞}-norm, the Hankel-norm and the H^2-norm. For certain applications it is a disadvantage of the H^{∞}-norm and the Hankel-norm that there is no inner product associated with it, like in the case of the H^2-norm. For families of stable systems with a given finite order and without direct feedthrough term, the H^{∞}-norm and the Hankel-norm are equivalent, but the H^2-norm is not, it is a topologically different norm. A norm that does have a corresponding inner product and is equivalent to the H^{∞}-norm and the Hankel-norm on such a family is the Hilbert-Schmidt-Hankel norm (HSH-norm). It is defined as the Hilbert-Schmidt norm of the

Hankel operator that is associated with the system. It is well-known (see e.g. [5]) that the square of this norm is equal to the sum of squares of the Hankel singular values of the system.

The H^∞-norm and the H^2-norm of a system have a direct interpretation in terms of its transfer function restricted to the stability boundary, i.e. to the unit circle in the discrete time case and to the imaginary axis in the continuous time case. In fact the H^∞-norm only depends on the image of the transfer function on the stability boundary: the Nyquist diagram. The purpose of this note is to give an interpretation of the HSH-norm in terms of the Nyquist diagram. Our main result is that π times the square of the HSH-norm is equal to the area enclosed by the oriented Nyquist diagram, multiplicities included. This will be established in section 2. In section 3 a generalization to systems with anti-stable part will be presented. Several implications of these results will also be treated.

2 THE HILBERT-SCHMIDT-HANKEL NORM AND THE NYQUIST DIAGRAM

Consider single input single output (SISO) linear dynamical systems with a state space representation

$$(\Sigma^d) \qquad \begin{aligned} x_{t+1} &= Ax_t + bu_t, \quad t\in\mathbb{Z},\ x_t\in\mathbb{R}^n,\ u_t\in\mathbb{R} \\ y_t &= cx_t + du_t,\ y_t\in\mathbb{R} \end{aligned} \qquad (2.1)$$

in the discrete time case or

$$(\Sigma^c) \qquad \begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \quad t\in\mathbb{R},\ x(t)\in\mathbb{R}^n,\ u(t)\in\mathbb{R} \\ y(t) &= cx(t) + du(t), \quad y(t)\in\mathbb{R}, \end{aligned} \qquad (2.2)$$

in the continuous time case. Clearly there is one input and one output and the state space dimension is n; A, b, c are real matrices and vectors of the sizes n×n, n×1, 1×n, respectively and d is scalar. Two state space representations will be considered to be input-output equivalent ("i/o-equivalent") if they have the same input-output behaviour. Without loss of generality it will be assumed that (A,b,c,d) is a minimal representation. It is well-known that two such quadruples (A,b,c,d), $(\tilde{A},\tilde{b},\tilde{c},\tilde{d})$ are i/o-equivalent if they have the same transfer function

$$T(z) = c(zI-A)^{-1}b + d = \tilde{c}(zI-\tilde{A})^{-1}\tilde{b} + \tilde{d}$$

from which $d=\tilde{d}$ follows immediately.

In this section we will consider systems which have no direct feed-through term, i.e. $d=0$, and which are i/o-stable (also called "asymptotically stable"), i.e. the spectrum $\sigma(A)$ of the matrix A lies:

(i) in the open unit disk (discrete time case)

(ii) in the open left half plane (continuous time case).

Let $h_k=cA^{k-1}b$, $k=1,2,\ldots$ denote the Markov parameters of the system. The Hankel operator of a system is given by:

(i) the Hankel matrix

$$\mathcal{H}=\begin{bmatrix} h_1 & h_2 & h_3\cdots \\ h_2 & h_3 & h_4\cdots \\ h_3 & h_4 & h_5\cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \tag{2.3}$$

in the discrete time case,

(ii) the Hankel integral operator $\mathcal{H}$ with kernel

$$k(t,s)=c\ e^{A(t+s)}b \tag{2.4}$$

in the continuous time case.

The set of all finite dimensional i/o-stable systems can be considered to be a linear vector space with the following definitions of scalar multiplication and addition:

If Σ_1 is represented by (A_1,b_1,c_1) and Σ_2 by (A_2,b_2,c_2), then $\lambda\Sigma_1$, $\lambda\in\mathbb{R}$, is the system which is represented by (e.g) $(A_1,\lambda b_1,c_1)$ and $\Sigma_1+\Sigma_2$ is the system which is represented (possibly non-minimally) by

$$\left[\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},\begin{pmatrix} b_1 \\ b_2 \end{pmatrix},(c_1,c_2)\right].$$

Equivalently one can express this with the transfer functions or the Hankel operators:

If T_1 resp. T_2 are the transfer functions of Σ_1 resp. Σ_2, then $\lambda\Sigma_1$ has the transfer function λT_1 and $\Sigma_1+\Sigma_2$ has the transfer function T_1+T_2; if Σ_1 resp. Σ_2 has the Hankel operator H_1 resp. H_2, then $\lambda\Sigma_1$, $\lambda\in\mathbb{R}$, has the Hankel operator λH_1 and $\Sigma_1+\Sigma_2$ has the Hankel operator H_1+H_2.

The Hilbert-Schmidt norm of the Hankel operator can be considered as a norm on the linear space of i/o-stable systems. In the discrete time case it is given by

$$\| \Sigma \|^2_{HSH} = \mathrm{tr}\ HH^T = \sum_{k=1}^{\infty} k\ h_k^2 . \tag{2.5}$$

In the continuous time case it is given by:

$$\| \Sigma \|^2_{HSH} = \int_{\sigma=0}^{\infty} \int_{\tau=0}^{\infty} k(\tau,\sigma)^2\ d\tau\ d\sigma . \tag{2.6}$$

It is well-known that both in the discrete time case and the continuous time case the square of the Hilbert-Schmidt norm is equal to the sum of squares of the singular values of the Hankel operator, which provides the means to calculate the HSH-norm explicitly as:

$$\| \Sigma \|^2_{HSH} = \mathrm{tr}\ PQ \tag{2.7}$$

where P and Q are the controllability Grammian, resp. the observability Grammian of the system. The positive definite symmetric matrices P and Q can be obtained as the unique solution of the Lyapunov equation

$$P - A\ P\ A^T = b\ b^T \tag{2.8a}$$

resp.

$$Q - A^T\ Q\ A = c^T\ c \tag{2.8b}$$

in the discrete time case, and

$$A\ P + P\ A^T = -b\ b^T \tag{2.9a}$$

resp.

$$Q\ A + A^T\ Q = -c^T\ c \tag{2.9b}$$

in the continuous time case.

The HSH-norms in the discrete time case and in the continuous time case can be related to one another by the following well-known bijective transformation (cf. [1], p.1119-1120):

$$A = (I+\tilde{A})^{-1} (\tilde{A}-I) \tag{2.10a}$$

$$b = \sqrt{2}\ (I+\tilde{A})^{-1} \tilde{b} \tag{2.10b}$$

$$c = \sqrt{2}\ \tilde{c}\ (I+\tilde{A})^{-1} \tag{2.10c}$$

where $(\tilde{A},\tilde{b},\tilde{c})$ represents an i/o-stable discrete time system $\tilde{\Sigma}$ and (A,b,c) a corresponding i/o-stable continuous time system Σ. This transformation leaves the controllability Grammians invariant and therefore also the HSH-norm. Furthermore it is a <u>linear</u> mapping from the space of discrete time, i/o-stable, strictly proper linear systems to the linear space of continuous time, i/o-stable, strictly proper systems. This can be seen by considering the same transformation in terms of the corresponding transfer functions:

$$T(s) = \tilde{T}\left(\frac{1+s}{1-s}\right) - \tilde{T}(-1) \tag{2.11}$$

where T is the transfer function of Σ and $\tilde{T}$ the transfer function of $\tilde{\Sigma}$. Note that

$$T(\infty) = \tilde{T}(-1) - \tilde{T}(-1) = 0 \tag{2.12}$$

and therefore T is strictly proper indeed. Our main result will be a characterization of the HSH-norm in terms of the associated oriented Nyquist diagram of the system, i.e. the image that the transfer function produces of the unit circle (with anti-clockwise orientation) in the discrete-time case, and of the imaginary axis (with the orientation that is obtained by going from $-i\infty$ to 0 to $+i\infty$) in the continuous time case.

By the area enclosed by such a closed oriented curve γ in the complex plane will be meant the following. For each point $z_0 \in \mathbb{C}\setminus\gamma$, let

$$n_\gamma(z_0) = \frac{1}{2\pi} \operatorname{Im} \oint_\gamma \frac{1}{z-z_0}\, dz \in \mathbb{Z} \tag{2.13}$$

denote the winding number of γ with respect to z_0, i.e. the number of times γ winds around z_0 in the anti-clockwise direction minus the number of times γ winds around z_0 in the clockwise direction. For an n-th order transfer function it is well-known that

$$|n_\gamma(z_0)| \le n \quad \text{for all} \quad z_0 \in \mathbb{C} \ . \tag{2.14}$$

The area enclosed by such an oriented closed curve, multiplicities included, is usually defined as the integral

$$A_a(\gamma) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n_\gamma \ (x+iy) \ dx \ dy \ . \tag{2.15}$$

Of course it follows that if γ winds in clockwise direction $A_a(\gamma)$ will be negative. To avoid confusion we will therefore define $A_c(\gamma)$ to be the area enclosed by γ with respect to clockwise orientation, i.e.

$$A_c(\gamma) = -A_a(\gamma). \tag{2.16}$$

Because the curve is closed, $n_\gamma(z)$ vanishes outside some closed bounded set and therefore the integral is well-defined.
We can now state our main result:

2.1 Theorem

The HSH-norm squared of an i/o-stable, strictly proper system Σ is equal to π^{-1} times the area enclosed by its oriented Nyquist diagram:

$$\| \Sigma \|^2_{HSH} = \frac{1}{\pi} A_c(\gamma) \tag{2.17}$$

where

(i) $\gamma(\theta) = T(e^{i\theta})$, $\theta \in [0,2\pi)$ running from 0 to 2π, in the discrete time case,

or

(ii) $\gamma(\omega) = T(i\omega)$, $\omega \in \mathbb{R}$ running from $-\infty$ to $+\infty$ in the continuous time case.

See Figure 1 (p.15)

Proof It will suffice to show the result in the discrete time case, because the bijective transformation presented in (2.10a, b, c) leaves the HSH-norm invariant and, as can be seen from (2.11), shifts the oriented Nyquist diagram, which implies that the area enclosed by the oriented Nyquist diagram remains invariant as well.

In the discrete time case the HSH-norm squared of an i/o-stable strictly proper system Σ with Markov parameters $\{h_k\}_{k=1}^{\infty}$ can be written as

$$\| \Sigma \|^2_{HSH} = \sum_{k=1}^{\infty} k h_k^2 = \frac{1}{2\pi i} \oint_{|z|=1} \left(\sum_{k=1}^{\infty} h_k z^k \right) \left(\sum_{k=1}^{\infty} k h_k z^{-k} \right) \frac{dz}{z} =$$

$$= \frac{1}{2\pi i} \oint_{|z|=1} \left\{ T(z^{-1}) \right\} \left\{ -z \frac{dT(z)}{dz} \right\} \frac{dz}{z} = - \frac{1}{2\pi i} \oint_{|z|=1} \overline{T(z)} \frac{dT(z)}{dz} =$$

$$= \frac{-1}{2\pi i} \oint_{|z|=1} \overline{T(z)}\, dT(z) = \frac{-1}{2\pi i} \int_{T \in \gamma} \overline{T}\, dT,$$

where γ now denotes the oriented Nyquist diagram, i.e.

$$\gamma = \left\{ T(e^{i\theta}) \mid \theta: 0 \to 2\pi \right\} .$$

Consider

$$\frac{1}{2i} \int_{T \in \gamma} \overline{T}\, dT = \frac{1}{2i} \int_{\substack{T = x + iy \in \gamma \\ x, y \in \mathbb{R}}} (x - iy)\, d(x + iy) = \frac{1}{2i} \int_{T = x + iy \in \gamma} (xdx + ydy) +$$

$$\frac{1}{2i} \int_{T = x + iy \in \gamma} i(xdy - ydx) \qquad (2.18)$$

The first term on the right hand side is zero; one has

$$\int xdx + ydy = \frac{1}{2} x^2 + \frac{1}{2} y^2 + \text{const} \qquad (2.19)$$

and therefore this integral over the closed curve γ vanishes. The second term on the right hand side of (2.18) is equal to

$$\frac{1}{2} \int_{T=x+iy\in\gamma} xdy - ydx = A_a(\gamma) \tag{2.20}$$

according to Green's theorem. Therefore

$$\| \Sigma \|^2_{HSH} = \frac{1}{\pi} A_c(\gamma), \tag{2.21}$$

where γ is the oriented Nyquist diagram . Q.E.D.

2.2 Remarks

(i) For an n-th order system Σ with i/o-stable strictly proper transfer function T(z), such that T(z)+d is all-pass for some suitable choice of $d \in \mathbb{R}$, i.e. $|T(z)+d|=1$ for all z in the stability boundary, it follows that

$$\| \Sigma \|^2_{HSH} = \frac{1}{\pi} A_c(\gamma) = n \tag{2.22}$$

because the oriented Nyquist curve γ of an i/o all-pass transfer function winds n times around the unit circle in clockwise direction. (The fact that such an all-pass function is usually not strictly proper does not matter here, because adding or substracting a constant term to a transfer function means only a translation along the real axis of the oriented Nyquist diagram, which clearly does not affect $A_c(\gamma)$, nor does it affect the HSH-norm).

As it is well-known that the n positive Hankel singular values σ_k of such a system are all one,

$$\| \Sigma \|^2_{HSH} = \sum_{k=1}^{n} \sigma_k^2 = n \tag{2.23}$$

and the two ways of calculating the HSH-norm squared indeed give the same result.

(ii) It follows immediately from this result that, for a system Σ of order n,

$$\| \Sigma \|^2_{HSH} \le n \, \| \Sigma \|^2_{\infty} , \tag{2.24}$$

where $\|\Sigma\|_{\infty}$ denotes the H^{∞}-norm of Σ, i.e.

$$\| \Sigma \|_{\infty} = \max_{|z|=1} |T(z)|$$

in the discrete time case and

$$\| \Sigma \|_{\infty} = \max_{Im(z)=0} |T(z)|$$

in the continuous time case,

because $|n_{\gamma}(z)| \le n$ for all $z \in \mathbb{C}$, and

$n_{\gamma}(z)=0$ for all $z \in \mathbb{C}$ with $|z| > \|\Sigma\|_{\infty}$.

See Figure 2 (p.15)

(iii) Consider the group of all automorphisms φ of the unit disk (in the discrete time case) that map the reals to the reals. It is well-known (cf. [4]) that such an automorphism φ can be written as:

$$\varphi(z) = \pm \frac{az+b}{bz+a} , \qquad a,b \in \mathbb{R}, \quad a^2 - b^2 = 1 \tag{2.25}$$

Because φ maps the unit circle to the unit circle, it is clear that the system with strictly proper transfer function $T \circ \varphi(z) - T \circ \varphi(\infty)$ has the same HSH-norm as the system with transfer function T. In other words, the mapping of systems given in terms of their transfer functions, by

$$T \longmapsto T \circ \varphi - T \circ \varphi(\infty)$$

is an isometry with respect to the HSH-norm, and the set of these isometries forms a group (of course). It is easy to see that these isometries leave the McMillan degree invariant. This group of isometries was found before, in a completely different way in [4], see also [2], [3].

A similar group of isometries exists for the continuous time case. It corresponds to the automorphisms of the left half plane.

(iv) It follows from the theorem, that if for a discrete time system Σ_1 with transfer function T(z) we denote by Σ_k the system with transfer function $T(z^k)$, k=1,2,..., then

$$\|\Sigma_k\|^2_{HSH} = k\ \|\Sigma_1\|^2_{HSH}\ . \tag{2.26}$$

This can be concluded from the fact that if $z=e^{i\theta}$ runs once through to unit circle (θ: $0 \to 2\pi$), then z^k runs k times through the unit circle ($k\theta$: $0 \to 2k\pi$) and therefore the area enclosed by the oriented Nyquist diagram of Σ_k, multiplicities included, will be k times the area enclosed by the oriented Nyquist diagram of Σ_1. More generally, consider an all-pass (rational) function b(z) which leaves the unit disk invariant. Then clearly all the zeros of b(z) must lie in the open unit disk and therefore all its poles must lie outside the unit circle. (It is easy to see that this is also a sufficient condition.) If b(z) is of order k, then the i/o-stable system Σ_b with transfer function T(b(z))-T(b(∞)) has HSH-norm given by

$$\|\ \Sigma_b\|^2_{HSH} = k\ \|\ \Sigma_1\|^2_{HSH}\ . \tag{2.27}$$

(v) Of course the Nyquist diagram relates the (i/o-stable) system to its behaviour under output feedback. If one uses output feedback $u = \frac{y}{\kappa} + v$ then the closed loop system will have a number of unstable poles equal to the winding number of the Nyquist diagram w.r.t. the point κ. If we allow for complex κ then the Lebesgue measure of the set of κ's in $\mathbb{C}$ which lead to an unstable closed loop system, where the κ's are counted with <u>multiplicity equal to the number of unstable poles</u> in the corresponding closed-loop system, is equal to π times the HSH-norm squared. So, loosely speaking, in this interpretation the HSH-norm is a measure of how easy the system can be destabilized under complex output feedback (multiplicities of the unstable poles included). This is "easy" if the HSH-norm is large.

Using (2.27) we will now give an alternative proof of the known fact that the minimum phase factors in spectral factorization are also minimum HSH-norm factors. The proof that is given here does not involve Nehari's theorem. (More generally one can show that minimum phase factors have the smallest singular values; this can be shown using Nehari's theorem, but we will not go into that here.)

2.3 Corollary

Let $\psi(z)$ be a rational spectral density function bounded on the unit circle and consider spectral factorization

$$\psi(z) = T(z)^* \, T(z) \tag{2.28}$$

where the "root" T(z) is the transfer function of an i/o-stable strictly proper discrete time system. Of all such roots the minimum phase roots $\pm T_m(z)$ have minimal HSH-norm.

Remark

A similar result holds in the continuous time case; it can be derived directly from the result in the discrete-time case using the usual transformation (see (2.11)).

Proof of 2.3 First note that if Σ has transfer function T and $\tilde{\Sigma}$ has transfer function $\frac{1}{z}$ T(z), then

$$\| \tilde{\Sigma} \|^2_{HSH} = \| \Sigma \|^2_2 + \| \Sigma \|^2_{HSH} \tag{2.29}$$

(Warning: this holds only for the discrete-time case.)

This follows easily from formula (2.5), from the fact that $T(z)=\sum_{k=1}^{\infty} h_k z^{-k}$ and from the formula $\|\Sigma\|_2^2 = \sum_{k=1}^{\infty} h_k^2$.

Now consider the all-pass first order factor

$$z = b(s) = \frac{cs+d}{\bar{d}s+\bar{c}}, \quad c,d\in\mathbb{C}, \quad |c|^2-|d|^2=1,$$

which maps the unit disk to the unit disk and therefore has its <u>zero</u> $(\frac{-d}{c})$ inside the unit disk. Then

$$s = b^{-1}(z) = \frac{\bar{c}z-d}{-\bar{d}z+c}$$

and this is also an element of the group of automorphisms of the unit disk. Now consider, in an obvious notation:

$$\left\| \frac{1}{b(s)} T(s) \right\|^2_{HSH} = \left\| \frac{1}{z} T(b^{-1}(z)) \right\|^2_{HSH}$$

according to (2.27) with k=1, applied with $s=b^{-1}(z)$ instead of b. Note that $\frac{1}{z} T(b^{-1}(z))$ is strictly proper. Applying (2.29) one obtains

$$\left\|\frac{1}{z} T(b^{-1}(z))\right\|^2_{HSH} = \left\|T(b^{-1}(z))\right\|^2_2 + \left\|T(b^{-1}(z))-T(b^{-1}(\infty))\right\|^2_{HSH} \geq$$

$$\geq \left\|T(b^{-1}(z)) - T(b^{-1}(\infty))\right\|^2_{HSH} = \left\|T(s)\right\|^2_{HSH} ,$$

again due to (2.27) with k=1, now applied with z=b(s). So we conclude that

$$\left\| \frac{1}{b(s)} T(s)\right\|^2_{HSH} \geq \left\|T(s)\right\|^2_{HSH}$$

and the inequality is strict if $T\not\equiv 0$.

Now if the zero $s = \frac{-d}{c}$ happens to coincide with a zero of the transfer function T(s), then $\frac{1}{b(s)} T(s)$ is the transfer function that is obtained by replacing the factor $cs+d = c(s-(-d/c))$ by $\bar{d}s+\bar{c} = \bar{d}\left(s-(-\frac{\bar{c}}{\bar{d}})\right)$. Of course $-\overline{(\frac{c}{d})}$ is the (usual) reflection of $(-\frac{d}{c})$ w.r.t. the unit circle. By repeating this argument for each unstable zero in the transfer functions, it follows that the spectral factor with all its zeroes in the open unit disk has smaller HSH-norm than a spectral factor with one or more zeroes outside the unit circle. Q.E.D.

3 GENERALIZATION TO UNSTABLE SYSTEMS

Let T(z) be a transfer function which has no poles on the stability boundary. Let

$$T(z) = T_1(z) + T_2(z) + d \tag{3.1}$$

be the additive decomposition in the strictly proper stable and the strictly proper anti-stable part of the transfer function and a constant term d. Then $T_1(z)$ and $T_2(z^{-1})$ (in the discrete time case) resp. $T_2(-z)$ (in the continuous time case) are strictly proper transfer functions of stable systems. In the discrete time case let $\{h_k^{(1)}\}_{k=1}^{\infty}$ denote the impulse response of $T_1(z)$ and $\{+h_k^{(2)}\}_{k=1}^{\infty}$ the impulse response of $T_2(1/z)$.

Define the following impulse response function:

$$\tilde{h}_k = \begin{cases} h_k^{(1)}, & k>0 \\ d, & k=0 \\ h_{-k}^{(2)}, & k<0 \end{cases} . \tag{3.2}$$

Analogously, in the continuous time case, define the impulse response function

$$\tilde{h}(t) = \begin{cases} h^{(1)}(t), & t>0 \\ \text{arbitrary}, & t=0 \\ h^{(2)}(t), & t<0 \end{cases} . \tag{3.3}$$

Then the following generalization of theorem 2.1 holds:

$$\frac{1}{\pi} A_c(\gamma) = \sum_{k\in\mathbb{Z}} k\, \tilde{h}_k^2 = \|\Sigma_1\|^2_{HSH} - \|\Sigma_2\|^2_{HSH} \tag{3.4}$$

in the discrete time case, where Σ_1 and Σ_2 are the stable systems that correspond to $T_1(z)$ resp. $T_2(\frac{1}{z})$;

$$\frac{1}{\pi} A_c(\gamma) = \int_{t\in\mathbb{R}} t\, \tilde{h}^2(t) = \|\Sigma_1\|^2_{HSH} - \|\Sigma_2\|^2_{HSH} \tag{3.5}$$

in the continuous time case, where Σ_1 and Σ_2 are the stable systems that correspond to $T_1(z)$ resp. $T_2(-z)$.

Proof Completely analogous to the proof of theorem (2.1).

Remarks

For an all-pass function with k stable and n-k unstable poles it follows that

$$\|\Sigma_1\|^2_{HSH} - \|\Sigma_2\|^2_{HSH} = \frac{1}{\pi} A_c(\gamma) = n-2k , \tag{3.6}$$

because the total increase in argument of the all-pass function along the unit circle is $-2\pi k$ (due to the stable poles) plus $2\pi(n-k)$ (due to the stable zeroes), which makes $2\pi(n-2k)$. Therefore the total area enclosed by the Nyquist diagram, multiplicities included, is $\pi(n-2k)$ and (3.6) follows.

This result can also be shown using the techniques of [1], by applying Theorem 5.1 of that paper to the sum of the balanced state space representations of T_1 and T_2. However, the proof given here is shorter and it gives more geometrical insight.

References

[1] Glover, K., "All optimal Hankel-norm approximations and their L^∞-error bounds", Int. J. Control 39, pp.1115-1193.

[2] Hanzon, B., "A geometric approach to system identification using model-reduction techniques", pp.695-700 in: H-F Chen (ed.), Proceedings of 8th IFAC/IFORS Symposium, Beijing, China, August 1988, Pergamon Press (1988).

[3] Hanzon, B., "Riemannian geometry on families of linear systems, the deterministic case", Report 88-62, Fac. Tech. Math. and Inf., Delft University of Technology, Delft (1988), submitted to MCSS.

[4] Hanzon, B., "Identifiability, Recursive Identification and Spaces of Linear Dynamical Systems", CWI Tracts 63 and 64, Centre for Mathematics and Computer Science, Amsterdam, 1989.

[5] Partington, J.R., "An Introduction to Hankel Operators", London Math. Soc. Student Texts 13, Cambridge Univ. Press, Cambridge, 1988.

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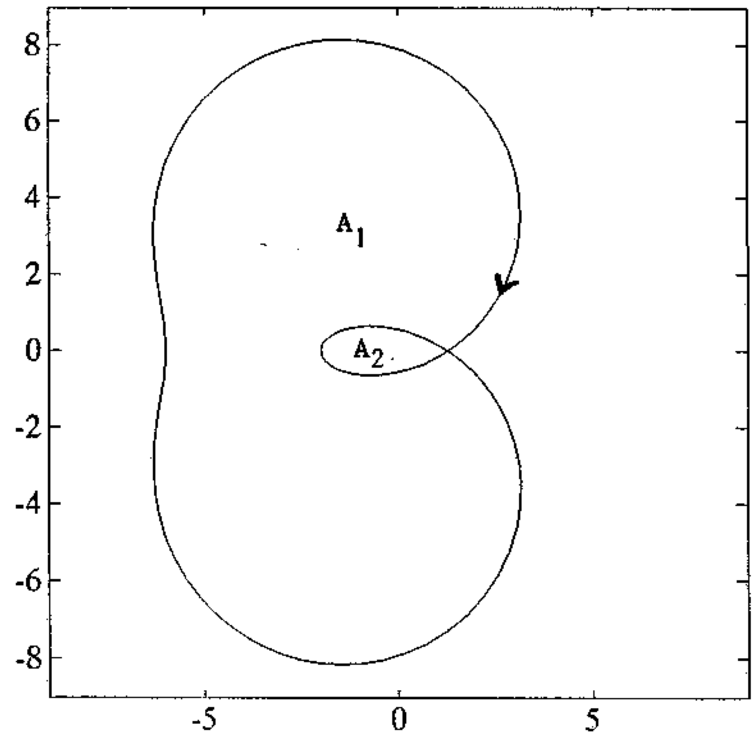

Figure 1 $A(\gamma) = A_1 + 2A_2 = \pi\|\Sigma\|^2_{HSH}$

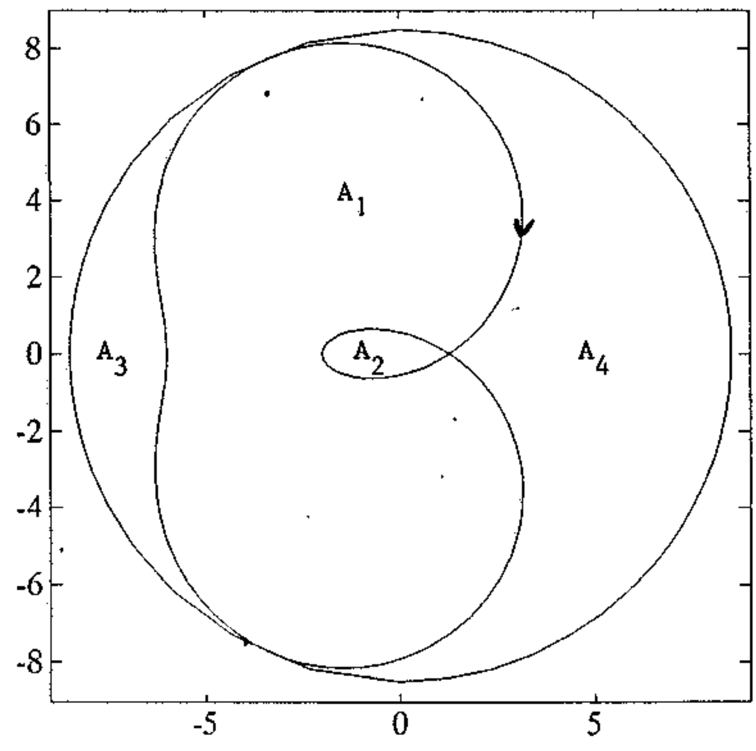

Figure 2 $n=2;\ \pi\|\Sigma\|^2_{HSH} = A_1 + 2A_2 \le 2(A_1+A_2+A_3+A_4) = 2\pi\|\Sigma\|^2_\infty$

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