

1990-41

ET

05348

SERIE RESEARCH MEMORANDA

TRANSIENT PRODUCT FORM DISTRIBUTIONS IN
QUEUEING NETWORKS

by

Richard J. Boucherie
Peter G. Taylor

Research Memorandum 1990-41
August 1990



vrije Universiteit *amsterdam*

Faculteit der Economische Wetenschappen en Econometrie
A M S T E R D A M



Transient product form distributions in queueing networks

Richard J. Boucherie* and Peter G. Taylor†

July 31, 1990

Abstract

In this paper it is shown that a necessary and sufficient condition for a Markovian queueing network to have transient product form is that all queues are infinite server queues.

Keywords: Queueing network, Transient behaviour, Product form, Infinite server queues, Necessary condition, Canonical product form.

1 Introduction

There is a very large literature which studies product forms in queueing networks (see, for example [7], [13], [14], and the references therein). However, most of these authors consider equilibrium behaviour only. For the transient behaviour of queueing networks there seem to be very few analytical results although the transient behaviour of single queues has been studied extensively (see, for example [9]). Here, the time dependent probability distribution is expanded in terms of the eigenvalues and eigenvectors of the transition matrix. For networks of queues, however, this method seems to be of little value. In this case the number of eigenvectors and eigenvalues would be almost impossible to handle. Therefore, it seems of great interest to give closed form (e.g. product form) expressions for the transient probability distribution.

The transient behaviour of open networks of $M/G/\infty$ queues with homogeneous Poisson input is studied in [8]. In [4] these results are extended to a tandem network of $M/G/\infty$ queues with non-homogeneous Poisson input. [5] studies the transient behaviour of closed queueing networks of $M/G/\infty$ queues and open networks of $M/G/\infty$ queues with non-homogeneous Poisson input. It is shown in these references that a network of $M/G/\infty$ queues has a transient product form distribution. Furthermore, it is shown that this distribution converges for open queueing networks

*Department of Econometrics, Free University, Amsterdam, The Netherlands

†Department of Mathematics, University of Western Australia, Nedlands, Western Australia, 6009

with a homogeneous Poisson input process and for closed queueing networks to the well known product form equilibrium distribution. The method used in these references to derive the transient product form distribution is an independence argument. It relies on the fact that the sample paths for individual jobs in a queueing network consisting of $M/G/\infty$ queues only are independent and therefore the transient probability distribution has product form. It is assumed in these references that the network is initially empty in the open network case and that all jobs start at the same queue in the closed network case. Up to now, however, networks consisting only of $M/G/\infty$ queues seem to be the only networks studied in great detail.

Based on the experience with equilibrium product form distributions one would expect that networks in which non $M/G/\infty$ queues appear may also possess a product form transient distribution. However, in this paper we show that for a Markovian network to have a transient product form distribution all queues must be $M/M/\infty$ queues. This states the main result of this paper:

- For a network to have product form transient distribution it is necessary that all queues are infinite server queues.

In the remaining part of the introduction we first describe the model. Then we introduce canonical forms for the transient product form distribution. In section 2 we give sufficient conditions for the transient distribution to be of product form. In sections 3 and 4 respectively we use the canonical form to give necessary conditions for an open resp. closed queueing network to have product form. It then turns out to be the case that it is necessary and sufficient for the network to have a transient product form distribution that all queues be infinite server queues.

Consider a continuous-time queueing network consisting of N stations, labelled $1, 2, \dots, N$ in which one type of job can move between the stations. A state of the queueing network is a vector $\bar{n} = (n_1, \dots, n_N)$, where n_i denotes the number of jobs at queue i , $i = 1, \dots, N$. Assume that the queueing network can be represented by a continuous-time Markov chain with state space $V \subset N_0^N = \{0, 1, 2, \dots\}^N$. The transition rate from state \bar{n} to state \bar{n}' is denoted by $q(\bar{n}, \bar{n}')$.

Assume that the transition rates are of the form

$$q(\bar{n}, \bar{n} - e_i + e_j) = p_{ij} \frac{\psi(\bar{n} - e_i)}{\phi(\bar{n})} \quad i, j = 0, \dots, N, \quad \bar{n} \in V, \quad (1.1)$$

where p_{ij} denotes the probability that a job leaving queue i routes to queue j , $i, j = 1, \dots, N$, p_{0j} denotes the probability that an entering job routes to queue j , $j = 1, \dots, N$ and p_{i0} denotes the probability that a job leaving queue i leaves the network, $i = 1, \dots, N$ and where e_i denotes the i th unit vector, $i = 1, \dots, N$, $e_0 = 0$, the vector consisting of zeros only and $\psi(\cdot)$, $\phi(\cdot)$ are arbitrary functions such that $\phi(\cdot) > 0$, $\psi(\cdot) \geq 0$, and $\psi(\bar{m}) = 0$ if $m_i < 0$ for some i . It is known (cf. [1], [6], [12]) that for transition rates of the form stated in (1.1) a necessary and sufficient condition for the process to have product form equilibrium distribution

$$\pi(\bar{n}) = B\phi(\bar{n}) \prod_{k=1}^N c_k^{n_k} \quad \bar{n} \in V \quad (1.2)$$

is that the coefficients $\{c_k\}_{k=1}^N$ satisfy the traffic equations

$$c_j = \sum_{i=1}^N c_i p_{ij} + p_{0j} \quad j = 1, \dots, N. \quad (1.3)$$

The form (1.3) for the traffic equations varies slightly from that given by many authors. The usual form for the traffic equations has p_{0j} replaced by λp_{0j} with λ the parameter of a Poisson arrival process. The fact that p_{0j} appears here is a consequence of the fact that we have written all transitions (including arrivals) in the form (1.1). This point is discussed in more detail in section 1.1. Also, note that the function $\psi(\cdot)$ does not appear in the equilibrium distribution. This is a direct consequence of the form of the global balance equations.

In this paper we will show that a necessary and sufficient condition for the time-dependent distribution $P(\bar{n}, t)$ of a queueing network with homogeneous transition rates given in (1.1) and initial conditions

$$P(\bar{n}, 0) = B_0\phi(\bar{n}) \prod_{k=1}^N \xi_k^{n_k}$$

to be of product form, defined by

$$P(\bar{n}, t) = B(t)\phi(\bar{n}) \prod_{k=1}^N c_k(t)^{n_k} \quad \bar{n} \in V \quad (1.4)$$

is that the network consist of $M/M/\infty$ queues only and that the coefficients $\{c_k(t)\}_{k=1}^N$ satisfy the time-dependent version of the traffic equations

$$\frac{1}{\mu_k} \frac{dc_k(t)}{dt} = \sum_{i=1}^N \{c_i(t)p_{ik} - c_k(t)p_{ki}\} + p_{0k} - c_k p_{k0} \quad k = 1, \dots, N \quad (1.5)$$

with initial conditions

$$c_k(0) = \xi_k,$$

where for a closed queueing network we set $p_{0k} = p_{k0} = 0$, $k = 1, \dots, N$ in the relations above and where $\{\mu_k\}_{k=1}^N$ is a set of coefficients defined in terms of the transition rates between specified states.

Note that, for a given physical model, there is freedom in setting the parameters in the description above. This can be immediately seen by observing (1.1), where we are allowed to multiply both $\phi(\cdot)$ and $\psi(\cdot)$ by an arbitrary constant without affecting the transition rates, or by observing (1.5) for a closed queueing network where we are allowed to multiply the $c_k(t)$ by an arbitrary constant. Therefore, since we wish to give necessary and sufficient conditions for the time-dependent distribution to be of

product form, we first transform the model such that all freedom in the coefficients appearing in the model is removed. This gives slightly different formulations for the open and closed model and also changes the form of the time-dependent version of the traffic equations for the open queueing network. Therefore, in the discussion below, we will consider the open and closed network separately.

1.1 Canonical form for the open network

In this paper we consider an open queueing network with state independent Poisson input with parameter λ . The transition rate $q(\bar{n}, \bar{n} + e_j)$ for an arriving job is then given by

$$q(\bar{n}, \bar{n} + e_j) = \lambda p_{0j} \quad j = 1, \dots, N.$$

From (1.1) we then obtain

$$\psi(\bar{n}) = \lambda \phi(\bar{n}) \quad \forall \bar{n} \in V.$$

In the following lemma we summarize all changes this implies for $P(\bar{n}, t)$. This lemma provides a unique form for the transition rates already obtained in [2] and [6]. However, since it also determines the parameters of the time-dependent queueing network and also gives a transformation for the coefficients $c_k(t)$ we have inserted it here.

Lemma 1.1 (Canonical form) *Let $P(\bar{n}, t)$ be a time-dependent probability distribution on $V = \{\bar{n} : \bar{n} = (n_1, \dots, n_N), n_i \geq 0, i = 1, \dots, N\}$ of the form*

$$P(\bar{n}, t) = \hat{B}(t) \hat{\phi}(\bar{n}) \prod_{k=1}^N \hat{c}_k(t)^{n_k} \quad \bar{n} \in V, t \geq 0$$

and let $f(\bar{n}, i)$ be a function on the set V of the form

$$f(\bar{n}, i) = \frac{\hat{\psi}(\bar{n} - e_i)}{\hat{\phi}(\bar{n})} \quad \bar{n} \in V, i = 0, \dots, N \tag{1.6}$$

satisfying

$$f(\bar{n}, 0) = \lambda \quad \bar{n} \in V. \tag{1.7}$$

Then by defining

$$B(t) = \hat{B}(t) \hat{\phi}(0) \tag{1.8}$$

$$\phi(\bar{n}) = \frac{\hat{\phi}(\bar{n})}{\hat{\phi}(0)} \left[\prod_{k=1}^N \lambda^{n_k} \right]^{-1} \tag{1.9}$$

$$c_k(t) = \lambda \hat{c}_k(t) \tag{1.10}$$

$P(\bar{n}, t)$ and $f(\bar{n}, i)$ can be written in the canonical form

$$P(\bar{n}, t) = B(t)\phi(\bar{n}) \prod_{k=1}^N c_k(t)^{n_k} \quad \bar{n} \in V, t \geq 0 \quad (1.11)$$

$$f(\bar{n}, i) = \frac{\phi(\bar{n} - e_i)}{\phi(\bar{n})} \quad \bar{n} \in V, i = 1, \dots, N \quad (1.12)$$

where

$$\phi(0) = 1. \quad (1.13)$$

Proof Comparison of (1.6) and (1.7) gives for all $\bar{n} \in V$

$$\hat{\psi}(\bar{n}) = \lambda \hat{\phi}(\bar{n}). \quad (1.14)$$

Substitution of (1.8), (1.9) and (1.10) into the right-hand side of (1.11) gives

$$\begin{aligned} B(t)\phi(\bar{n}) \prod_{k=1}^N c_k(t)^{n_k} &= \hat{B}(t)\hat{\phi}(0) \frac{\hat{\phi}(\bar{n})}{\hat{\phi}(0)} \left[\prod_{k=1}^N \lambda^{n_k} \right]^{-1} \prod_{k=1}^N (\lambda \hat{c}_k(t))^{n_k} \\ &= \hat{B}(t)\hat{\phi}(\bar{n}) \prod_{k=1}^N \hat{c}_k(t)^{n_k}. \end{aligned}$$

Substitution of (1.14) and (1.9) into the right-hand side of (1.6) immediately gives the right-hand side of (1.12) and from (1.9) it immediately follows that $\phi(0) = 1$. \square

Inserting the transformation (1.10) in the time-dependent traffic equations (1.5) gives

$$\frac{1}{\mu_k} \frac{dc_k(t)}{dt} = \sum_{i=1}^N \{c_i(t)p_{ik} - c_k(t)p_{ki}\} + \lambda p_{0k} - c_k p_{k0} \quad k = 1, \dots, N, \quad (1.15)$$

with initial conditions

$$c_k(0) = \lambda \xi_k.$$

This is the time-dependent version for the traffic equations used by many authors when the arrival process is a state independent Poisson process. Note that for the original process with transition rates (1.1) and stationary distribution (1.2) this is not the appropriate form. However, for the case of the transformed model, the traffic equations (1.15) are equivalent to (1.5).

The condition (1.7) imposed on $f(\cdot)$ implies that the arrival process to the open queueing network is a state-independent Poisson process with parameter λ . With the transformation defined in the lemma above the transition rates for the open network are given by

$$q(\bar{n}, \bar{n} - e_i + e_j) = p_{ij} \frac{\phi(\bar{n} - e_i)}{\phi(\bar{n})} \quad i = 1, \dots, N, j = 0, \dots, N$$

$$q(\bar{n}, \bar{n} + e_j) = \lambda p_{0j} \quad j = 1, \dots, N.$$

which is in agreement with the transition rates obtained in [2] and [6].

1.2 Canonical form for the closed network

Note that ψ does not appear in the time-dependent probability distribution defined in (1.4). In the stationary case this is a consequence of the global balance equations. Also, it was shown in section 1.1 that to model an open queueing network with state independent Poisson arrivals it is necessary that ψ be a constant multiple of ϕ . In the closed case, however, it is intuitively obvious that the function $\psi(\cdot)$ should appear explicitly in the time-dependent distribution. This can be argued as follows: For fixed \bar{n} and i the role of $\psi(\bar{n} - e_i)$ is to speed up or delay transitions out of state \bar{n} due to the departure of a job at station i or transitions into state \bar{n} due to the arrival of a job at station i . If $\psi(\bar{n} - e_i)$ is changed then the arrival rate to and from state \bar{n} is changed. While this does not affect the equilibrium probability of being in state \bar{n} it will affect the time-dependent probability for most initial conditions. Thus there is no hope that a closed queueing network with general $\psi(\cdot)$ could have a time-dependent distribution of the form (1.4). Therefore, we assume from the start that for the closed network $\psi(\cdot) = \phi(\cdot)$. This agrees with the condition that is imposed on the open network by assumption of state independent Poisson arrivals.

Note that there still remain some degrees of freedom in the parameters of the process. In the following lemma we give a transformation which removes all freedom from the model.

Lemma 1.2 (Canonical form) *Let $P(\bar{n}, t)$ be a time-dependent probability distribution on $V = \{\bar{n} : \bar{n} = (n_1, \dots, n_N), n_i \geq 0, i = 1, \dots, N, \sum_{j=1}^N n_j = M\}$ of the form*

$$P(\bar{n}, t) = \widehat{B}(t) \widehat{\phi}(\bar{n}) \prod_{k=1}^N \widehat{c}_k(t)^{n_k} \quad \bar{n} \in V, t \geq 0$$

and let $f(\bar{n}, i)$ be a function on the set V of the form

$$f(\bar{n}, i) = \frac{\widehat{\phi}(\bar{n} - e_i)}{\widehat{\phi}(\bar{n})} \quad \bar{n} \in V, i = 1, \dots, N.$$

Define

$$\mu_i = \frac{f(Me_i, i)}{M} \quad i = 1, \dots, N$$

then by defining

$$c_k(t) = \widehat{c}_k(t) \left[\sum_{i=1}^N \frac{\widehat{c}_i(t)}{\mu_i} \right]^{-1} \quad k = 1, \dots, N, t \geq 0 \quad (1.16)$$

$$B(t) = \widehat{B}(t) \widehat{\phi}(0) \left[\sum_{i=1}^N \frac{\widehat{c}_i(t)}{\mu_i} \right]^M \quad t \geq 0 \quad (1.17)$$

$$\phi(\bar{n}) = \frac{\widehat{\phi}(\bar{n})}{\widehat{\phi}(0)} \quad (1.18)$$

$P(\bar{n}, t)$ and $f(\bar{n}, i)$ can be written in the canonical form

$$P(\bar{n}, t) = B(t)\phi(\bar{n}) \prod_{k=1}^N c_k(t)^{n_k} \quad \bar{n} \in V, t \geq 0 \quad (1.19)$$

$$f(\bar{n}, i) = \frac{\phi(\bar{n} - e_i)}{\phi(\bar{n})} \quad \bar{n} \in V, i = 1, \dots, N \quad (1.20)$$

where

$$\phi(0) = 1 \quad (1.21)$$

and

$$\sum_{i=1}^N \frac{c_i(t)}{\mu_i} = 1 \quad \forall t \geq 0. \quad (1.22)$$

Proof For fixed t $P(\bar{n}, t)$ is a probability distribution over V , therefore for all k we have that $0 \leq \hat{c}_k(t) < \infty$ for all t and for all t there exists a k such that $\hat{c}_k(t) > 0$. By assumption we have that $\phi(\cdot) > 0$, which implies

$$0 < \sum_{i=1}^N \frac{\hat{c}_i(t)}{\mu_i} < \infty$$

from which we obtain that (1.16) is well-defined.

Inserting (1.16), (1.17) and (1.18) into the right-hand side of (1.19) gives by using that $\sum_{i=1}^N n_i = M$

$$\begin{aligned} B(t)\phi(\bar{n}) \prod_{k=1}^N c_k(t)^{n_k} &= \hat{B}(t)\hat{\phi}(0) \left(\sum_{i=1}^N \frac{\hat{c}_i(t)}{\mu_i} \right)^M \frac{\hat{\phi}(\bar{n})}{\hat{\phi}(0)} \prod_{k=1}^N \hat{c}_k(t)^{n_k} \left[\prod_{k=1}^N \left(\sum_{i=1}^N \frac{\hat{c}_i(t)}{\mu_i} \right)^{n_k} \right]^{-1} \\ &= P(\bar{n}, t). \end{aligned}$$

Inserting (1.16) into the left-hand side of (1.22) gives

$$\sum_{i=1}^N \frac{c_i(t)}{\mu_i} = \sum_{i=1}^N \frac{1}{\mu_i} \hat{c}_i(t) \left[\sum_{i=1}^N \frac{\hat{c}_i(t)}{\mu_i} \right]^{-1} = 1$$

which completes the proof. \square

Substitution of the transformation (1.16) into the time-dependent traffic equations (1.5) gives the following form

$$\frac{1}{\mu_k} \frac{dc_k(t)}{dt} = \sum_{i=1}^N c_i(t)p_{ik} - c_k(t) \quad k = 1, \dots, N.$$

In this paper we will assume henceforth that all queueing networks, closed or open, are represented in the appropriate canonical form.

2 Sufficient conditions

In this section we give a sufficient condition for the transient probability distribution to be of product form.

The transition rates for the open queueing network are given by

$$q(\bar{n}, \bar{n} - e_i + e_j) = p_{ij} \frac{\phi(\bar{n} - e_i)}{\phi(\bar{n})} \quad i, j = 1, \dots, N \quad (2.1a)$$

$$q(\bar{n}, \bar{n} + e_j) = \lambda p_{0j} \quad j = 1, \dots, N \quad (2.1b)$$

$$q(\bar{n}, \bar{n} - e_i) = p_{i0} \frac{\phi(\bar{n} - e_i)}{\phi(\bar{n})} \quad i = 1, \dots, N. \quad (2.1c)$$

Thus, $q(\bar{n}, \bar{n} - e_i + e_j)$ can be written as $p_{ij} f(\bar{n}, i)$ where the $f(\bar{n}, i)$ are given for the open and closed models respectively by (1.12) and (1.20). We thus assume that (1.13) resp. (1.21) and (1.22) are also satisfied. For a closed queueing network the transition rates are given by (2.1a), obtained from the transition rates for the open network by setting $p_{i0} = p_{0i} = 0$, $i = 1, \dots, N$. Note that we have assumed that $\phi(\bar{n}) > 0$ for all $\bar{n} \in N_0^N$ and $\phi(\bar{n}) = 0$ if $n_i < 0$ for some i .

A probability distribution $P(\bar{n}, t)$ is said to be the time-dependent probability distribution for the process with transition rates $q(\bar{n}, \bar{n}')$ if $P(\bar{n}, t)$ satisfies the forward Kolmogorov equations (cf. [14])

$$\frac{dP(\bar{n}, t)}{dt} = \sum_{\bar{n}'} \{P(\bar{n}', t)q(\bar{n}', \bar{n}) - P(\bar{n}, t)q(\bar{n}, \bar{n}')\} \quad (2.2)$$

with initial conditions

$$P(\bar{n}, 0) = P_0(\bar{n}).$$

The product form transient distribution (1.4) satisfies the forward Kolmogorov equations if and only if the following relation holds for all $\bar{n} \in V$ and for all $t > 0$

$$\begin{aligned} \frac{1}{B(t)} \frac{dB(t)}{dt} + \sum_{i=1}^N \frac{n_i}{c_i(t)} \frac{dc_i(t)}{dt} &= \sum_{i,j=1}^N \frac{\phi(\bar{n} - e_i)}{c_i(t)\phi(\bar{n})} \{c_j(t)p_{ji} - c_i(t)p_{ij}\} \\ &+ \sum_{i=1}^N \{c_i(t)p_{i0} - \lambda p_{0i}\} \\ &+ \sum_{i=1}^N \frac{\phi(\bar{n} - e_i)}{c_i(t)\phi(\bar{n})} \{\lambda p_{0i} - c_i(t)p_{i0}\} \end{aligned} \quad (2.3)$$

as can easily be verified by substituting (1.4) and (2.1) into (2.2).

Remark 2.1 (Lévy's dichotomy) In order to derive (2.3) from (2.2) we implicitly used the fact that $P(\bar{n}, t) > 0$ for all $\bar{n} \in V$ for all $t > 0$. This can be justified as follows. If $P(\bar{n}, t) = 0$ for some t , then, from Lévy's dichotomy (cf. [3, page 126]), $P(\bar{n}, t) = 0$ for all $t > 0$. This implies that for some k_0 for which $n_{k_0} > 0$ we have that $c_{k_0}(t) = 0$ for all $t > 0$. However, if this is the case, then $P(\bar{n}, t) = 0$ for all \bar{n} such that $n_{k_0} > 0$. Therefore, for all $t > 0$ queue k_0 cannot contain any jobs and we may remove queue k_0 from the network. Hence we may assume that $P(\bar{n}, t) > 0$.

Remark 2.2 (State space V) If the product form transient distribution (1.4) satisfies the forward Kolmogorov equations then for the open queueing network we obtain

$$P(0, t) = B(t)\phi(0) = B(t)$$

which is non-zero. Thus the state $\bar{n} = 0$ must have non-zero probability for all time t .

Similarly, for the closed network,

$$P(Me_i, t) = B(t)\phi(Me_i)c_i(t)^M \quad i = 1, \dots, N$$

which is non-zero unless the number of customers in queue i is zero for all t , which, as remarked above, we can discard.

The following theorem gives a sufficient condition for the transient probability distribution to be of product form. For a network of $M/G/\infty$ queues, with non-homogeneous Poisson input in the open network case, this product form was obtained in [5]. However, the formulation used in that paper is rather different from the formulation used in the theorem below. The transient form for the traffic equations does not appear in [5]. Furthermore, the initial conditions in the theorem below are more general than the initial conditions in [5]. Therefore, we give the theorem in terms of our formulation and also include the proof of the theorem.

Theorem 2.3 (Sufficient conditions) *Consider a queueing network with product form initial distribution*

$$P_0(\bar{n}) = B_0\phi(\bar{n}) \prod_{k=1}^N \xi_k^{n_k} \quad \bar{n} \in V$$

where $\phi(\cdot)$ satisfies

$$\frac{\phi(\bar{n} - e_k)}{\phi(\bar{n})} = \mu_k n_k \quad k = 1, \dots, N. \quad (2.4)$$

Then

$$P(\bar{n}, t) = B(t)\phi(\bar{n}) \prod_{k=1}^N c_k(t)^{n_k} \quad (2.5)$$

where the $\{c_k(t)\}_{k=1}^N$ satisfy the traffic equations

$$\frac{1}{\mu_k} \frac{dc_k(t)}{dt} = \sum_{i=1}^N c_i(t) p_{ik} + \lambda p_{0k} - c_k(t) \quad k = 1, \dots, N \quad (2.6)$$

with initial conditions

$$c_k(0) = \xi_k \quad (2.7)$$

and the normalization constant $B(t)$ satisfies

$$\frac{1}{B(t)} \frac{dB(t)}{dt} = \sum_{i=1}^N \{c_i(t) p_{i0} - \lambda p_{0i}\} \quad (2.8)$$

with initial conditions

$$B(0) = B_0. \quad (2.9)$$

Proof Insertion of (2.4) into (2.3) gives

$$\begin{aligned} & \frac{1}{B(t)} \frac{dB(t)}{dt} - \sum_{i=1}^N \{c_i(t) p_{i0} - \lambda p_{0i}\} \\ &= \sum_{i=1}^N \frac{n_i \mu_i}{c_i(t)} \left\{ \sum_{j=1}^N \{c_j(t) p_{ji} - c_i(t) p_{ij}\} + \lambda p_{0i} - c_i(t) p_{i0} - \frac{1}{\mu_i} \frac{dc_i(t)}{dt} \right\} \end{aligned}$$

which is obviously satisfied if (2.6) and (2.8) hold. The initial conditions (2.7) and (2.9) give

$$P(\bar{n}, 0) = P_0(\bar{n})$$

which implies that (2.5) gives the transient probability distribution of the process with transition rates satisfying (2.4). \square

Remark 2.4 (Explicit form for $P(\bar{n}, t)$) Note that the above theorem applies to both the open and the closed network case. In the closed case interpret $p_{k0} = p_{0k} = 0$ in (2.6) and (2.8).

From (2.4) we obtain

$$\phi(\bar{n}) = \prod_{k=1}^N \frac{1}{n_k!} \left(\frac{1}{\mu_k} \right)^{n_k}. \quad (2.10)$$

For the closed network case we find from (2.8) and (2.9) that $B(t) = B_0$ for all $t \geq 0$. From (1.22) we then obtain

$$P(\bar{n}, t) = M! \prod_{k=1}^N \frac{1}{n_k!} \left(\frac{c_k(t)}{\mu_k} \right)^{n_k} \quad \bar{n} \in V, t \geq 0.$$

For the open network case we find from (2.8)

$$\begin{aligned} \frac{d \log B(t)}{dt} &= \sum_{i=1}^N \{c_i(t)p_{i0} - \lambda p_{0i}\} \\ &= \sum_{i=1}^N \left\{ c_i(t) \left(1 - \sum_{j=1}^N p_{ij} \right) - \lambda p_{0i} \right\} \\ &= - \sum_{i=1}^N \frac{1}{\mu_i} \frac{dc_i(t)}{dt} \end{aligned}$$

which implies

$$B(t) = B_0 \prod_{k=1}^N \exp \left[-\frac{c_k(t) - \xi_k}{\mu_k} \right] = \prod_{k=1}^N \exp \left[-\frac{c_k(t)}{\mu_k} \right].$$

From (2.10) we then obtain

$$P(\bar{n}, t) = \prod_{k=1}^N \exp \left[-\frac{c_k(t)}{\mu_k} \right] \frac{1}{n_k!} \left(\frac{c_k(t)}{\mu_k} \right)^{n_k} \quad \bar{n} \in V, t \geq 0.$$

Remark 2.5 (Mixed open and closed networks) Theorem 2.3 applies to open and closed networks separately, but it also applies to mixed queueing networks where, for example, part of the network is closed and part of the network is open. This is a direct consequence of the product form initial distribution which guarantees that separate parts of the network are independent at $t = 0$. However, if these parts do not interact then they remain independent and the product form holds for all $t \geq 0$. This observation is not mentioned in [5].

Remark 2.6 (Time-dependent input process) Note that, in the proof above, we may replace λ by $\lambda(t)$. This implies that the results from [5] are generalized in the theorem above to product form initial conditions.

Remark 2.7 (General initial conditions) Note that, since the forward Kolmogorov equations are a set of linear differential equations, if $P^{(i)}(\bar{n}, t)$ is a solution of the forward Kolmogorov equations with initial conditions $P_0^{(i)}(\bar{n})$, $i = 1, \dots, I$ then $\sum_{i=1}^I k_i P^{(i)}(\bar{n}, t)$ is a solution of the forward Kolmogorov equations with initial conditions $\sum_{i=1}^I k_i P_0^{(i)}(\bar{n})$. This allows us to further extend the possible initial distributions to non product form initial conditions.

3 Necessary conditions for the open network

In this section we consider a Markovian open queueing network. We will show that a necessary condition for the network to have a transient product form distribution is that all queues are $M/M/\infty$ queues. First we give a general lemma considering the solution of the traffic equations. This lemma will be used in the proof of the necessity result stated in Theorem 3.3.

Lemma 3.1 *Let $\{\mu_i\}_{i=1}^N$ be a set of positive numbers, $\{\lambda_i\}_{i=1}^N$ and $\{\xi_i\}_{i=1}^N$ be sets of non-negative numbers and $P = [p_{ij}]$, $i, j = 1, \dots, N$ be a stochastic matrix whose essential submatrices are strictly substochastic and such that*

$$\forall j = 1, \dots, N \exists \text{ a sequence } i_{j_1}, \dots, i_{j_k} \text{ such that } \lambda_{i_{j_1}} p_{i_{j_1} i_{j_2}} p_{i_{j_2} i_{j_3}} \cdots p_{i_{j_k} j} > 0. \quad (3.1)$$

Let $\{c_i(t)\}_{i=1}^N$ be a solution to the set of differential equations

$$\frac{1}{\mu_i} \frac{dc_i(t)}{dt} = -c_i(t) + \sum_{j=1}^N c_j(t) p_{ji} + \lambda_i \quad i = 1, \dots, N \quad (3.2)$$

such that

$$c_i(0) = \xi_i \quad i = 1, \dots, N. \quad (3.3)$$

Then if there exists $\bar{n} \in N_0^N \setminus \{\bar{0}\}$ such that

$$\prod_{i=1}^N c_i(t)^{\bar{n}_i} \quad (3.4)$$

is independent of t , it must be the case that the initial conditions are such that the $c_i(t)$ are in equilibrium. That is, the ξ_i satisfy

$$\xi_i = \sum_{j=1}^N \xi_j p_{ji} + \lambda_i \quad i = 1, \dots, N.$$

Proof Define the vectors $\bar{c}(t)$, $\bar{\lambda}$ and $\bar{\xi}$ such that $[\bar{c}(t)]_i = c_i(t)$, $[\bar{\lambda}]_i = \lambda_i$ and $[\bar{\xi}]_i = \xi_i$ and the matrix \mathcal{M} such that

$$[\mathcal{M}]_{ij} = \begin{cases} \frac{1}{\mu_i} & \text{if } i = j \\ \mu_i & \text{otherwise.} \end{cases}$$

The system of equations (3.2) with initial conditions (3.3) can now be written

$$\frac{d\bar{c}(t)}{dt} = \bar{c}(t)[P - I]\mathcal{M} + \bar{\lambda}\mathcal{M} \quad (3.5)$$

such that

$$\bar{c}(0) = \bar{\xi}. \quad (3.6)$$

The assumption that the essential submatrices of P are strictly substochastic implies that all the eigenvalues of $(P - I)\mathcal{M}$ have negative real parts (see [11]) and so the solution of (3.5) can be written in the form (see, e.g. [10])

$$\bar{c}(t) = \bar{\lambda}(I - P)^{-1} + \sum_{j=1}^J \sum_{l=0}^{m(j)} A_{jl} t^l e^{-\alpha_j t} \bar{w}_{jl} \quad (3.7)$$

where the set $\{-\alpha_j\}_{j=1}^J$ contains the distinct eigenvalues of $(P - I)\mathcal{M}$, which have algebraic multiplicity $m(j) + 1$, ordered so that $Re(\alpha_{m+1}) \geq Re(\alpha_m)$ and $Im(\alpha_{m+1}) > Im(\alpha_m)$ if $Re(\alpha_m) = Re(\alpha_{m+1})$ and the \bar{w}_{jl} are linearly independent vectors. The A_{jl} are constants determined by the initial conditions (3.6) and the factor $\bar{\lambda}(I - P)^{-1}$ arises as a particular solution to (3.5). The matrix $(I - P)^{-1}$ contains only non-negative entries (see Theorem 2.6 of [11]). Moreover, assumption (3.1) implies that $\bar{\lambda}(I - P)^{-1}$ contains only positive entries.

The individual components of (3.7) can be written

$$c_i(t) = A_{00}^{(i)} e^{-\alpha_0 t} + \sum_{j=1}^J \sum_{l=0}^{m(j)} A_{jl}^{(i)} t^l e^{-\alpha_j t} \quad (3.8)$$

where, for convenience, we have denoted $\alpha_0 = 0$ and $A_{00}^{(i)} = [\bar{\lambda}(I - P)^{-1}]_i$. For fixed i

$$J(i) = \max\{j \geq 0 : \exists l \text{ with } A_{jl}^{(i)} \neq 0\}$$

and for fixed i and $j \leq J(i)$

$$L_{ji} = \begin{cases} \max\{l \geq 0 : A_{jl}^{(i)} \neq 0\} & \text{if such an } l \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

$J(i)$ is well defined because $A_{00}^{(i)} \neq 0$. Using (3.8) and the ordering of the α_i it follows that the coefficient of

$$[t^{L_{J(i)i}} e^{-\alpha_{J(i)} t}]^{n_i}$$

in $c_i(t)^{n_i}$ is $B(i) \equiv (A_{J(i)L_{J(i)i}})^{n_i}$ which is non-zero. The coefficient of

$$t^{\sum_i n_i L_{J(i)i}} \exp \left[- \sum_i n_i \alpha_{J(i)} t \right] \quad (3.9)$$

in the expansion of (3.4) is

$$\prod_{i=1}^N B(i)^{n_i}.$$

Now for $\bar{n} \neq 0 \in N_0^N$ (3.9) depends on t unless $\alpha_{J(i)} = 0 \forall i$. If this is not the case (3.9) is linearly independent of the other terms in the expansion of (3.4). It follows that, for (3.4) to be independent of t , $\alpha_{J(i)} = 0 \forall i$ which in turn implies $J(i) = 0 \forall i$.

Thus, from (3.8)

$$c_i(t) = A_{00}^{(i)} = [\bar{\lambda}(I - P)^{-1}]_i$$

and the system is in equilibrium. \square

Remark 3.2 In the following theorem ξ_i , λ_i and p_{ij} will be interpreted as initial conditions, arrival parameters and routing parameters for an open network of queues. In this context the ij th entry of $(I - P)^{-1}$ gives the expected number of visits of a job to station j per sojourn time in the network conditional on it entering the network in station i . Thus the j th entry of

$$\frac{1}{\sum_{i=1}^N \lambda_i} \bar{\lambda} (I - P)^{-1}$$

is the expected number of visits of a job to station j per sojourn in the network. By Assumption (3.1) this is non-zero. If this assumption is not satisfied there exist queues which cannot be reached from outside the network. Assume queue i is such a queue. Then in (3.8) above $A_{00}^{(i)} = 0$. There are now two cases to be considered:

- (i) There exist j, l such that $A_{jl}^{(i)} > 0$, in which case $B(i) > 0$ and an argument similar to that used above leads to the conclusion that (3.4) is dependent of t .
- (ii) $c_i(t) = 0 \forall t$, which implies $\xi_i = 0$ and queue i started off empty, as did all other queues j with $p_{ji} > 0$. In this case these queues never have any jobs and can be removed from the model.

Conversely, if queue i is such that it can be reached from outside the network but customers can never depart the network having reached queue i the assumption that the essential submatrices of $(I - P)$ are strictly substochastic no longer holds, and 0 is an eigenvalue of $(I - P)$. In this case the i th component of the particular solution of (3.5) has the form

$$A_{00}^{(i)} t^j \quad j > 0$$

where $A_{00}^{(i)} \neq 0$. It follows that $c_i(t)^{n_i}$ and $\prod_{i=1}^N c_i(t)^{n_i}$ can never be independent of t . This is to be expected since, in this case, there exists no equilibrium distribution for the queueing network.

The last case to be considered is that in which there exist queues to which jobs can neither arrive from outside the network nor depart to outside the network. In such a situation there must exist an irreducible closed network which is isolated from any other nodes in the network. In this case we need a result slightly different to Lemma 3.1. This is given in Lemma 4.1 in section 4.

Theorem 3.3 *Assume the network has an initial distribution of the form*

$$P(\bar{n}, 0) = B_0 \phi(\bar{n}) \prod_{k=1}^N \xi_k^{n_k}. \quad (3.10)$$

Then it is necessary for the time-dependent probability distribution to be of product form

$$P(\bar{n}, t) = B(t) \phi(\bar{n}) \prod_{k=1}^N c_k(t)^{n_k} \quad (3.11)$$

that for the set of numbers $\{\mu_k\}_{k=1}^N$ defined as

$$\mu_k = \frac{1}{\phi(e_k)} \quad (3.12)$$

the $\{c_k(t)\}_{k=1}^N$ satisfy

$$\frac{1}{\mu_k} \frac{dc_k(t)}{dt} = \sum_{i=1}^N c_i(t) p_{ik} + \lambda p_{0k} - c_k(t) \quad (3.13)$$

with initial condition

$$c_k(0) = \xi_k \quad (3.14)$$

and that the normalization constant $B(t)$ satisfies

$$\frac{1}{B(t)} \frac{dB(t)}{dt} = \sum_{i=1}^N \{-\lambda p_{0i} + c_i(t) p_{i0}\} \quad (3.15)$$

with initial condition

$$B(0) = B_0. \quad (3.16)$$

Furthermore, if the network is not in equilibrium, then it is necessary for the existence of a product form transient distribution that

$$\frac{\phi(\bar{n} - e_k)}{\phi(\bar{n})} = \mu_k n_k. \quad (3.17)$$

Proof Relation (2.3) must hold for all \bar{n} , therefore in the light of Remark 2.2 insertion of $\bar{n} = 0$ is allowed. Using the fact that $\phi(\bar{n}) = 0$ if $n_i < 0$ for some i this gives (3.15). From (3.10) and (3.11) we obtain for $\bar{n} = 0$ that $B(0) = B_0$ which proves (3.16). Insertion of (3.15) into (2.3) gives

$$\sum_{i=1}^N \frac{n_i}{c_i(t)} \frac{dc_i(t)}{dt} = \sum_{i=1}^N \frac{1}{c_i(t)} \frac{\phi(\bar{n} - e_i)}{\phi(\bar{n})} \left\{ \sum_{j=1}^N \{c_j(t) p_{ji} - c_i(t) p_{ij}\} - c_i(t) p_{i0} + \lambda p_{0i} \right\}. \quad (3.18)$$

Insertion of $\bar{n} = e_i$ into this relation and use of (1.13) and (3.12) gives the traffic equations. From (3.10) and (3.11) we obtain for $t = 0$ and $\bar{n} = e_i$ that the initial condition for $c_k(t)$ is given by $c_k(0) = \xi_k$. Insertion of (3.13) in (3.18) gives

$$0 = \sum_{i=1}^N \frac{1}{c_i(t)} \frac{dc_i(t)}{dt} \left\{ n_i - \frac{\phi(\bar{n} - e_i)}{\mu_i \phi(\bar{n})} \right\} \quad (3.19)$$

which must hold for all \bar{n} . We now show by an inductive argument that for all \bar{n} $\phi(\cdot)$ is given by

$$\phi(\bar{n}) = \prod_{k=1}^N \frac{1}{n_k!} \left(\frac{1}{\mu_k} \right)^{n_k}. \quad (3.20)$$

If $\bar{n} = 0$ then, by (1.13) $\phi(0) = 1$ which satisfies (3.20). Now assume that (3.20) holds for all \bar{n} such that $\sum_{i=1}^N n_i \leq M - 1$, then for \bar{n} such that $\sum_{i=1}^N n_i = M$ we obtain by insertion of (3.20) for $\bar{n} - e_i$ into (3.19)

$$\begin{aligned} 0 &= \sum_{i=1}^N \frac{1}{c_i(t)} \frac{dc_i(t)}{dt} \left\{ n_i - \frac{1}{\mu_i \phi(\bar{n})} \prod_{k=1}^N \frac{1}{(n_k - \delta_{ki})!} \left(\frac{1}{\mu_k} \right)^{n_k - \delta_{ki}} \right\} \\ &= \sum_{i=1}^N \frac{n_i}{c_i(t)} \frac{dc_i(t)}{dt} \left\{ 1 - \frac{1}{\phi(\bar{n})} \prod_{k=1}^N \frac{1}{n_k!} \left(\frac{1}{\mu_k} \right)^{n_k} \right\}. \end{aligned} \quad (3.21)$$

From Lemma 3.1 and Remark 3.2 we obtain, unless the process is in equilibrium, that for at least some $t > 0$

$$\sum_{i=1}^N \frac{n_i}{c_i(t)} \frac{dc_i(t)}{dt} = \frac{d}{dt} \log \left[\prod_{k=1}^N c_k(t)^{n_k} \right] \neq 0.$$

Therefore, from (3.21) it follows that (3.20) holds for \bar{n} , which completes the induction step.

Taking the quotient of (3.20) for $\bar{n} - e_i$ and \bar{n} immediately implies that (3.17) holds. \square

4 Necessary conditions for the closed network

In this section we turn to closed queueing networks. Because the state space is restricted to $V = \{\bar{n} : \sum_{i=1}^N n_i = M\}$ the induction argument used for open queueing networks cannot be applied. We are able only to prove that if one queue of a network with time-dependent product form is an $M/M/\infty$ queue then, unless the network is in equilibrium, all the queues must be $M/M/\infty$ queues. The queues can always be re-labelled so that this "special" queue is queue number 1. Lemma 4.1, below, gives a preliminary result needed for the theorem. In the analysis below we assume that the network is irreducible. Note that this is not a restriction since we can analyse the irreducible subnetworks separately if the network is reducible.

Lemma 4.1 *Let $\{\mu_i\}_{i=1}^N$ be a set of positive numbers and $\{\xi_i\}_{i=1}^N$ be a set of non-negative numbers, $P = [p_{ij}]$ be an irreducible stochastic matrix and $\{c_i(t)\}_{i=1}^N$ be a solution to the set of differential equations*

$$\frac{1}{\mu_i} \frac{dc_i(t)}{dt} = -c_i(t) + \sum_{j=1}^N c_j(t) p_{ji} \quad i = 1, \dots, N \quad (4.1)$$

such that

$$c_i(0) = \xi_i \quad i = 1, \dots, N$$

and

$$\sum_{i=1}^N \frac{c_i(t)}{\mu_i} = 1 \quad \forall t. \quad (4.2)$$

Then if there exist $\bar{n} \in V = \{\bar{n} \in N_0^N : \sum_{i=1}^N n_i = M\}$ other than $M e_1$ such that

$$\prod_{i=2}^N c_i(t)^{n_i} \quad (4.3)$$

is independent of t , it must be the case that the initial conditions are such that the $c_i(t)$ are in equilibrium, i.e. the ξ_i satisfy

$$\xi_j = \sum_{i=1}^N \xi_i p_{ij} \quad j = 1, \dots, N.$$

Proof Equation (4.1) can be written

$$\frac{d\bar{c}(t)}{dt} = \bar{c}(t)(P - I)\mathcal{M}$$

such that

$$\bar{c}(0) = \bar{\xi}$$

and the individual components $c_i(t)$ still have the form (3.8) except that the term $A_{00}^{(i)} e^{-\alpha_0 t}$ arises not as a particular solution to the inhomogeneous differential equation, but because $\alpha_0 = 0$ is now an eigenvalue (of multiplicity 1) of $(P - I)\mathcal{M}$. It follows that the $A_{00}^{(i)}$ are the equilibrium values for the closed network. The $c_i(t)$ can be chosen such that (4.2) holds by the lemma on canonical form.

Using similar arguments to the proof of Lemma 3.1 we can conclude that the coefficient of

$$t^{\sum_{i=2}^N n_i L_{J(i)i}} \exp \left[- \sum_{i=2}^N n_i \alpha_{J(i)} t \right]$$

in the expansion of (4.3) is $\prod_{i=2}^N B(i)$ which is non-zero.

Thus (4.3) is independent of t only if $A_{ji}^{(i)} = 0$ in (3.8) for all $j > 0$. In this case $c_i(t) = A_{00}^{(i)} = \xi_i$, $i = 2, \dots, N$. However, by (4.2), for all t

$$\begin{aligned} c_1(t) &= \mu_1 \left(1 - \sum_{i=2}^N \frac{c_i(t)}{\mu_i} \right) \\ &= \xi_1. \end{aligned}$$

The network is thus in equilibrium. \square

Theorem 4.2 Assume the network has an initial distribution of the form

$$P(\bar{n}, 0) = B_0 \phi(\bar{n}) \prod_{k=1}^N \xi_k^{n_k}.$$

Then it is necessary for the time-dependent probability distribution to be of product form

$$P(\bar{n}, t) = B(t)\phi(\bar{n}) \prod_{k=1}^N c_k(t)^{n_k} \quad (4.4)$$

that for the set of numbers $\{\mu_k\}_{k=1}^N$ defined as

$$\mu_k = \frac{\phi((M-1)e_k)}{M\phi(Me_k)}$$

the $\{c_k(t)\}_{k=1}^N$ satisfy

$$\frac{1}{\mu_k} \frac{dc_k(t)}{dt} = \sum_{i=1}^N \{c_i(t)p_{ik} - c_k(t)p_{ki}\} \quad (4.5)$$

with initial condition

$$c_k(0) = \xi_k \quad (4.6)$$

and that

$$B(t) = B_0 \quad t \geq 0. \quad (4.7)$$

Furthermore, if queue 1 is an independent $M/M/\infty$ queue, and if the network is not in equilibrium, then it is necessary for the existence of a product form transient distribution that

$$\frac{\phi(\bar{n} - e_k)}{\phi(\bar{n})} = \mu_k n_k. \quad (4.8)$$

Proof Insertion of $\bar{n} = Me_i$ into (2.3) gives after rearranging the terms

$$\frac{1}{B(t)} \frac{dB(t)}{dt} \frac{c_i(t)}{\mu_i} + M \frac{1}{\mu_i} \frac{dc_i(t)}{dt} = M \sum_{j=1}^N \{c_j(t)p_{ji} - c_i(t)p_{ij}\}. \quad (4.9)$$

Summation of (4.9) over all i gives by using (4.2)

$$\frac{dB(t)}{dt} = 0$$

which implies (4.7). Insertion of (4.7) into (4.9) implies that the traffic equations (4.5) hold. Insertion of $\bar{n} = Me_i$ into (4.4) for $t = 0$ gives (4.6). Insertion of (4.5) and (4.7) into (2.3) gives

$$0 = \sum_{i=1}^N \frac{1}{c_i(t)} \frac{dc_i(t)}{dt} \left\{ n_i - \frac{\phi(\bar{n} - e_i)}{\mu_i \phi(\bar{n})} \right\}. \quad (4.10)$$

Now assume that queue 1 is an independent $M/M/\infty$ queue with service rate $n_1\mu_1$ if n_1 jobs are at station 1, then the transition rates can be rewritten as

$$q(\bar{n}, \bar{n} - e_i + e_j) = \left(\frac{\hat{\phi}(\bar{v} - e_i)}{\hat{\phi}(\bar{v})} \mathbf{1}(i \neq 1) + n_i \mu_i \mathbf{1}(i = 1) \right) p_{ij} \quad (4.11)$$

where \bar{v} is the vector which is obtained from \bar{n} by removing the 1st component and $\hat{\phi}$ is the function which is obtained from ϕ by removing the 1st component from the argument of ϕ and $\mathbf{1}(A)$ is the indicator function of event A . Note that $\hat{\phi}$ is defined up to a multiplicative constant only, therefore, without loss of generality we may assume that $\hat{\phi}(0) = 1$. Insertion of (4.11) into (4.10) gives

$$0 = \sum_{i=2}^N \frac{1}{c_i(t)} \frac{dc_i(t)}{dt} \left\{ v_i - \frac{\hat{\phi}(\bar{v} - e_i)}{\mu_i \hat{\phi}(\bar{v})} \right\}$$

which must hold for all \bar{v} in the set $\hat{V} = \{\bar{v} : v_i \geq 0, \sum_{i=2}^N v_i \leq M\}$. The remaining part of the proof is very similar to the proof of Theorem 3.3. By using Lemma 4.1 we obtain by induction that, unless the process is in equilibrium

$$\frac{\hat{\phi}(\bar{v} - e_i)}{\hat{\phi}(\bar{v})} = v_i \mu_i. \quad \square$$

5 Discussion and general remarks

The major contribution of this paper is a proof of the fact that a Markovian queueing network which is not in equilibrium can have the transient product form

$$P(\bar{n}, t) = B(t) \phi(\bar{n}) \prod_{k=1}^N c_k(t)^{n_k} \quad (5.1)$$

only if it is a network of infinite server queues.

In addition we prove the result that a network of $M/M/\infty$ queues with an initial product form distribution

$$P(\bar{n}, 0) = B_0 \phi(\bar{n}) \prod_{k=1}^N \xi_k^{n_k}$$

has the transient product form distribution (5.1) where the $c_k(t)$ satisfy the time dependent traffic equations

$$\frac{1}{\mu_k} \frac{dc_k(t)}{dt} = \sum_{i=1}^N c_i(t) p_{ik} + \lambda p_{0k} - c_k(t) \quad (5.2)$$

subject to

$$c_k(0) = \xi_k.$$

These two results can be combined to show that it is necessary and sufficient for a network of queues, not in equilibrium, to have a product form transient distribution that it be a network of infinite server queues.

For more specific initial conditions, but more general arrival characteristics and service time distributions the sufficiency part of this result has been previously established in [4], [5] and [8]. However, the time dependent traffic equations (5.2) appear to be new, as is the expression of the open and closed models in canonical form.

Our results have the advantage of reducing the derivation of the transient distribution of the network to the solution of a set of linear differential equations with a number of variables equal to the number of queues in the network.

There are several observations that can immediately be made from our results. It is clear from the form of (3.7) that as $t \rightarrow \infty$ $P(\bar{n}, t)$ approaches the equilibrium distribution $\pi(\bar{n})$. For open networks it is of interest to also consider the case where the arrival rate to queue i is a function of time (say $\lambda p_{0i} = \lambda_i(t)$). This was considered in [4] and [5] where it was shown that networks of queues with non-homogeneous Poisson input have a product form transient distribution. It is easy to show, using methods similar to ours, that if a Markovian network of queues with non-homogeneous Poisson input has product form then (3.13) - (3.16) must be satisfied. However, it is unclear how to argue, in general, results analogous to Lemma 3.1 and Remark 3.2. These results are used in the proof of Theorem 3.3 only to show that if a network is not in equilibrium then for some $t > 0$

$$\sum_{i=1}^N \frac{n_i}{c_i(t)} \frac{dc_i(t)}{dt} \neq 0. \quad (5.3)$$

If (5.3) can be assumed then (3.17) also holds for networks with non-homogeneous Poisson input.

A similar observation can be made with respect to the proof of Theorem 4.2. Statements (4.5) - (4.7) can be shown to follow from product form even if ϕ is replaced by an arbitrary ψ in the numerator of (2.1a) and if it is not assumed that one queue is an infinite server queue. These assumptions are used only in the proof of (4.8).

If a network of queues is to have product form for arbitrary service time distributions then it follows from our analysis that it must be a network of infinite server queues merely by observing that exponential service times are special cases of general service times.

Acknowledgements

This research was done during a visit of Richard J. Boucherie to the University of Western Australia, which was made possible by a Grant from the LNMB (Landelijk Netwerk Mathematische Besliskunde), the Dutch graduate school for operations research.

References

- [1] Boucherie, R. J. and Van Dijk, N. M. (1989). "Product forms for queueing networks with state dependent multiple job transitions." *Research Report. Free University, Amsterdam, The Netherlands*.
- [2] Chandy, K. M. and Martin, A. J. (1983). "A characterization of product-form queueing networks." *J. A. C. M.* **30** 286-299.
- [3] Chung, K. L. (1967). *Markov chains with stationary transition probabilities*. 2nd ed. Springer-Verlag.
- [4] Foley, R. D. (1982). "The non-homogeneous $M/G/\infty$ queue." *Opsearch* **19**, 40-48.
- [5] Harrison, J. H. and Lemoine, A. J. (1981). "A note on networks of infinite-server queues." *J. Appl. Prob.* **18**, 561-567.
- [6] Henderson, W. and Taylor, P. G. (1990). "Networks of queues with batch arrivals and batch services." *Queueing Systems* **6**, 71-88.
- [7] Kelly, F. P. (1979). *Reversibility and stochastic networks*. Wiley.
- [8] Kingman, J. F. C. (1969). "Markov population processes." *J. Appl. Prob.* **6**, 1-18.
- [9] Riordan, J. (1962). *Stochastic service systems*.
- [10] Sanchez, D. A. (1968). *Ordinary differential equations and stability theory: an introduction*. W. H. Freeman and Company.
- [11] Seneta, E. (1981). *Non-negative matrices and Markov chains*. Springer-Verlag.
- [12] Serfozo, R. F. (1988). "Markovian network processes with system-dependent transition rates." preprint.
- [13] Walrand, J. (1988). *An introduction to queueing networks*. Prentice Hall.
- [14] Whittle, P. (1986). *Systems in stochastic equilibrium*. Wiley.

1990-1	B. Vogelvang	Testing For Co-Integration with Spot Prices of Some Related Agricultural Commodities
1990-2	J.C.J.M. van den Bergh P. Nijkamp	Ecologically Sustainable Economic Development Concepts and Model Implications
1990-3	J.C.J.M. van den Bergh P. Nijkamp	Ecologically Sustainable Economic Development in a Regional System: A Case Study in Agricultural Development Planning in the Netherlands
1990-4	C.Gorter P.Nijkamp P.Rietveld	Employers' Recruitment Behaviour and Re-Employment Probabilities of Unemployed
1990-5	K.Burger	Off-farm income and the farm-household the case of Kenyan smallholders
1990-6	H. Visser	Crowding out and the Government Budget
1990-7	P. Rietveld	Ordinal Data in Multicriteria Decision Making, a Stochastic Dominance Approach to Siting Nuclear Power Plants
1990-8	G. van der Laan P.H.M. Ruys D.J.J. Talman	Signaling devices for the supply of semi-public goods
1990-9	F.A.G. den Butter	Labour Productivity Slowdown and Technical Progress: An empirical analysis for The Netherlands
1990-10	R.W. van Zijp	Neo-Austrian Business Cycle Theory
1990-11	J.C. van Ours	Matching Unemployment and Vacancies: The Efficiency of the Dutch Labour Market
1990-12	B. Vogelvang	Hypotheses Testing Concerning Relationships between Spot Prices of Various Types of Coffee
1990-13	A.F. de Vos I.J. Steyn	Stochastic Nonlinearity: A Firm Basis for the Flexible Functional Form
1990-14	Y.H. van Emmerik D. de Jong W.W.A. Zuurmond D.N. Dukkervan Emden	Opereren in overleg: geprotocolleerde samenwerking 1e-2e-lijn bij dagchirurgie
1990-15	T.J.J.B. Wolters	Mediation and Collective Bargaining: A Diagnostic Approach
1990-16	E.M.A. Scholten J. Koelewijn	Financieringsproblematiek van startende ondernemingen: een mogelijke verklaring op basis van empirisch onderzoek.
1990-17	E. Hüner H.P. Smit	Saturation and Model Specification of Passenger car Ownership
1990-18	F.A.G. den Butter	Sociale zekerheid, de wig en economische groei