SERIE RESEARCH MEMORANDA

CHAOS THEORY AND SPATIAL DYNAMICS

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Abstract

The theory of chaos is nowadays receiving a great deal of attention among social scientists. Many attempts are being made to offer a meaningful interpretation and application of notions of chaos in social systems.

The present paper aims to link chaos theory to spatial interaction analysis by focusing attention on the conditions under which a general utility function related to a dynamic logit model for spatial interaction analysis will exhibit chaotic behaviour.

In addition, the present paper will also analyze the impact of this dynamic logit model upon a more general spatial system, notably a Lotka-Volterra system in the context of a transportation network including congestion phenomena. Time lags will also be incorporated in order to account for non-instantaneous effects in prey-predator type of interactions.

Finally, it will be shown that under certain conditions on the parameters of the spatial system concerned a so-called Hopf bifurcation will take place. In other words, unstable system's behaviour may emerge for particular lag values reflecting the influence from the past.

The theoretical analysis in the paper will be illustrated by means of various simulation experiments.

1. Introduction

It is increasingly realized that the evolutionary patterns of many spatial phenomena require dynamic modes of analysis (see Dendrinos and Mullally, 1985, and Nijkamp and Schubert, 1985). Conventional macro types of dynamic spatial models however, have not offered a satisfactory potential for replicating and predicting spatial evolution at a sufficiently detailed level.

In the recent past, much attention has been given to micro-based behavioural models of choice. Especially the disaggregate models of choice based on logit or probit analysis have gained much popularity, not only in a theoretical respect (see McFadden, 1974, and Domencich and McFadden, 1975), but also in an empirical respect (see among others Van Lierop, 1986, and Van Wissen and Rima, 1988).

Logit (and probit) models belong to the family of discrete models which have received a great deal of attention among social scientists. These models are essentially based on the principle of utility maximization: it is assumed that an individual decision-maker's preferences regarding available choice options can be described by means of a utility function and that each individual decision-maker will choose the alternative with the highest utility level for him or her.

Fields of application of such discrete choice models are inter alia migration analysis, residential choice analysis, industrial location analysis, travel mode choice analysis etc (see the special issue of Regional Science and Urban Economics, edited by Nijkamp, 1987, and Golledge and Timmermans, 1988).

However, a full dynamic analysis of disaggregate choice models - at both the micro level of individual behaviour and the meso level of groups or spatial entities - has not yet been developed, although considerable progress has been made in the area of longitudinal data analysis (see in particular Heckman, 1981, and for a review also Fischer and Nijkamp, 1987). Thus so far contributions to <u>dynamic</u> logit models are very scarce. Some examples can be found in studies by Ben-Akiva and De Palma (1986), De Palma and Lefevre (1983), Haag (1986), Leonardi (1983), and Weidlich and Haag (1988). All these authors show that the logit model can be considered as a particular case of a more general Markovian model describing the probability that an individual moves from i to j during an infinitesimal interval of time. In the review article of Fischer and Nijkamp (1987), the authors point out various flaws in the use of dynamic discrete models of choice, such as the problem of structural state dependence, of serial correlation (or spurious state dependence), of multi-actor decisions, and of heuristic choice styles. In addition, the econometrics of dynamic choice modelling (including stationarity and attrition bias) and the transferability of results towards other groups or spatial entities still deserves more attention.

The present paper draws on previous recent contributions to dynamic choice models from the present authors, where they showed the formal equivalence and compatibility between (macro) spatial interaction models and (micro)logit models, in both a static and a dynamic context. In the present paper we will in particular develop a further dynamic extension of a logit model by focusing attention on the conditions under which a dynamic logit model can exhibit chaotic behaviour. Here we will analyse a dynamic logit model - taking as a frame of reference modal split choice in a transportation network - by incorporating dynamic congestion effects. For this purpose a special type of Lotka-Volterra's predator-prey model with time delays will be designed. It will be shown in the paper that under given conditions upon the parameters of the system (in particular when the lag parameter reflecting the influence from the past exceeds a critical value) a so-called Hopf bifurcation - and consequently unstable system's behaviour - may emerge.

2. Dynamic Logit Models and Chaotic Behaviour

Dynamic discrete models of choice may be derived as an extension of static models (see Heckman, 1981). The dynamic logit model can however also emerge as a solution to an optimal control problem whose objective function is a cumulative entropy function. We can easily demonstrate this by taking as an illustration a dynamic system's model for a transportation network in which all origin-destination flows are time dependent (see Nijkamp and Reggiani, 1988a):

$$\max \omega = \int_0^T - T_{ij} \, (\ln T_{ij} - 1) \, dt$$

s.t.

$$\dot{O}_i = \rho_i \, O_i + \delta_i \, (\sum_j T_{ji} - \sum_j T_{ij})$$

$$\sum_j T_{ij} = O_i \tag{2.1}$$

$$\sum_i T_{ij} = D_j$$

$$\sum_i \sum_j T_{ij} \, c_{ij} = C$$

where O_i, i.e. the total volume of flows from origin i, can be regarded as a state variable, whilst the flows T_{ij} may be considered as control variables (e.g. by regulating the network capacity on a given sector from i to j). D_j is supposed to be a certain given attraction indicator for place j; c_{ij} is the unit transportation cost between i and j, and C is the total cost; ρ_i reflects the transition rate of the origin system whilst δ_i denotes the spinoff effects upon i caused by inflows and outflows. The justification and further analysis of problem (2.1) can be found in Nijkamp and Reggiani (1988b; 1989a), where it has also been shown that the solution of (2.1) is a dynamic spatial interaction model of the following type:

$$P_{j,t} = \frac{T_{ij,t}}{O_{i,t}} = \frac{W_{j,t} \exp(-\kappa \, c_{ij,t})}{\sum_j W_{j,t} \exp(-\kappa \, c_{ij,t})} \tag{2.2}$$

where $P_{j,t}$ can be considered as the probability of choosing destination j at time t; $W_{j,t}$ is a weight (balancing) factor and κ a distance decay parameter.

Equation (2.2) can be easily transformed (by supposing for the sake of simplicity $W_{j,t}=1$) into:

$$P_{j,t} = \frac{\exp(u_{j,t})}{\sum_j \exp(u_{j,t})} \tag{2.3}$$

where $u_{j,t}$ represents the utility of moving to j at time t.

Equation (2.3) is clearly the standard form of a logit model in its multi-period generalisation. This is still however, a comparative static model. Given the theoretical framework of (2.3) let us therefore now consider the rate of change of $P_{j,t}$ with respect to time (i.e., $\frac{d\,P_{j,t}}{dt}$):

$$\frac{dP_{j,t}}{dt} = \dot{P}_{j,t} = \frac{d}{dt} \left(\frac{\exp\,(u_{j,t})}{\sum_j \exp\,(u_{j,t})}\right) \qquad (2.4)$$

It is easy to see that expression (2.4) leads to:

$$\dot{P}_j = \dot{u}_j \frac{\exp\,(u_j)}{\sum_j \exp\,(u_j)} \left(1 - \frac{\exp\,(u_j)}{\sum_j \exp\,(u_j)}\right) \qquad (2.5)$$

or:

$$\dot{P}_j = \dot{u}_j\, P_j\, (1-P_j) \qquad (2.6)$$

In (2.5) and (2.6) we have omitted the symbol t for the sake of simplicity. Relationship (2.6) has some very interesting properties, as the right hand side is partly a Verhulst equation. By assuming that the utility of travelling increases linearly with time, through a fixed parameter α, we would have:

$$\dot{u}_j = \text{const} = \alpha \qquad (2.7)$$

so that then the final expression (2.6) becomes:

$$\dot{P}_j = \alpha\, P_j\, (1-P_j) \qquad (2.8)$$

The right hand side of (2.8) is now apparently a Verhulst expression, which is often used in a description of a dynamic system marked by saturation phenomena. Equation (2.8) is then clearly a logistic expression for the derivative of a logit model, under assumption (2.7) (see also Figure 1). The logistic evolutionary model has been adopted in dynamic spatial interaction models among others by Allen and Sanglier (1978) and Wilson (1981). However, the lack of a microeconomic foundation in these models has been criticized (see Haag, 1989). Therefore our result (2.8) is highly interesting, as it represents the dynamic equation of the logit model based on economic random-utility theory.

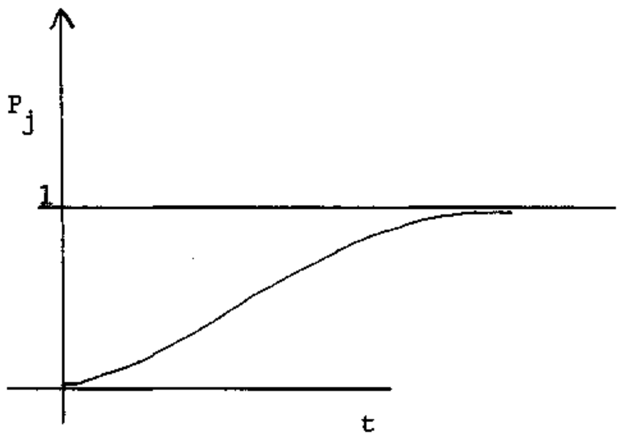

Figure 1. Logistic growth curve of choice probabilities

If we assume now a unit time period (i.e., in discrete terms), then equation (2.8) can be approximated as follows:

$$P_{j,t+1} - P_{j,t} = \alpha P_{j,t} (1-P_{j,t}), \qquad (2.9)$$

while (2.9) can be rearranged in the following form:

$$P_{j,t+1} = (\alpha+1) P_{j,t} - \alpha P_{j,t}^2 \qquad (2.10)$$

Expression (2.10) takes one of the forms of the difference equations that represent logistic growth in a difference equation system (see also Wilson and Bennett, 1985).

We will now compare expression (2.10) with the standard equation for logistic growth, i.e.,

$$X_{t+1} = N X_t (1-X_t) \qquad (2.11)$$

where N is a constant. The equivalent of the constant N in (2.10) is α+1, so that we can rewrite (2.10) as follows:

$$P_{j,t+1} = N P_{j,t} (1 - \frac{N-1}{N} P_{j,t}) \qquad (2.12)$$

It is evident that if we make the transformation $X_{j,t} = P_{j,t} (N-1)/N$, equation (2.12) can be written in the canonical form (2.11) (by following Wilson's (1981) procedure). However, in our case it is interesting to explore whether X_j exceeds 1, in order to avoid unrealistic behaviour in the dynamic logit model. The previous Verhulst dynamic equation

(2.11) was more thoroughly investigated by May (1976). May (1976) showed for system (2.11) that, if:

$$1 < N < 3 \qquad (2.13)$$

X_t tends to a stable equilibrium value as t becomes large. For the range:

$$3 < N < 4 \qquad (2.14)$$

there are various kinds of periodic solutions, but as soon as $N > 3.8495$ the behaviour is oscillatory but chaotic, i.e., with no discernible regularity. Thus our simple dynamic logit model may in principle embody chaotic behaviour as soon as the model is specified in difference equation form (see for a broad review of chaotic models also Nijkamp and Reggiani, 1988c).

Simulation experiments show that sometimes P_j may assume values smaller than 0 or greater than 1. In our model on travel behaviour such negative values would reflect unfeasible system's behaviour. Such a situation would according to Wilson and Bennett (1985) have to be interpreted as 'negative divergence'. In such cases, which may occur for values of N larger than 3.8495 (or for $2.8495 < \alpha < 3$), one would have to switch in simulation experiments to a non-negative value of P_j or to values of P_j<1 (thus causing sudden jumps in the system's trajectory). In the next section the stability of logit results will be investigated by means of various simulation experiments.

3. Simulation Experiments

In this section we will present results of some simulation experiments in order to compare the behaviour of our dynamic logit model (2.12) (in the context of travel behaviour) with the dynamic structure of the well-known May model (2.11) (in the context of the behaviour of biological populations). Two simulations runs will be described.

3.1 Regular motion pattern

In our simulation it will be assumed that $N = 3$ in the equations (2.11) and (2.12), whilst the initial value is $X=P_j=0.1$. The results are presented in Figures 2 and 3.

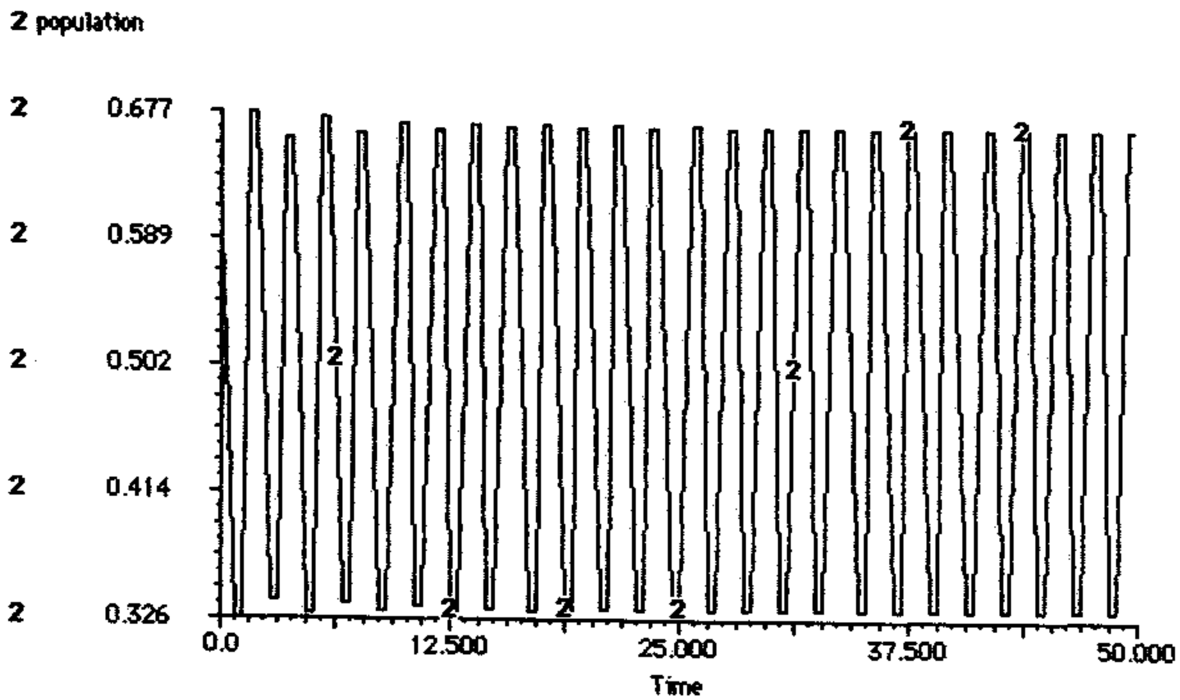

Figure 2. May's law for N = 3

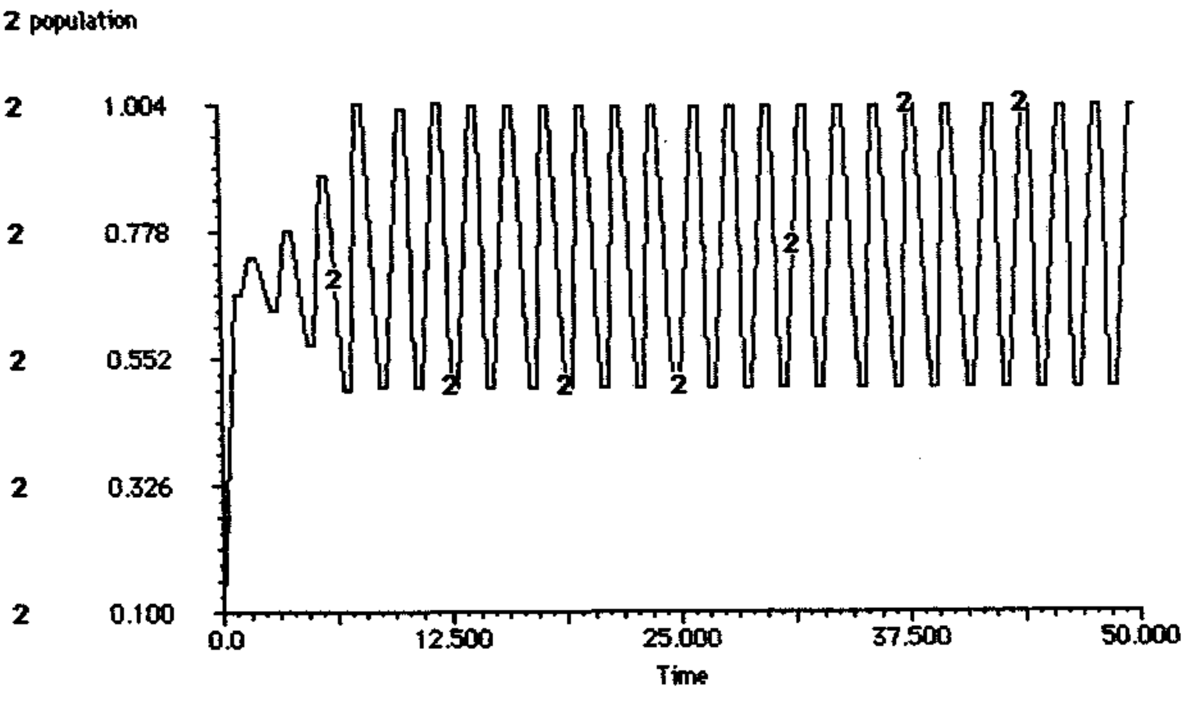

Figure 3. Logit model for N = 3.

It is evident that in both cases a regular pattern emerges. However, the dynamic logit model illustrated in Fig. 3 shows unrealistic values of P_j (in particular $P_j>1$), and therefore other values of N have to be considered.

3.2 Irregular motion pattern

Here we assume, in accordance with our exposition in the previous section, a parameter value N = 3.9. The initial values of the variables are again the same. The results for both types of models can be seen in Figures 4 and 5.

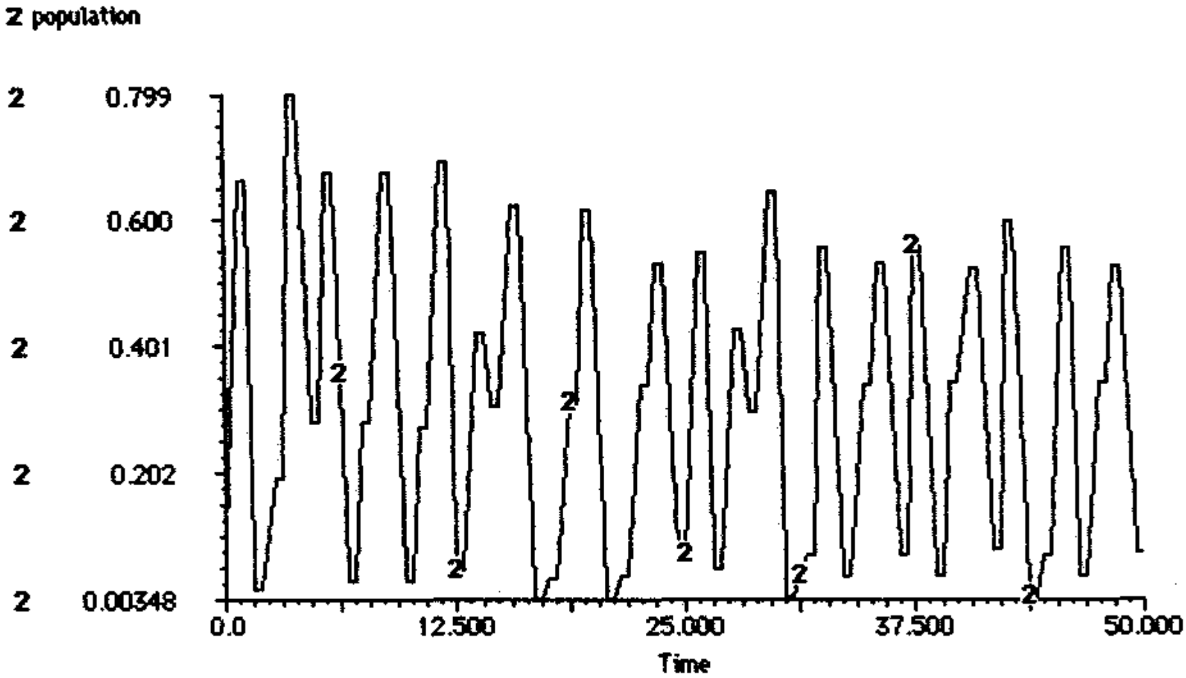

Figure 4. May's law for N = 3.9

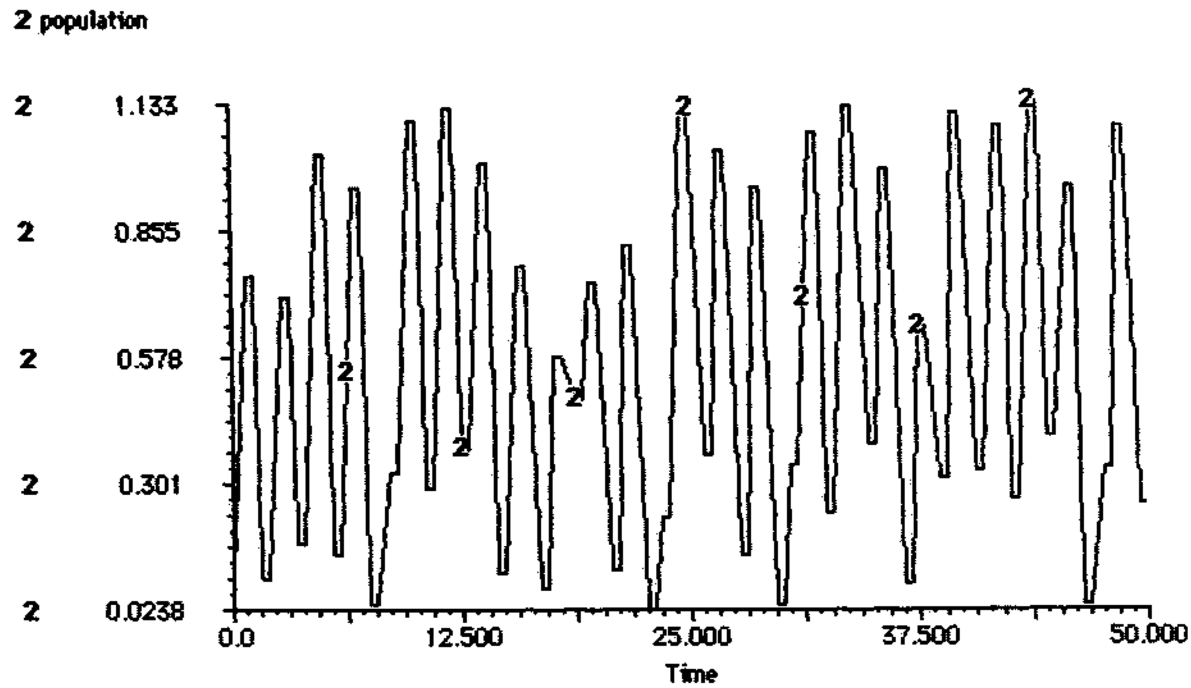

Figure 5. Logit model for $N = 3.9$

It is clear that in both cases an oscillatory, though irregular motion appears, the only difference being the amplitude of the two phenomena. Therefore we can conclude that our dynamic logit model (2.12) belongs to the family of May models; it exhibits chaotic behaviour for certain values of the parameter N in particular for $3.8..<N<4$. However, also for $N>3$, model (2.12) shows values of $P_j>1$ which are less plausible. These values can also be verified by means of the related bifurcation diagrams (see Nijkamp and Reggiani, 1989b). In conclusion, when the marginal utility function α related to the dynamic logit model (2.4) is less than $\simeq 2$ (note $\alpha = N-1$), we have stable solutions in choosing alternative j; when $2<\alpha<3$ unfeasible movements may arise. Consequently, in the latter case the dynamic logit model may result in unrealistic values.

4. Dynamic Logit Models with Time Lags for Congestion Effects

4.1 Introduction

So far we gave assumed away the impact of capacity constraints (except for the implicit inference of a fixed transport cost c_{ij}), but no real congestion phenomenon was included (see also Rietveld, 1988).

For the sake of simplicity we consider now - without loss of generality - a choice situation in which P_j represents the probability of choosing the mode of transport j (instead of the destination like in section 2). Consequently we also assume that the variation of probability is, according to (2.8), not incorporating the impact of congestion.

Congestion can be included in (2.8) by considering a capacity constraint in case of a high travel demand as a specific type of 'predator' which has a negative impact on growth of mobility ('prey') for mode j. Thus capacity limits are assumed to 'absorb' excess mobility. Furthermore we will take into consideration the fact that, for mode j, congestion decreases with a reduction of flows, and increases with mobility.

We can write the previous considerations in a formal way as the following prey-predator type of model:

$$\begin{aligned} \dot{P}_j &= \alpha\, P_j\, (1-P_j) - \beta\, P_j Q_j \\ & \qquad\qquad\qquad\qquad j = 1,\ldots,J \qquad\qquad (4.1) \\ \dot{Q}_j &= -\gamma\, Q_j + \epsilon\, Q_j\, P_j \end{aligned}$$

In (4.1) Q_j represents the congestion phenomena measured by the lack of capacity in the mode of transport j; $\alpha>0$ is the growth rate of mobility in the absence of congestion; $\gamma>0$ is the indigenous lack of congestion in the absence of travellers; $\beta>0$ is the rate at which mobility decreases in mode j, owing to congestion in j; $\epsilon>0$ is the rate at which congestion increases.

System (4.1) is a special type of the well-known Lotka-Volterra model with limited prey (see Volterra, 1931). Clearly system (3.1) has a non-trivial equilibrium point for the following values of P_j and Q_j:

$$P_j^* = \frac{\gamma}{\epsilon} \quad , \quad Q_j^* = \frac{\alpha}{\beta}\,(1 - \frac{\gamma}{\epsilon}) \qquad (4.2)$$

It can be shown that equilibrium point (3.2) is a stable focus (see e.g., MacDonald, 1975; Nijkamp and Reggiani, 1987).

Let us now introduce a time delay in system (4.1) in order to take into account also the case of a non-instantaneous effect of congestion on P_j. This requires the inclusion of time lags in Lotka-Volterra dynamics. Lotka-Volterra equations with delays have been largely discussed in the literature but usually with a time lag incorporated in the predator equation (see e.g., Cushing, 1977; Farkas, 1984; MacDonald, 1976, 1977; Volterra, 1931). It is interesting to observe however, that in our specific case of traffic congestion, we have evidently a delayed effect of Q_j on the prey P_j. Consequently, the first equation has to include past values of Q_j. Therefore we can replace Q_j in the first equation of (3.1) by the following expression:

$$R_j\ (t) = \int_{-\infty}^{t} Q_j\ (\tau)\ G\ (t-\tau)\ d\tau \tag{4.3}$$

where $G(t) = a \exp(-at)$, with $a>0$, is a weight kernel function satisfying the normalisation condition $|G| = 1$, i.e.,

$$\int_{-\infty}^{t} G(t-\tau)\ d\tau = \int_{0}^{\infty} G(s)ds = 1 \tag{4.4}$$

Mathematically, G serves to describe the importance of lag effects before period t (see also Cushing, 1977). Thus the per capita growth of mobility P_j depends on the past history of Q_j.
It should be noted that other forms of G can be used as well (see MacDonald, 1977). The (simple) exponential form of G is easy to handle as we can then easily define the cumulative congestion impact from the past as follows:

$$\dot{R}_j = a\ (Q_j - R_j) \tag{4.5}$$

where 1/a can be interpreted as "a measure of the influence of the past" (see also El-Owaidy and Ammar, 1988).
Therefore, system (4.1) can be replaced by the following three-dimensional ordinary differential system:

$$\begin{aligned} \dot{P}_j &= \alpha\ P_j\ (1-P_j) - \beta\ P_j\ R_j \\ \dot{Q}_j &= -\gamma\ Q_j + \epsilon\ Q_j\ P_j \\ \dot{R}_j &= a\ (Q_j - R_j) \end{aligned} \tag{4.6}$$

where the dot symbol denotes a differentiation of the pertaining variable with respect to t.

4.2 Equilibrium Analysis

In this section we assume that $E_{.}$ (., ., .) represents an expression for equilibrium values pertaining to the system of 3 equations with 3 variables in (4.6). It is now clear that E_0 (0,0,0) and E_1 (1,0,0) are two trivial equilibrium points of a system (4.6).
Then we also find:

$$E_2\ (P_j^*,\ Q_j^*,\ R_j^*) = \left(\frac{\gamma}{\epsilon}\ ;\ \frac{\alpha}{\beta}\left(1 - \frac{\gamma}{\epsilon}\right)\ ;\ \frac{\alpha}{\beta}\left(1 - \frac{\gamma}{\epsilon}\right)\right) \qquad (4.7)$$

E_2 is an equilibrium point in the positive octant iff

$$\frac{\gamma}{\epsilon} < 1 \qquad (4.8)$$

The variational matrix (i.e., the Jacobian system of first-order derivatives) of system (4.6) is:

$$V\ (P_j,\ Q_j,\ R_j) = \begin{bmatrix} \alpha - 2\alpha\ P_j - \beta\ R_j & 0 & -\beta\ P_j \\ \epsilon\ Q_j & -\gamma + \epsilon P_j & 0 \\ 0 & a & -a \end{bmatrix} \qquad (4.9)$$

Then we can easily derive that matrix (4.9) for the equilibrium point E_2 is equal to:

$$V_2 = \begin{bmatrix} -\alpha\frac{\gamma}{\epsilon} & 0 & -\beta\frac{\gamma}{\epsilon} \\ \frac{\epsilon\alpha}{\beta} - \frac{\alpha\gamma}{\beta} & 0 & 0 \\ 0 & a & -a \end{bmatrix} \qquad (4.10)$$

V_2 has the following characteristic equation:

$$\lambda^3 + \left(a + \alpha\frac{\gamma}{\epsilon}\right)\lambda^2 + a\ \alpha\frac{\gamma}{\epsilon}\ \lambda + a\ \alpha\gamma - a\ \alpha\frac{\gamma^2}{\epsilon} = 0 \qquad (4.11)$$

By using the Routh-Hurwitz criteria (see Gandolfo, 1983), we know that E_2 is asymptotically stable if (4.8) holds and

$$a > \frac{\epsilon^2 - \gamma\epsilon - \alpha\gamma}{\epsilon} \qquad (4.12)$$

The critical value

$$a^{*} = \frac{\epsilon^{2} - \gamma\epsilon - \alpha\gamma}{\epsilon} \tag{4.13}$$

is clearly positive under the condition:

$$\epsilon > \frac{\gamma + \sqrt{\gamma^{2} + 4\,\alpha\,\gamma}}{2} \tag{4.14}$$

By following Marsden and McCracken (1976) it can also be shown that at a^{*} a Hopf bifurcation (i.e. an unstable equilibrium point surrounded by a stable limit cycle) takes place. It means that the time delay has a destabilizing effect on system (4.1): in particular the stability of system (4.1) is lost when the measure of the influence of the past (i.e., $\frac{1}{a}$) surpasses the positive value $\frac{1}{a^{*}}$, i.e., when the influence of the past increases.

Having achieved this plausible result, a further step could be to examine whether the Hopf bifurcation at a^{*} is supercritical (i.e., the point loses its stability by expelling a stable periodic orbit) or subcritical (i.e., the point loses its stability by absorbing a non-stable periodic orbit) (see Sparrow, 1982). If the bifurcation is subcritical a chaotic behaviour of the 'Lorenz type' may emerge (see again Marsden and McCracken, 1976) in our dynamic system (4.1). Such phenomena would require much more thorough research. In a final section we will present various simulation results for our dynamic congestion model.

5. Simulation Experiments

In this section some results from various simulation experiments, related to the previous section on congestion effects, will be described. In particular it will be shown how the parameter a, related to the time delay, influences the stability of system (4.6). Especially when a surpasses the critical value a^{*}, the system becomes stable; when a decreases (below a^{*}), we have a destabilizing effect due to the increased influence of the past.

For this simulation the following parameter values will be assumed:

$$\gamma = 0.1 \qquad \alpha = 2.4 \qquad \beta = 0.5 \tag{5.1}$$

When according to condition (4.14) we choose:

$$\epsilon = 0.75 \tag{5.2}$$

the following critical value a^* results from (4.13):

$$a^* = 0.33 \tag{5.3}$$

The initial values are assumed to be:

$$P_j = Q_j = R_j = 0.4 \tag{5.4}$$

The results for different values of a are printed in Figures 6, 7, 8, 9 and 10. After the reference pattern (Figure 6), two simulation runs for both increasing and decreasing values of a will be presented, respectively.

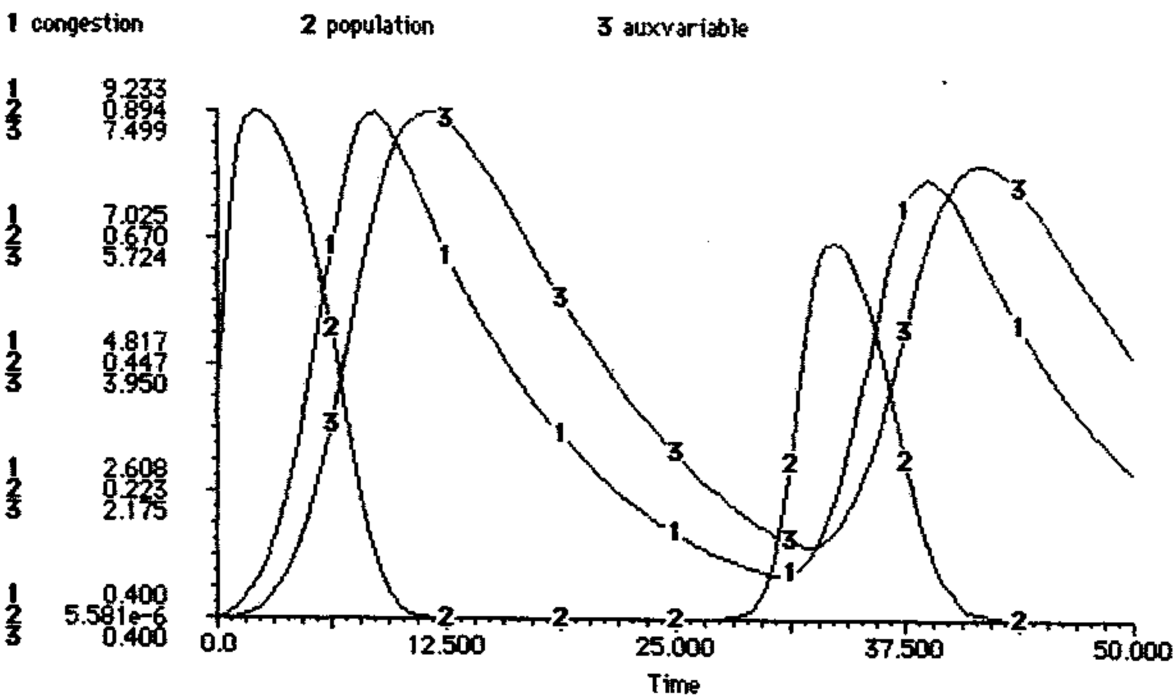

Figure 6. Results for $a = a^* = 0.33$

(a) Simulations for increasing values of $a > a^*$

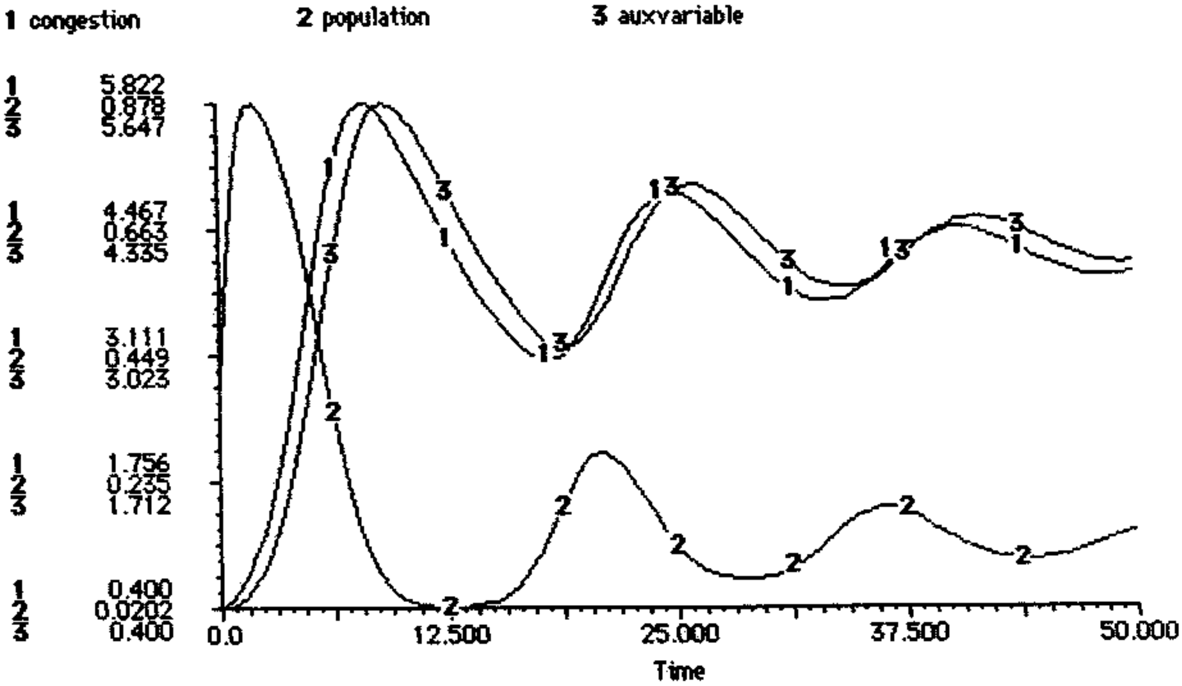

Figure 7. Results for a = 1.

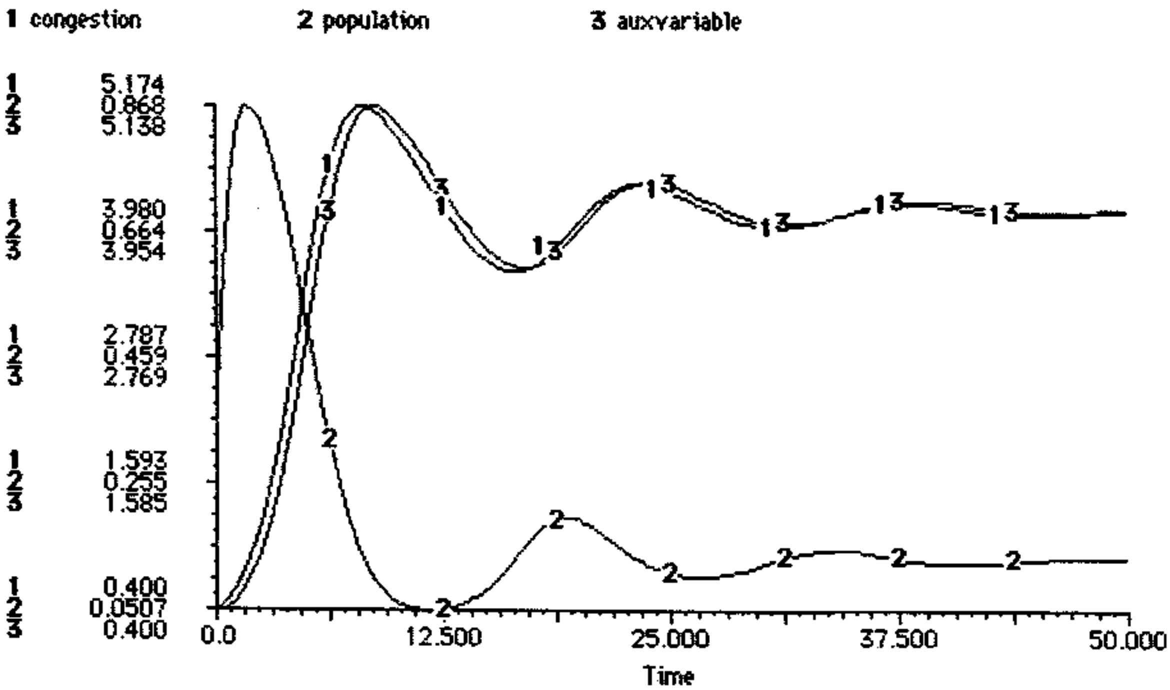

Figure 8. Results for a = 1.8

(b) Simulations for decreasing values of $a < a^*$

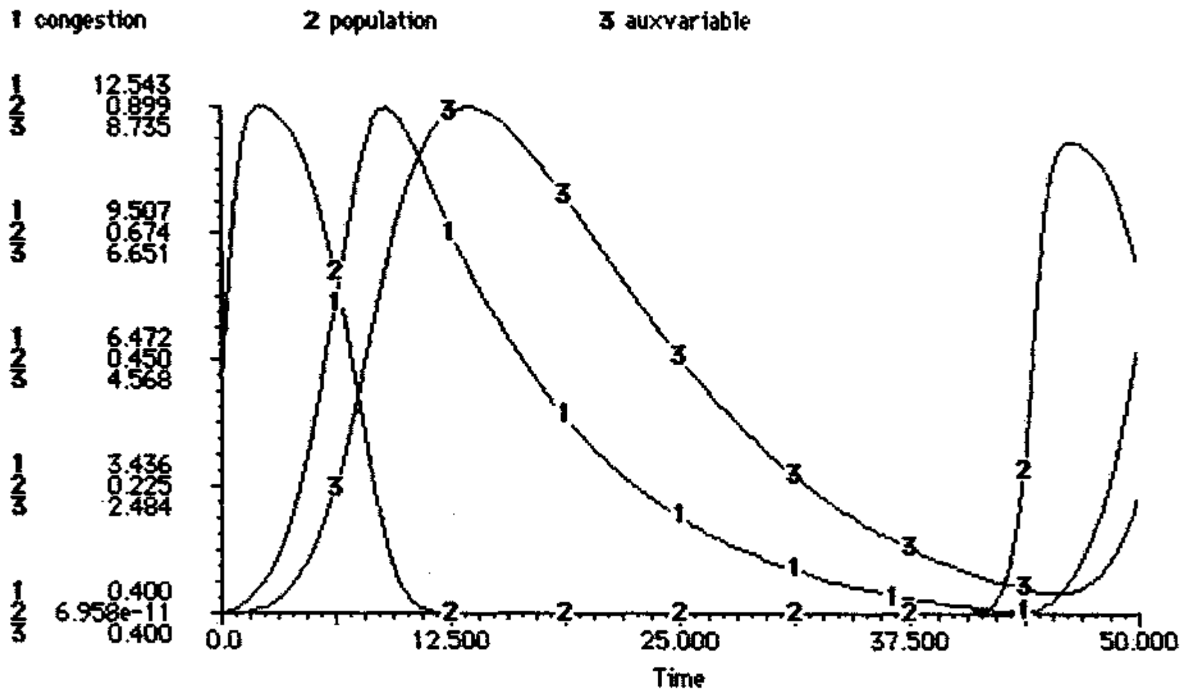

Figure 9. Results for a = 0.2

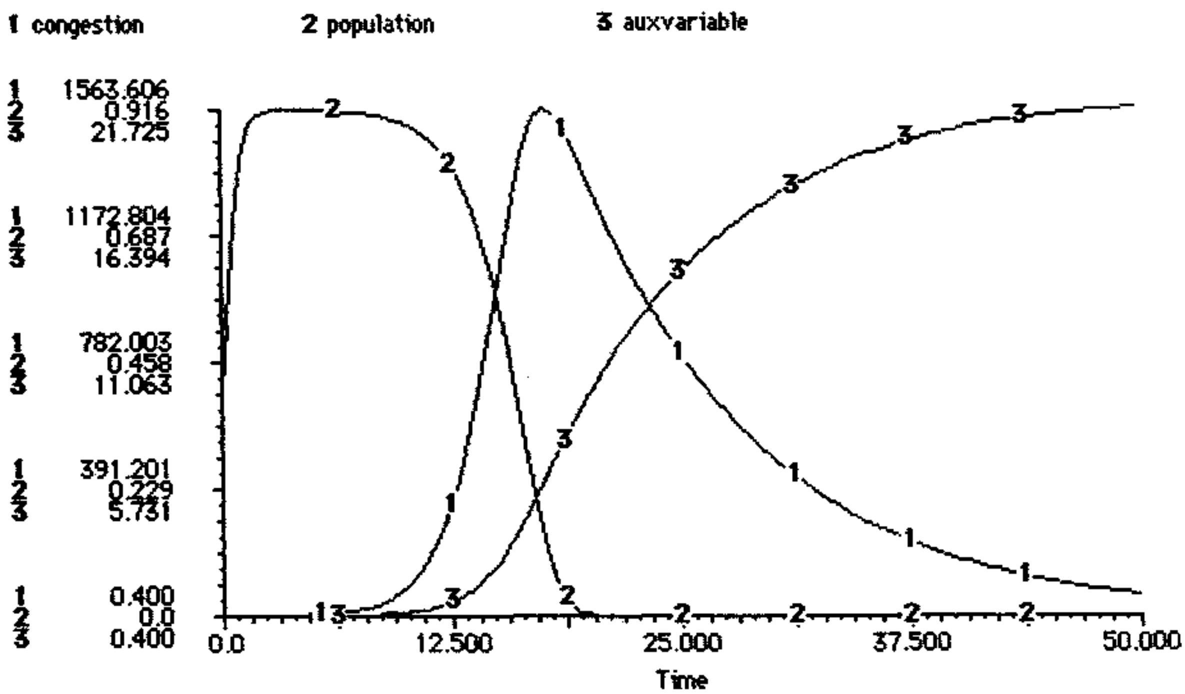

Figure 10. Results for a = 0.001

It should be noted that in the second case (Figure 10) we have chosen a very long time delay ($a \to 0$) which is in general less meaningful (see also MacDonald, p.321), but which shows clearly the formal behaviour of the model.

Thus our simulation results clarify the theoretical observations made in section 4. Other simulation experiments with different parameter values invariably show that time delays can induce instability: in particular under certain conditions imposed upon the parameters, a Hopf bifurcation takes places at the critical value a^* and the amplitude of the population oscillations increases for values of the parameter a below a^*.

6. Conclusions

In this paper the link between chaos theory and spatial interaction has been examined.

Firstly, it has been shown that under certain conditions a dynamic logit model - in the context of mobility flows - may exhibit irregular motion. Secondly, the dynamic logit model can be incorporated in a more general spatial system, in particular a Lotka-Volterra system with time lags, which takes into account non-instantaneous congestion phenomena. It has been shown that time delay has a destabilizing effect on the previous system, in particular when the influence of the past exceeds a critical value, at which a Hopf bifurcation emerges. Finally, simulation experiments illustrate the above mentioned theoretical observations.

The previous expositions on the behaviour of relatively simple dynamic non-linear models appear to embody a wealth of new theoretical insights. In a practical context, such models have so far hardly been tested because of lack of time series data. This is a major flaw in current dynamic transportation modeling. Clearly, current attempts at dealing with longitudinal data sets may remove part of these shortcomings, but there is still a clear need for more rigorous research endeavours in this field. For the time being, simulation experiments have to be used as substitutes for missing information. Nevertheless, also such simulation models demonstrate clearly that the presence of a chaotic structure in a dynamic system may generate unforeseeable system's behaviour. Thus our results also incorporate caveats concerning the use of straightforward extrapolation and prediction methods in case of non-linear dynamic models.

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