1989-80

SERIE RESEARCH MEMORANDA

SOLUTION OF GENERAL EQUILIBRIUM MODELS WITH NONCONVEX TECHNOLOGIES

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Research Memorandum 1989-88

December 1989

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AMSTERDAM

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Abstract

In this paper a procedure is presented to compute equilibria in a general equilibrium model with nonconvex technologies. For this purpose a general equilibrium model is formulated which makes direct application of a simplicial (fixed point) algorithm possible. Besides giving a computational procedure the algorithm also gives rise to an existence proof.

Acknowledgement

The authors wish to thank Dolf Talman for his helpful comments.

1. Introduction

A lot of research has been done on proving the existence of equilibria in models where production exhibits increasing returns to scale (see for example Bonnisseau (1987), Bonnisseau and Cornet (1987), and Kamiya (1988)). For various equilibrium concepts the existence of such equilibria has indeed been proved under rather mild assumptions. However, the computation of such equilibria has not received much attention yet. In Kamiya (1986) a procedure is proposed which is based on the well-known fixed point algorithm as originated by Scarf.

In Kamiya's algorithm, a kind of two step procedure is executed. In the first step a homotopy path is followed ending with a set of production vectors such that all firms set the same prices and in the second step a homotopy path is followed starting with the end point of the homotopy path of the first step and ending with an equilibrium.

In this paper we propose an alternative procedure. In our procedure we use the variable dimension restart algorithm on $S^n \times R^m_+$ as described in Hofkes (1989). A function is formulated which is such that a nonlinear complementarity point of this function corresponds with an equilibrium of a general equilibrium model with nonconvex technologies. The market prices and individual inputs, which serve directly to compute the function value, are simultaneously adjusted, ending up with a marginal pricing equilibrium. Furthermore, for a certain class of models we do not need assumption A2(ii) of Kamiya (1986). This assumption says that the pricing rule must guarantee the profits per unit of production to be nonnegative when the scale of production becomes infinite and consequently rules out production sets like $y = x^2$ which are allowed in our model.

The above described approach seems to be very natural and elegant. Besides giving a computational procedure our algorithm also gives rise to an alternative existence proof. One of the conditions for the existence of an equilibrium is that the above mentioned function has to be upper semi continuous, which corresponds with a condition of upper semi continuity of the producer's pricing rules. It must be noted that both marginal cost pricing and average cost pricing satisfy this condition under the (standard) assumptions on production (see also Cornet (1989)). The producer's pricing rules we will use will be such that the standard first order conditions for optimality are met. In the increasing returns to scale case, now, losses can occur, which will be covered by the consumers who are supposed to be the owners of the firms. The efficiency of these equilibria will not be considered in this paper, but it is clear that there exist efficient equilibria at which some firms may have deficits (see also Brown and Heal (1979) who give conditions which ensure that there exists at least one efficient equilibrium).

This paper is organized as follows. Section 2 describes the model. In section 3 the algorithm is exposed. Finally, section 4 gives some concluding remarks.

2. The model

In this section we will expose the model. Let us assume that we have an economy with m consumers, n producers, and k commodities. The vector of market prices of the k commodities is denoted by $p \in R^k_+\backslash\{0\}$. Consumer i has a consumption set $X^i \subset R^k_+$, a utility function $u^i(x^i)$ on X^i, and initial endowments $w^i \in R^k_+$, $i=1,..,m$. Firm j has a production possibilities set $Y^j \subset R^k$, where inputs are measured negatively and outputs are measured positively, and sets prices according to a pricing rule $\pi^j(y^j,p)$ for each production vector on the boundary of the production possibilities set ($y^j \in \partial Y^j$) and a vector of market prices $p \in R^k_+\backslash\{0\}$. Furthermore, revenues (or losses) accrue to the consumers according to some (fixed) distribution rule $r^i(p,y^1,\ldots,y^n) = \Sigma_j\vartheta_{ij}p.y^j$, $i=1,\ldots,m$, with $\Sigma_i\vartheta_{ij} = 1$ for all j. (The inner product of two vectors x and y, $x,y \in R^\ell$, is denoted by $x.y = \Sigma_i x_i y_i$). Finally, we define O_h as the set of all firms which have commodity h as output and I_h as the set of all firms which use commodity h as input. We will use the following assumptions on production.

<u>Assumption 2.1</u>

(i) Y^j is closed and contains 0, $j=1,\ldots,n$.

(ii) The free disposal assumption holds, i.e. $Y^j - R^k_+ \subset Y^j$, $j=1,\ldots,n$.

(iii) There is no free production, i.e. $Y^j \cap R^k_+ = \{0\}$, $j=1,\ldots,n$.

(iv) The irreversibility assumption holds, i.e. $\Sigma Y^j \cap (-\Sigma Y^j) = \{0\}$.

Note that we do not require that the (individual) production possibilities sets have a smooth boundary. In order to be able to define marginal pricing even at (inward) kinks of the production possibilities sets we have to generalize the notion of marginal rates of transformation. For this purpose we will use the concepts of tangent cones and cones of normals as developed by Clarke (1975) (see also Cornet (1989)).

Definition 2.2

The Clarke tangent cone of $Y \subset R^k$ at $y \in Y$, $T_C(Y,y)$, is the set $\{x \in R^k \mid$ for any sequence (t^h,y^h) in $R_+ \times Y$ with $t^h>0$ and tending to $(0,y)$, there exists a sequence (x^h) tending to x such that $(y^h+t^hx^h) \in Y$ for large enough $h\}$.

Definition 2.3

The Clarke normal cone is the polar cone to the Clarke tangent cone, i.e. the set $N_C(Y,y) = \{ x \in R^k \mid x.z \le 0 \text{ for all } z \in T_C(Y,y)\}$.

Note that $T_C(Y,y)$ is identical to the tangent cone originally defined by Clarke (see Brown et al. (1986)). If Y is a C^1-manifold, then $N_C(Y,y)$ corresponds with the usual space of normals at y. Furthermore, $T_C(Y,y)$ is nonempty and convex (see Brown et al. (1986)). Finally, Cornet (1989) proves that under assumptions 2.1(i) and 2.1(ii) $N_C(Y^j,y^j) \subset R^k_+$ and $N_C(Y^j,y^j) \neq \{0\}$ for all $y^j \in \partial Y^j$.

We will use the following assumptions on consumption.

Assumption 2.4

(i) X^i, $i=1,\dots,m$, is a closed, convex subset of R^k_+, containing 0.

(ii) $u^i(x^i)$ is a quasi-concave function, $i=1,\dots,m$.

(iii) $u^i(x^i)$ satisfies local nonsatiation, i.e. for all $x^i \in X^i$, for all $\epsilon > 0$, there is a consumption bundle $x \in X^i \cap B(x^i,\epsilon)$ such that $u^i(x^i) < u^i(x)$, $i=1,\dots,m$.

(iv) $w = \Sigma_i w^i > 0$.

(v) For all $p \in R^k_+\backslash\{0\}$, for all $y^j \in \partial Y^j$, $j=1,\dots,n$, if $p \in N_C(Y^j,y^j)$ for $j=1,\dots,n$, then $p.w^i + r^i(p,y^1,\dots,y^n) > 0$ for $i=1,\dots,m$.

Assumption 2.1 and assumptions 2.4(i) - 2.4(iv) are the standard assumptions on production and consumption. Assumption 2.4(v) needs further clarification. This assumption amounts to saying that whenever there is a price-equilibrium, i.e. all producers set the same prices, income of each consumer should be positive. In other words, in each price-equilibrium, which can be seen as a potential equilibrium of the economy, the shares each consumer has in the losses and profits of the firms cannot be that large that his income becomes negative.

Definition 2.5

A marginal pricing equilibrium is an (m+n+1)-tuple $((x^{i*})$ $i=1,\ldots,m$, (y^{j*}) $j=1,\ldots,n$, $p^*)$ such that:

(i) for all $i = 1,\ldots,m$, x^{i*} maximizes $u^i(x^i)$ on X^i subject to the budget constraint $p^*.x^i \le p^*.w^i + r^i(p^*,y^{1*},\ldots,y^{n*})$.

(ii) for all $j = 1,\ldots,n$, $y^{j*} \in \partial Y^j$ and $p^* \in N_C(Y^j,y^{j*})$.

(iii) $\Sigma_i x^{i*}{}_h \le \Sigma_j y^{j*}{}_h + w_h \perp p^*{}_h \ge 0$, $h = 1,\ldots,k$.

($a \le c \perp b \ge d$ means $a \le c$, $b \ge d$ and $(a-c)(b-d) = 0$)

Finally we make the following assumption.

Assumption 2.6

(i) For all j, there is just one h such that $j \in O_h$.

(ii) If $I_h \neq \emptyset$ then $O_h = \emptyset$ and if $O_h \neq \emptyset$ then $I_h = \emptyset$, $h=1,\ldots,k$.

Assumption 2.6 (i) says that each firm has just one output. Assumption 2.6 (ii) says that we consider an economy which has no intermediate goods. The level of aggregation is such that all intermediate goods industries drop out. Now, let the number of inputs in the economy be given by κ ($\le k-1$) and the number of outputs by $(k-\kappa)$. Furthermore, all commodities are indexed such that the first κ are inputs and the last $(k-\kappa)$ are the outputs of the economy. So the outputs are indexed by $i = \kappa+1,\ldots,k$. Finally, note that $N_C(Y^j,y^j)$ is such that for some $u \in N_C(Y^j,y^j)$, $\exists i \in \{h|j\in O_h$ or $j\in I_h\}$ such that $u_i > 0$.

Let us now define the pricing rule firm j is supposed to follow. The market price of the output will serve as a normalization factor for the individual producer prices of the inputs. Given a vector of market prices p, $p \in S^{k-1}$, with S^{k-1} the k-dimensional unit simplex given by $S^{k-1} = \{x \in R^k{}_+ |\ \Sigma x_i=1,\ x_i\ge 0,\ i=1,\ldots,k\}$, let $B(p) = \{p' \in R^k{}_+ |\ p'_i = p_i,\ i = \kappa+1,\ldots,k\}$.

<u>Definition 2.7</u>
The mapping $\pi^j : \partial Y^j \times S^{k-1} \rightarrow R^k$ is called a <u>marginal pricing rule</u> if

$$\pi^j(y^j,p) = \text{trunc}(N_C(Y^j,y^j) \cap B(p)) \text{ if } N_C(Y^j,y^j) \cap B(p) \neq \emptyset,$$
$$\pi^j(y^j,p) = \{p' \in R^k_+ \mid p'_i = p_i \text{ for } i = \kappa+1,\ldots,k, \quad p'_h = 1 \text{ if } \exists\, u \in N_C(Y^j,y^j) : u_h > 0,\ j \in I_h, \quad p'_h = 0 \text{ if } \forall\, u \in N_C(Y^j,y^j) : u_h = 0,\ j \in I_h\} \quad \text{if } N_C(Y^j,y^j) \cap B(p) = \emptyset.$$

where trunc(A) means that for every <u>a</u> $\in$ A each component of <u>a</u> larger than 1 is set equal to 1.

According to definition 2.7 the individual prices the firm sets for the output commodities will per definition be equal to the market prices of these commodities, i.e. for all $p^j \in \pi^j(y^j,p)$, $p^j_i = p_i$ (≤ 1), $i=\kappa+1, \ldots, k$. The individual input prices $(p^j_h,\ j \in I_h)$ then follow from the pricing rule. The closed set B(p) serves to normalize the input prices of firm j with respect to the market price of firm j's output, as the normal cone just gives the price ratios. The individual input prices are truncated at one when the input/output price ratio is larger than one. Whenever $N_C(Y^j,y^j) \cap B(p) = \emptyset$, which actually means that at least one of the input/output price ratios is infinite, all the input prices for which this ratio is indeed infinite are set equal to one, while all the other input prices are equal to zero. The second part of the definition just serves to ensure that $\pi^j(y^j,p)$ is upper semi continuous. However, as we will see in the sequel, in equilibrium the first part of the definition will always be operative.

Now, for $q^j \in R^\kappa_+$ a vector denoting the amount of inputs used by firm j, we define the following transformation to get an efficient corresponding production vector, i.e. $y^j = y^j(q^j) \in \partial Y^j$:

$$y^j_h = -q^j_h \text{ if } I_h \neq \emptyset,$$
$$y^j_h = 0 \text{ if } I_h = \emptyset \text{ and } j \notin O_h,$$
$$y^j_h = \max\{\tilde{y}^j_h \mid \tilde{y}^j \in Y^j,\ \tilde{y}^j_i = y^j_i \text{ if } j \notin O_i\} \text{ if } j \in O_h.$$

So, for each vector of inputs we have a corresponding production vector $y^j(q^j)$ on the boundary of the production set. In the following $y^j = y^j(q^j)$. Finally, let $p = (p_I^T, p_O^T)^T$ be a vector of prices, with p_I the κ-dimensional vector of input prices and p_O the $(k-\kappa)$-dimensional vector of output prices. Then, for $j = 1,\ldots,n$, the mapping $\tilde{\gamma}^j : S^{k-1} \times (R^\kappa_+)^n \to R^\kappa_+$ is defined by:

$$\tilde{\gamma}^j(p,q^1,\ldots,q^n) = \{\tilde{p}_I \in R^\kappa_+ | ((\tilde{p}_I^T, p_O^T)^T \in \pi^j(y^j,p)\}.$$

So, $\tilde{\gamma}^j(p,q^1,\ldots,q^n)$ restricts the price vectors in the set $\pi^j(y^j,p)$ to the vectors of individual input prices of firm j.

Now, let the mapping $\phi : S^{k-1} \times (R^\kappa_+)^n \to R^{k+\kappa n}$ be given by:

$$\phi(p,q^1,\ldots,q^n) = \begin{pmatrix} \zeta(p,q^1,\ldots,q^n) \\ \gamma^1(p,q^1,\ldots,q^n) \\ \cdot \\ \cdot \\ \cdot \\ \gamma^n(p,q^1,\ldots,q^n) \end{pmatrix},$$

where

$$\zeta(p,q^1,\ldots,q^n) = \Sigma_i d^i(p,y^1,\ldots,y^n) - \{\Sigma_j y^j\} - \{w\}$$

and

$$\gamma^j(p,q^1,\ldots,q^n) = \tilde{\gamma}^j(p,q^1,\ldots,q^n) - \{p_I\}, \quad j=1,\ldots,n,$$

with

$d^i(p,y^1,\ldots,y^n)$ such that:

if $p.w + \Sigma_j p.y^j < 0$ then
$d^i(p,y^1,\ldots,y^n) = \{0\}$ for all i

if $p.w + \Sigma_j p.y^j = 0$ then
$d^i(p,y^1,\ldots,y^n) = \{0\}$ if $p.w^i + r^i(p,y^1,\ldots,y^n) < 0$ and
$d^i(p,y^1,\ldots,y^n) = \{x^i \in X^i |\ p.x^i \le 0\}$ if $p.w^i + r^i(p,y^1,\ldots,y^n) \ge 0$

if $p.w + \Sigma_j p.y^j > 0$ then

$d^i(p,y^1,\ldots,y^n) = \{0\}$ if $p.w^i + r^i(p,y^1,\ldots,y^n) < 0$ and

$d^i(p,y^1,\ldots,y^n) = \{x^i \in \tilde{X}^i \mid p.x^i \le 0\}$ if $p.w^i + r^i(p,y^1,\ldots,y^n) = 0$ and

$d^i(p,y^1,\ldots,y^n) = \{\text{argmax } u^i(x^i) \mid x^i \in \tilde{X}^i,\ p.x^i \le \chi^i\}$ with

$$\chi^i = \frac{p.w^i + r^i(p,y^1,\ldots,y^n)}{\Sigma_{i\in I^+}(p.w^i + r^i(p,y^1,\ldots,y^n))}(p.w + \Sigma_j p.y^j), \text{ if } p.w^i + r^i(p,y^1,\ldots,y^n) > 0$$

where I^+ is the set of consumers with income greater than zero and where $\tilde{X}^i = F \cap X^i$ with F some compact convex set which contains $\hat{X}^i$ in its interior, with $\hat{X}^i$ the projection of the set of attainable states $A = \{(x^i),(y^j) \mid \Sigma_i x^i - \Sigma_j y^j \le w,\ x^i \in X^i,\ y^j \in Y^j\}$ on X^i.

Now ϕ is upper semi continuous, since ζ is upper semi continuous (see e.g. Kamiya (1988)) and γ^j, $j=1,\ldots,n$, is upper semi continuous by the upper semi continuity of $\pi^j(y^j,p)$.

<u>Definition 2.8</u>

Let ψ be a mapping from (a subset of) R^ℓ_+ into R^ℓ. $(x^*,f^*) \in R^\ell_+ \times R^\ell$ is called a <u>nonlinear complementarity pair</u> of ψ when $f^* \in \psi(x^*)$, $f^*_i = 0$ if $x^*_i > 0$, and $f^*_i \le 0$ if $x^*_i = 0$.

<u>Theorem 2.9</u>

A nonlinear complementarity pair of ϕ yields a marginal pricing equilibrium.

<u>Proof</u>

Let $((p^{*T},q^{1*T},\ldots,q^{n*T})^T,(z^{*T},c^{1*T},\ldots,c^{n*T})^T)$ be a nonlinear complementarity pair of ϕ, with $z^* = \Sigma_i x^{i*} - \Sigma_j y^{j*} - w$, with $x^{i*} \in d^i(p^*,y^{1*},\ldots,y^{n*})$, $i=1,\ldots,m$, and $c^{j*} = p^{j*} - p_I^*$, with $p^{j*} \in \tilde{\gamma}^j(p^*,q^{1*},\ldots,q^{n*})$, $j=1,\ldots,n$. By definition 2.8, $z^*_h \le 0 \perp p^*_h \ge 0$, so condition (iii) of definition 2.5 is satisfied. Furthermore, $c^{j*}_h = p^{j*}_h - p^*_h \le 0$, $j=1,\ldots,n$, $h=1,\ldots,\kappa$. Now, either $N_C(Y^j,y^{j*}) \cap B(p^*) = \emptyset$ or $N_C(Y^j,y^{j*}) \cap B(p^*) \ne \emptyset$. As for each j there is exactly one i such that $j \in O_i$ we can only have that $N_C(Y^j,y^{j*}) \cap B(p^*) = \emptyset$ if for all $u \in N_C(Y^j,y^{j*})$: $u_i = 0$ and $p^*_i > 0$ for $j \in O_i$. But then there is at least one h, $j \in I_h$, with $u_h > 0$ for some $u \in N_C(Y^j,y^{j*})$. So, for this h: $p^{j*}_h = 1$. Now, we have a contradiction as $p^*_i > 0$ and $p^*_h \ge p^{j*}_h = 1$, so $p^*_i + p^*_h > 1$, while $p^* \in S^{k-1}$. So we must have that $N_C(Y^j,y^{j*}) \cap B(p^*) \ne \emptyset$. But then $\pi^j(y^{j*},p^*) = \text{trunc}(N_C(Y^j,y^{j*}) \cap B(p^*))$

and consequently $(p^{j*T}, p_0^{*T})^T \in N_C(Y^j, y^{j*})$. Furthermore, for all $p^j \in \pi^j(y^j, p)$, $p^j_i = p_i$ for $i=\kappa+1,\ldots,k$, and for $h=1,\ldots,\kappa$ we have $p^*_h = p^{j*}_h$ if $q^{j*}_h > 0$ and if $q^{j*}_h = 0$ then $p^{j*}_h - p^*_h \leq 0$, $j=1,\ldots,n$. However, if $p^{j*}_h \leq p^*_h$ and $q^{j*}_h = 0$ for some h, then $p^* \in N_C(Y^j, y^{j*})$ and condition (ii) of definition 2.5 is satisfied.

Finally, $p^* \in N_C(Y^j, y^{j*})$ and we have by the survival assumption that $p^*.w^i + r^i(p^*, y^{1*}, \ldots, y^{n*}) > 0$ for all i and hence that $\chi^i = p^*.w^i + r^i(p^*, y^{1*}, \ldots, y^{n*})$ for all i. But then $d^i(p^*, y^{1*}, \ldots, y^{n*}) = \{\text{argmax } u^i(x^i) | x^i \in X^i, p^*.x^i \leq p^*.w^i + r^i(p^*, y^{1*}, \ldots, y^{n*})\}$ and since $z^* \leq 0$ we have $x^{i*} \in \hat{X}^i$ and $x^{i*} \in \{\text{argmax } u^i(x^i) | x^i \in X^i, p^*.x^i \leq p^*.w^i + r^i(p^*, y^{1*}, \ldots, y^{n*})\}$ (see also Debreu (1959)), so condition (i) of definition 2.5 is satisfied.

□

According to theorem 2.9 a nonlinear complementarity pair of ϕ corresponds with a marginal pricing equilibrium. Note that when ζ satisfies the weak desirability assumption for $p \in \pi^j(y^j, p)$, i.e $z_h \geq 0$ if $p_h = 0$ for all $z \in \zeta$, then each nonlinear complementarity pair yields an equilibrium with $z^*_h = 0$ for all h. In section 3 we will describe the algorithm which will enable us to find a nonlinear complementarity pair of ϕ.

3. The algorithm

The algorithm we will use is a simplicial variable dimension restart algorithm which operates in a simplicial subdivision of $S^{k-1} \times (R^\kappa_+)^n$ (for a detailed description see Hofkes (1989)). S^{k-1} represents the space of the market prices which are normalized as to sum to one and $(R^\kappa_+)^n$ represents the space of inputs of the n firms. Each firm's maximum/efficient output is uniquely determined by the value of its inputs. Let $q = (q^1, \ldots, q^n)$, with $q^j \in R^\kappa_+$ the vector of inputs of firm j. So, $q \in (R^\kappa_+)^n$.

A variable dimension restart algorithm is such that it generates a sequence of adjacent simplices of varying dimension, starting with a zero-dimensional simplex (the starting point which can be chosen arbitrarily) and ending within a finite number of steps with an approximating simplex, i.e. a simplex which yields an approximate solution.

Let $\sigma(w^1,\ldots,w^{t+1})$ denote a t-dimensional simplex in the simplicial subdivision of $S^{k-1} \times (R^{\kappa}_{+})^n$ with vertices $w^g = (p^g,q^g)$, $g=1,\ldots,t+1$. A piecewise linear approximation of ζ with respect to the simplicial subdivision of $S^{k-1} \times (R^{\kappa}_{+})^n$ is a function given by $Z(p,q) = \Sigma_g\lambda_g.\underline{z}(p^g,q^g)$ for (p,q) a point in some t-simplex $\sigma(w^1,\ldots,w^{t+1})$ given by $(p,q) = \Sigma_g\lambda_g.(p^g,q^g)$ and with $\underline{z}(p^g,q^g)$ some arbitrarily chosen value of $\zeta(p^g,q^g)$. Analogously, C^j is a piecewise linear approximation of γ^j given by $C^j(p,q) = \Sigma_g\lambda_g.\underline{c}^j(p^g,q^g)$ with $\underline{c}^j(p^g,q^g)$ some arbitrarily chosen value of $\gamma^j(p^g,q^g)$.

Now, for a subset T_1 of $\{1,\ldots,k\}$ with $|T_1| = t_1$, a subset T_2 of $\{k+1,\ldots,k+\kappa n)\}$ with $|T_2| = t_2$, and with $T=T_1\cup T_2$, $|T|=t=t_1+t_2$, a t-dimensional simplex $\sigma(w^1,\ldots,w^{t+1})$ is almost-complete if the system of $k+\kappa n+1$ linear equations:

$$\sum_{g=1}^{t+1} \lambda_g \begin{pmatrix} \underline{z}(w^g) \\ \underline{c}(w^g) \\ 1 \end{pmatrix} + \sum_{r\notin T} \mu_r \begin{pmatrix} e(r) \\ 0 \end{pmatrix} - \beta \begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (3.1)$$

has a solution $(\lambda^*,\mu^*,\beta^*)$ with $\lambda^*_g \geq 0$, $g = 1,\ldots,t+1$, $\mu^*_r \geq 0$, $r \notin T$, where $e(r)$ is the r-th $k+\kappa n$ dimensional unit vector, e is an $k+\kappa n$ dimensional-vector of ones, and $\underline{c}(w^g)=(\underline{c}^{1\top}(w^g),\ldots,\underline{c}^{n\top}(w^g))^\top$.

Observe that the number of variables λ^*,μ^*,β^* equals $t+1+k+\kappa n-t+1 = k+\kappa n+2$, while the number of equations equals $k+\kappa n+1$. Assuming standard regularity and nondegeneracy conditions an almost-complete simplex has two solutions in which just one of the variables is equal zero. These two solutions are called the basic solutions.

From the starting point the algorithm generates for varying T a path of adjacent almost-complete t-dimensional simplices by alternating replacement steps in the simplicial subdivision in order to move from one simplex to an adjacent simplex and by linear programming pivot steps in order to move from basic solution to basic solution in the system (3.1) of $k+\kappa n+1$ linear equations. The algorithm stops if a complete simplex σ^* is found.

Definition 3.1

For a subset T_1 of $\{1,\ldots,k\}$ with $|T_1| = t_1$, a subset T_2 of $\{k+1,\ldots,k+\kappa n\}$ with $|T_2| = t_2$, and with $T=T_1 \cup T_2$, $|T|=t=t_1+t_2$, a $(t-1)$-dimensional simplex $\sigma(w^1,\ldots,w^t)$ is complete if the system of $k+\kappa n+1$ linear equations:

$$\sum_{g=1}^{t} \lambda_g \begin{pmatrix} \underline{z}(w^g) \\ \underline{c}(w^g) \\ 1 \end{pmatrix} + \sum_{r \notin T} \mu_r \begin{pmatrix} e(r) \\ 0 \end{pmatrix} - \beta \begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.2}$$

has a solution $(\lambda^*,\mu^*,\beta^*)$ with $\lambda^*_g \geq 0$, $g = 1,\ldots,t$, $\mu^*_r \geq 0$, $r \notin T$, and for all $(p,q) \in \sigma(w^1,\ldots,w^t)$, $p_h = 0$ if $h \notin T_1$ and $q^j_h = 0$ if $K_{j,h} \notin T_2$, where $K_{j,h}:=k+(j-1)\kappa+h$.

Observe that a complete simplex $\sigma^*(w^1,\ldots,w^t)$ is either a facet of an almost-complete simplex $\sigma(w^1,\ldots,w^{t+1})$ with $\lambda^*_{t+1} = 0$ in one of the basic solutions or is an almost-complete $(t-1)$-dimensional simplex with respect to $\underline{T} = T\backslash\{i\}$ for some $i \in T$ with $\mu^*_i = 0$ at one of the basic solutions.

The sequence of adjacent almost-complete simplices traces in $S^{k-1} \times (R^{\kappa}_+)^n$ a piecewise linear path of points from the starting point to an approximate solution. This piecewise linear path is such that, with starting point $(\underline{p},\underline{q})$ $(0<\underline{p}_h<1,\ h=1,\ldots k)$, for any (p,q) on the path holds that (for some α and β, $0 \leq \alpha \leq 1$):

$$\begin{aligned} p_h &= \alpha.\underline{p}_h \text{ if } Z_h(p,q) < \beta,\ h=1,\ldots,k \\ p_h &\geq \alpha.\underline{p}_h \text{ if } Z_h(p,q) = \beta,\ h=1,\ldots,k \\ q^j_h &= \alpha.\underline{q}^j_h \text{ if } C^j_h(p,q) < \beta,\ j=1,\ldots,n,\ h=1,\ldots,\kappa \\ q^j_h &\geq \alpha.\underline{q}^j_h \text{ if } C^j_h(p,q) = \beta,\ j=1,\ldots,n,\ h=1,\ldots,\kappa \end{aligned} \tag{3.3}$$

Note that $\beta = \max(\max_h Z_h(p,q), \max_{j,h} C^j_h(p,q^g))$ and $\alpha = \min(\min_h p_h/\underline{p}_h, \min_{j,h} q^j_h/\underline{q}^j_h)$. In other words, p_h respectively q^j_h is increased whenever $Z_h(p,q)$ respectively $C^j_h(p,q)$ is equal to $\beta = \max(\max_h Z_h(p,q), \max_{j,h} C^j_h(p,q))$, while p_h respectively q^j_h are kept relatively constant if $Z_h(p,q)$ respectively $C^j_h(p,q)$ is less then β.

Now, since simplicial algorithms are such that they never visit one simplex twice, convergence is assured if we can restrict the algorithm to a compact subset of $S^{k-1} \times (R^{\kappa}_+)^n$, i.e. if there are for all

$j=1,\dots,n$, for all $h=1,\dots,\kappa$, $q^j{}_h{}^{max}$ such that the algorithm is confined to a compact set $S^{k-1} \times Q$, with $Q = \{q \in (R^\kappa{}_+)^n | \; q^j{}_h \le q^j{}_h{}^{max}, \; j = 1,\dots,n, \; h = 1,\dots,\kappa\}$.

<u>Theorem 3.2</u>

Let $\delta > 0$. Under assumption 2.6, the algorithm, when operating in a triangulation of $S^{k-1} \times (R^\kappa{}_+)^n$ with mesh size δ, is confined to the compact set $S^{k-1} \times Q^{max}$ with Q^{max} given by $q^j{}_h{}^{max} = w_h+1+\delta$ for $j=1,\dots,n$, $h=1,\dots,\kappa$.

<u>Proof</u>

The algorithm traces a path of points in $S^{k-1} \times (R^\kappa{}_+)^n$ on which $Z_i(p,q) = C^f{}_\ell(p,q) - \max(\max_h Z_h(p,q), \max_{j,h} C^j{}_h(p,q)) = \beta$ for $i \in T_1$, $K_{f,\ell} \in T_2$ and $Z_h(p,q) \le \beta$ for $h \notin T_1$, $C^j{}_h(p,q) \le \beta$ for $K_{j,h} \notin T_2$. First, note that $C^f{}_\ell$ is bounded from above since with $\underline{c}^f(p^g,q^g) = p^f(p^g,q^g) - p_I \in \gamma^f(p^g,q^g)$ it follows that $C^f{}_\ell = \Sigma_g\lambda_g\underline{c}^f{}_\ell(p^g,q^g) = \Sigma_g\lambda_g p^f{}_\ell(p^g,q^g) - p_\ell \le \Sigma_g\lambda_g p^f{}_\ell(p^g,q^g) \le 1$. Furthermore, $O_\ell = \emptyset$, and $x^i{}_\ell \ge 0$ for all $x^i \in d^i(p,q)$, so for all (p,q), $z_\ell(p,q) = \Sigma_i x^i{}_\ell + \Sigma_{j\in I\ell} q^j{}_\ell - w_\ell \ge \Sigma_j q^j{}_\ell - w_\ell$, for all $z(p,q) \in \zeta(p,q)$.

Suppose the algorithm visits a simplex σ which does not lie in $S^{k-1} \times Q^{max}$. So, σ has a vertex $w^r = (p^r, q^{1r}, \dots, q^{nr})$ with for some f for some ℓ, $q^{fr}{}_\ell > w_\ell + 1 + \delta$. But then $q^{fg}{}_\ell > w_\ell + 1$ for all g and consequently, $z_\ell(w^g) \ge \Sigma_j q^{jg}{}_\ell - w_\ell > 1$, for all $z(w^g) \in \zeta(w^g)$ for all vertices w^g. So, $Z_\ell(p,q) = \Sigma\lambda_g \underline{z}_\ell(w^g) > 1$. However, this yields a contradiction as $Z_\ell(p,q) \le \max(\max_h Z_h(p,q), \max_{j,h} C^j{}_h(p,q)) = \beta = C^f{}_\ell(p,q) \le 1$ does not hold any longer. Consequently, σ can never be visited by the algorithm and the algorithm is confined to $S^{k-1} \times Q^{max}$.

□

<u>Corollary 3.3</u>

The algorithm terminates under assumption 2.6 within a finite number of steps with a complete simplex.

<u>Proof</u>

According to theorem 3.2 the algorithm operates in a compact set. As the algorithm is such that it never visits a simplex more than once, the algorithm terminates within a finite number of steps with a complete simplex.

□

We have from corollary 3.3 that the algorithm terminates within a finite number of steps with a complete simplex. In theorem 3.5 we will show that the complete simplex σ^* with which the algorithm ends, yields an approximate nonlinear complementarity pair.

Assumption 3.4

If for two indices i and h, $j \in I_h$, $j \in O_i$ and $y^j{}_h < 0$, then for all $u \in N_C(Y^j, y^j)$: either $u_i > 0$ or $u_i = 0$ and $u_h = 0$.

Assumption 3.4 says that for a strictly positive amount of an input the marginal productivity of this input is finite.

Theorem 3.5

Let $\epsilon > 0$ be such that for all $u,s \in S^{k-1} \times (R^{\kappa}{}_+)^n$ in a complete simplex σ^*, $\max_i|v_i(u) - v_i(s)| < \epsilon$ for all $v(u) \in \phi(u)$ and $v(s) \in \phi(s)$ with $v(u) = (z^T(u), c^{1T}(u), \ldots, c^{nT}(u))^T$. Under assumptions 2.1, 2.4, 2.6 and 3.4, σ^* contains a point $(p^*, q^{1*}, \ldots, q^{n*})$ such that:

$$|\beta^*| < \epsilon$$

$$\left.\begin{array}{ll} \beta^* - \epsilon < z_h(p^*, q^{1*}, \ldots, q^{n*}) < \beta^* + \epsilon & \text{if } p^*{}_h > 0 \\ z_h(p^*, q^{1*}, \ldots, q^{n*}) < \beta^* + \epsilon & \text{if } p^*{}_h = 0 \end{array}\right\} \quad h=1,\ldots,k$$

$$\left.\begin{array}{ll} \beta^* - \epsilon < c^j{}_h(p^*, q^{1*}, \ldots, q^{n*}) < \beta^* + \epsilon & \text{if } q^{j*}{}_h > 0 \\ c^j{}_h(p^*, q^{1*}, \ldots, q^{n*}) < \beta^* + \epsilon & \text{if } q^{j*}{}_h = 0 \end{array}\right\} \quad j=1,\ldots,n,\ h=1,\ldots,\kappa$$

for all $z(p^*, q^{1*}, \ldots, q^{n*}) \in \zeta(p^*, q^{1*}, \ldots, q^{n*})$ and for all $c^j(p^*, q^{1*}, \ldots, q^{n*}) \in \gamma^j(p^*, q^{1*}, \ldots, q^{n*})$.

Proof

Let $(p^*, q^{1*}, \ldots, q^{n*})$ be given by $(p^*, q^{1*}, \ldots, q^{n*}) = \Sigma_g \lambda^*{}_g w^g$, with $w^1, \ldots, w^t$ the vertices of σ^*. Since $\Sigma_g \lambda^*{}_g = 1$, $(p^*, q^{1*}, \ldots, q^{n*})$ lies in σ^*. Furthermore, we have from (3.2) that $\Sigma_g \lambda^*{}_g \underline{z}_h(p^g, q^{1g}, \ldots, q^{ng}) = \beta^*$ if $h \in T_1$, $\Sigma_g \lambda^*{}_g \underline{z}_h(p^g, q^{1g}, \ldots, q^{ng}) = \beta^* - \mu^*{}_h$ if $h \notin T_1$, $\Sigma_g \lambda^*{}_g \underline{c}^j{}_h(p^g, q^{1g}, \ldots, q^{ng}) = \beta^*$ if $K_{j,h} \in T_2$, $\Sigma_g \lambda^*{}_g \underline{c}^j{}_h(p^g, q^{1g}, \ldots, q^{ng}) = \beta^* - \mu^*{}_{Kj,h}$ if $K_{j,h} \notin T_2$.

Finally, note that for all $z(p^*,q^{1*},\ldots,q^{n*}) \in \zeta(p^*,q^{1*},\ldots,q^{n*})$:

$p^*.z(p^*,q^{1*},\ldots,q^{n*}) = 0$ iff $p^*.w + \Sigma_j p^*.y^{j*} \geq 0$ and

$p^*.z(p^*,q^{1*},\ldots,q^{n*}) > 0$ iff $p^*.w + \Sigma_j p^*.y^{j*} < 0$.

Furthermore, if $p^*.z(p^*,q^{1*},\ldots,q^{n*}) = 0$ then:

$$|p^*.Z(p^*,q^{1*},\ldots,q^{n*})| = |p^*.(Z(p^*,q^{1*},\ldots,q^{n*}) - z(p^*,q^{1*},\ldots,q^{n*}))| =$$

$$|\Sigma p^*_h(\Sigma_g \lambda^*_g(\underline{z}_h(p^g,q^{1g},\ldots,q^{ng}) - z_h(p^*,q^{1*},\ldots,q^{n*})))| < \epsilon$$

and if $p^*.z(p^*,q^{1*},\ldots,q^{n*}) > 0$ then:

$$p^*.Z(p^*,q^{1*},\ldots,q^{n*}) > p^*.(Z(p^*,q^{1*},\ldots,q^{n*}) - z(p^*,q^{1*},\ldots,q^{n*})) =$$

$$\Sigma_h p^*_h(\Sigma_g \lambda^*_g(\underline{z}_h(p^g,q^{1g},\ldots,q^{ng}) - z_h(p^*,q^{1*},\ldots,q^{n*}))) > -\epsilon$$

since $|z_h(p^g,q^{1g},\ldots,q^{ng}) - z_h(p^*,q^{1*},\ldots,q^{n*})| < \epsilon$.

Now, in a complete simplex we have $p^*.Z(p^*,q^{1*},\ldots,q^{n*}) = \beta^*$. Hence, either $-\epsilon < \beta^* < \epsilon$ and $p^*.z(p^*,q^{1*},\ldots,q^{n*}) = 0$ or $\beta^* > -\epsilon$ and $p^*.z(p^*,q^{1*},\ldots,q^{n*}) > 0$.

Suppose, $\beta^* \geq \epsilon$. So, $p^*.Z(p^*,q^{1*},\ldots,q^{n*}) = \beta^* \geq \epsilon$ which implies that $p^*.z(p^*,q^{1*},\ldots,q^{n*}) > 0$ and thus $p^*.w + \Sigma_j p^*.y^{j*} < 0$. Consequently, $d^i(p^*,q^*) = \{0\}$, $i=1,\ldots,m$. Now, there are two possibilities:

(i) $\exists h : O_h \neq \emptyset$ and $p^*_h > 0$ or

(ii) $\forall h$ with $O_h \neq \emptyset : p^*_h = 0$.

ad (i) Now, $z_h(p^*,q^{1*},\ldots,q^{n*}) = -\Sigma q^j_h - w_h < 0$.

Furthermore, $|\underline{z}_h(p^g,q^{1g},\ldots,q^{ng}) - z_h(p^*,q^{1*},\ldots,q^{n*})| < \epsilon$.

Consequently, $\underline{z}_h(p^g,q^{1g},\ldots,q^{ng}) < z_h(p^*,q^{1*},\ldots,q^{n*}) + \epsilon < \epsilon$.

So, $Z_h(p^*, q^{1*}, \ldots, q^{n*}) < \epsilon$.

On the other hand, $Z_h(p^*, q^{1*}, \ldots, q^{n*}) = \Sigma_g \lambda^*_g \underline{z}_h(p^g, q^{1g}, \ldots, q^{ng}) = \beta^* \geq \epsilon$

This yields a contradiction.

ad (ii) In this case $\exists \ell : I_\ell \neq \emptyset$ and $p^*_\ell > 0$.

Now, $Z_\ell(p^*, q^*) = \Sigma_g \lambda^*_g \underline{z}_\ell(p^g, q^{1g}, \ldots, q^{ng}) = \beta^* \geq \epsilon$.

Furthermore, $|\underline{z}_\ell(p^g, q^{1g}, \ldots, q^{ng}) - z_\ell(p^*, q^{1*}, \ldots, q^{n*})| < \epsilon$. So:

$\underline{z}_\ell(p^g, q^{1g}, \ldots, q^{ng}) < z_\ell(p^*, q^{1*}, \ldots, q^{n*}) + \epsilon$.

But, then $z_\ell(p^*, q^{1*}, \ldots, q^{n*}) > 0$ must hold, as otherwise $\underline{z}_\ell(p^g, q^{1g}, \ldots, q^{ng}) < \epsilon$ and thus $Z_\ell(p^*, q^{1*}, \ldots, q^{n*}) < \epsilon$.

Now, as $z_\ell(p^*, q^{1*}, \ldots, q^{n*}) = \Sigma q^{f*}_\ell - w_\ell$ there must be some f such that $q^{f*}_\ell > 0$. But then, $C^f_\ell(p^*, q^{1*}, \ldots, q^{n*}) = \beta^* \geq \epsilon$.

On the other hand, if $q^{f*}_\ell > 0$ and $p^*_h = 0$ with $f \in O_h$, then $\underline{c}^f_\ell(p^g, q^{1g}, \ldots, q^{ng}) < c^f_\ell(p^*, q^{1*}, \ldots, q^{n*}) + \epsilon < \epsilon$ as $\underline{c}^f_\ell(p^*, q^{1*}, \ldots, q^{n*}) = p^{f*}_\ell - p^*_\ell = -p^*_\ell < 0$, as under assumption 3.4, $p^{f*}_\ell = 0$.

Consequently, $C^f_\ell(p^*, q^{1*}, \ldots, q^{n*}) = \Sigma \lambda^*_g \underline{c}^f_\ell(p^g, q^{1g}, \ldots, q^{ng}) < \epsilon$.

Again we have a contradiction.

Concluding, $|\beta^*| < \epsilon$ must hold.

Furthermore,

$|z_h(p^*, q^{1*}, \ldots, q^{n*}) - \beta^*| =$

$|\Sigma_g \lambda^*_g (z_h(p^*, q^{1*}, \ldots, q^{n*}) - \underline{z}_h(p^g, q^{1g}, \ldots, q^{ng}))| < \epsilon$ if $p^*_h > 0$,

$z_h(p^*, q^{1*}, \ldots, q^{n*}) - \beta^* \leq$

$\Sigma_g \lambda^*_g (z_h(p^*, q^{1*}, \ldots, q^{n*}) - \underline{z}_h(p^g, q^{1g}, \ldots, q^{ng})) < \epsilon$ if $p^*_h = 0$,

$|c^j{}_h(p^*,q^{1*},\ldots,q^{n*}) - \beta^*| =$

$|\Sigma_g\lambda^*{}_g(c^j{}_h(p^*,q^{1*},\ldots,q^{n*}) - \underline{c}^j{}_h(p^g,q^{1g},\ldots,q^{ng}))| < \epsilon$ if $q^{j*}{}_h>0$,

$c^j{}_h(p^*,q^{1*},\ldots,q^{n*}) - \beta^* \leq$

$\Sigma_g\lambda^*{}_g(c^j{}_h(p^*,q^{1*},\ldots,q^{n*}) - \underline{c}^j{}_h(p^g,q^{1g},\ldots,q^{ng})) < \epsilon$ if $q^{j*}{}_h=0$.

□

Note, that in case of a continuous function (which requires production to be smooth) for arbitrarily small ϵ there is a grid size δ such that $\max_i|v_i(u) - v_i(s)| < \epsilon$ and theorem 3.5 holds for arbitrarily small ϵ. However, in case of an upper semi continuous mapping an arbitrarily small ϵ cannot be guaranteed. In this case the only thing we can say is that if we have a sequence of triangulations with mesh size going to zero then the sequence of approximate solutions contains a subsequence converging to a nonlinear complementarity point of v as we will prove in theorem 3.6 (see also Van der Laan (1980)). In section 2 we have shown that a nonlinear complementarity pair of ϕ yields a marginal pricing equilibrium.

<u>Theorem 3.6</u>
Under assumptions 2.1, 2.4, 2.6 and 3.4, $\phi(p,q^1,\ldots,q^n)$ has a nonlinear complementarity pair.

<u>Proof</u>
Let $\{G_\tau,\tau=1,2,\ldots\}$ be a sequence of triangulations of $S^{k-1} \times Q^{max}$ with $q^j{}_h{}^{max} = w_h+1+\delta_\tau$ for $j=1,\ldots,n$, $h=1,\ldots,\kappa$, and δ_τ = mesh $G_\tau \to 0$ as $\tau \to \infty$. Let $V^\tau = (Z^\tau,C^{1\tau}, \ldots,C^{n\tau})$ be the piecewise linear approximation of ϕ with respect to G_τ. Now, according to corollary 3.3 and (3.2) there is a simplex σ^τ in G_τ with vertices $w^{1\tau},\ldots,w^{t\tau}$ containing a vector (p^τ,q^τ) such that:

$(p^\tau,q^\tau) = \Sigma_i\ \lambda^\tau{}_i.w^{i\tau}$ with

$\Sigma\lambda^\tau{}_i = 1,\ \lambda^\tau{}_i \geq 0$ and

$$\Sigma_i \lambda^\tau{}_i \underline{z}^\tau{}_h(w^{i\tau}) = \beta^\tau \text{ if } p^\tau{}_h > 0$$

$$\Sigma_i \lambda^\tau{}_i \underline{z}^\tau{}_h(w^{i\tau}) \leq \beta^\tau \text{ if } p^\tau{}_h = 0$$

$$\Sigma_i \lambda^\tau{}_i \underline{c}^{f\tau}{}_\ell(w^{i\tau}) = \beta^\tau \text{ if } q^{f\tau}{}_\ell > 0 \tag{3.4}$$

$$\Sigma_i \lambda^\tau{}_i \underline{c}^{f\tau}{}_\ell(w^{i\tau}) \leq \beta^\tau \text{ if } q^{f\tau}{}_\ell = 0.$$

Now, since $S^{k-1} \times Q^{max}$ is compact there is a subsequence τ_j, j=1,2,... such that $(p^{\tau j},q^{\tau j}) \to (p^*,q^*)$, $\lambda^{\tau j}{}_i \to \lambda^*{}_i$ and $\underline{v}^{\tau j}(w^{i\tau j}) \to \tilde{v}^i = (\tilde{z}^i,\tilde{c}^{1i\top},\ldots,\tilde{c}^{ni\top})^\top$ if $j \to \infty$, where $\underline{v}^{\tau j}(w^{i\tau j})$ is the element of $\phi(w^{i\tau j})$ underlying the piecewise linear approximation V^τ with respect to G_τ.

Furthermore, $\delta_\tau \to 0$ and consequently, $w^{i\tau j} \to (p^*,q^*)$ and as ϕ is upper semi continuous, $\tilde{v}^i \in \phi(p^*,q^*)$. As $\phi(p^*,q^*)$ is also convex $v^* = \Sigma\lambda^*{}_i.\tilde{v}^i \in \phi(p^*,q^*) = (\zeta(p^*,q^*),\gamma^1(p^*,q^*),\ldots,\gamma^n(p^*,q^*))$. Now, define $z^*{}_h = \Sigma\lambda^*{}_i.\tilde{z}^i{}_h$ and $c^{f*}{}_\ell = \Sigma\lambda^*{}_i.\tilde{c}^{fi}{}_\ell$ (so $z^* \in \zeta(p^*,q^*)$ and $c^{f*} \in \gamma^f(p^*,q^*)$), then taking limits in (3.4) yields:

$$z^*{}_h = \Sigma\lambda^*{}_i.\tilde{z}^i{}_h = \lim_{j\to\infty} \beta^{\tau j} \text{ if } p^*{}_h > 0$$

$$z^*{}_h = \Sigma\lambda^*{}_i.\tilde{z}^i{}_h \leq \lim_{j\to\infty} \beta^{\tau j} \text{ if } p^*{}_h = 0$$

$$c^{f*}{}_\ell = \Sigma\lambda^*{}_i.\tilde{c}^{fi}{}_\ell = \lim_{j\to\infty} \beta^{\tau j} \text{ if } q^{f*}{}_\ell > 0 \tag{3.5}$$

$$c^{f*}{}_\ell = \Sigma\lambda^*{}_i.\tilde{c}^{fi}{}_\ell \leq \lim_{j\to\infty} \beta^{\tau j} \text{ if } q^{f*}{}_\ell = 0.$$

Now, $p^*.z^* = \lim_{j\to\infty} \beta^{\tau j}$ and hence, $\lim_{j\to\infty} \beta^{\tau j} \geq 0$.

Suppose $\lim_{j\to\infty} \beta^{\tau j} > 0$. Now, there are two possibilities:

(i) $\exists h : O_h \neq \emptyset$ and $p^*{}_h > 0$ or

(ii) $\forall h$ with $O_h \neq \emptyset : p^*{}_h = 0$.

ad (i) Now, $z^*_h = \lim_{j\to\infty} \beta^{\tau j} > 0$.

Furthermore, $p^*.z^* = \lim_{j\to\infty} \beta^{\tau j} > 0$ which implies that:

$p^*.w + \Sigma_j p^*.y^{j*} < 0$. Consequently, $d^i_h(p^*,q^*) = 0$ and

$z^*_h = -\Sigma q^{j*}_h - w_h < 0$. But this yields a contradiction.

ad (ii) In this case $\exists \ell : I_\ell \neq \emptyset$ and $p^*_\ell > 0$.

Now, $z^*_\ell = \lim_{j\to\infty} \beta^{\tau j} > 0$.

Furthermore, $p^*.z^* = \lim_{j\to\infty} \beta^{\tau j} > 0$ which implies that

$p^*.w + \Sigma_j p^*.y^{j*} < 0$. Consequently, $d^i_\ell(p^*,q^*) = 0$ and $z^*_\ell = \Sigma q^{f*}_\ell - w_\ell$.

So, there must be some f such that $q^{f*}_\ell > 0$.

But then, $c^{f*}_\ell = \lim_{j\to\infty} \beta^{\tau j} > 0$.

On the other hand $c^{f*}_\ell = \Sigma\lambda^*_i.\bar{c}^{fi}_\ell = -p^*_\ell < 0$ as under assumption 3.4

$p^{f*}_\ell = 0$ if $q^{f*}_\ell > 0$ and $p^*_h = 0$ with $f \in O_h$.

Again we have a contradiction.

Concluding, $\lim_{j\to\infty} \beta^{\tau j} = 0$ must hold. Substituting this in (3.5) we have that $((p^*,q^*),\ \Sigma\lambda^*_i.\tilde{v}^i) = ((p^{*\top},q^{1*\top},\ldots,q^{n*\top})^\top,(z^{*\top},c^{1*\top},\ldots,c^{n*\top})^\top)$ is a nonlinear complementarity pair of ϕ.

□

Finally, some remarks on the (economic) interpretation of the algorithm have to be made. On the commodity markets (z(p,q)) equilibrium is established by the price mechanism. The so-called central planner sets the (market) prices such that demand equals supply. On the price markets ($c^j(p,q)$), on the other hand, equilibrium is established by quantity adjustments. If the difference between the individual price of an input and the market price has maximum value than the use of this input will increase.

4. Concluding remarks

In this paper we have described an algorithm which enables us to compute equilibria in general equilibrium models where production exhibits increasing returns to scale. All the information the algorithm needs is the function values of the vertices of the simplices which are visited by the algorithm. The determination of these function values is straightforward. Convergence of the algorithm is shown for the case where there are no intermediate goods industries. In practice, this means that, in order to guarantee convergence, we have to aggregate the production side of the economy at least up to the point where all intermediate goods industries drop out. This class of models can be useful when we are mainly interested in analyzing welfare effects of e.g. tax reforms for the consumers, i.e. when we are mainly interested in the consumption side of the economy.

Although, in this paper we have described a general equilibrium model with increasing returns to scale technologies it must be noted that actually also more general types of non-convexities are allowed in our model. Finally, it must be noted that by showing that the algorithm converges and finds an approximate equilibrium we have also proved the existence of an equilibrium.

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