SERIE RESEARCH MEMORANDA

ON THE EFFECT OF SOME JOCKEYING BETWEEN PARALLEL PROCESSORS

by

Nico M. van Dijk

Research Memorandum 1989-90

VRIJE UNIVERSITEIT
Faculteit der Economische Wetenschappen en Econometrie
AMSTERDAM

On the effect of some jockeying between parallel processors

Nico M. van Dijk
Free University, Amsterdam

Abstract A number of parallel processors is studied in which waiting jobs can jockey to a next processor at some jockeying rate. This model arises in distributed systems but has not been solved explicitly so far. An error bound will be provided on the effect of jockeying for small jockeying rates. Particularly, a simple throughput bound is hereby obtained.

Keywords Parallel processors * jockeying * throughput * error bound

1. Introduction.

Distributed systems are rapidly evolving with present day technology developments in telecommunication and computer networks and as such appear to become a major trend in the next decade. A price that one pays, however, for the improvement such as in reachability and performance that distributed systems provide are the various conflicts that they bring along such as due to selected accessing, common resources and synchronization requirements. An elegant overview of these aspects was recently presented in [1].

As a particular illustrative example of these conflicts, this reference mentions the system of a finite number of parallel processors with separate inputs so that jobs may have to wait at a processor while one of the other processors is free. An expression for the probability of such an event is given in [3]. A way to somewhat resolve this inefficiency conflict is to let waiting jobs jockey to a next processor at some jockeying rate. As of today, however, this system has proven to be intractible for establishing an explicit closed form solution.

This note therefore takes a different approach by providing error bounds on the effect of jockeying for different values of the jockeying rate. Particularly, by studying the extreme case without jockeying a simple bound on the throughput is hereby obtained.

The technique to this end is based on Markov reward arguments and the estimation of so-called bias-terms. This approach seems promising for extension to related issues such as the effect of propagation delays in communication networks.

2.1 Model

Consider a system of M separate finite and exponential single-server queues numbered 1,...,M, with at queue i a Poisson arrival input with intensity λ_i, exponential service times with parameter μ_i and a finite capacity buffer for at most N_i jobs, the one in service included. A job is lost upon arrival if it finds the buffer full. In addition, each job waiting in queue i, the one in service thus excluded, will jump over to try out the next queue i+1 at a rate α_i. If queue i+1 is full it will return to the position it came from at queue i. Otherwise it joins the end of queue i+1, where i+1=1 for i=N.

We are interested in the throughput L of this system, that is the total number of jobs that per unit of time actually enter one of the queues or, equivalently, leave one of the queues, when the system is in steady state. As of yet, there doesn't seem to be an explicit expression for L. For small rates α_i though, one might intuitively expect the case of completely separated queues, that is with all $\alpha_i=0$, to provide a reasonable bound on the throughput as given by

(1) $\bar{L} = \Sigma_i \ \lambda_i [1-E(\lambda_i, \mu_i, N_i)]$

where $E(\lambda_i, \mu_i, N_i)$ is Erlang's standard loss expression (e.g. [2]). In what follows we aim to formalize this intuition by providing an error bound for the effect of different α_i-values on the throughput. In particular, we will obtain an error bound on

(2) $|\bar{L} - L|$

2.2 Error bounds

Compare two systems as described in section 2.1 with α-values $\{\bar{\alpha}_i\}$ and $\{\alpha_i\}$ for systems 1 and 2 respectively. Here some or even all of the $\bar{\alpha}_i$'s or α_i's are allowed to be equal to 0.

Let state $\bar{k} = (k_1,\ldots,k_N)$ denote that currently k_i jobs are present in queue i, the one in service included, i=1,...,N. Then clearly, the underlying queueing processes for systems 1 and 2 constitute a continuous-time irreducible Markov chain at

$$S = \{\bar{k}\,|\,k_i \le N_i,\; i=1,\ldots,N\}.$$

From now on, we always denote an expression for system 1 with an upper bar symbol "-" while no symbol is used when corresponding to system 2. The symbol "(-)" is used when the expression is to be read for both systems. Further, the vectors $\bar{k}+e_i$ and $\bar{k}-e_i$ are equal to $\bar{k}$ up to one job more (+ sign) respectively less (- sign) at station i. Finally, let $1_{\{A\}}$ denote an indicator of an event A, i.e. $1_{\{A\}}=1$ if A is satisfied and $1_{\{A\}}=0$ otherwise, and read i+1=1 for i=N.

In order to compare the corresponding continuous-time models in a recursive manner we will apply the standard uniformization method (cf. [4], p.110) by letting

$$(3)\qquad Q \ge \Sigma_i\;[\lambda_i + \mu_i + \overset{(-)}{\alpha_i} N_i]$$

and defining discrete-time Markov chains $\{\bar{X}(t)\,|\,t=0,1,2,\ldots\}$ and $\{X(t)\,|\,t=0,1,2,\ldots\}$ with one-step transition probabilities for a transition from state $\bar{k}$ in $\bar{k}'$ by:

$$(4)\qquad \overset{(-)}{p}(\bar{k},\bar{k}') = \begin{cases} \lambda_i 1_{\{k_i<N_i\}} Q^{-1}, & (\bar{k}'=\bar{k}+e_i),\ (i\le N) \\ \mu_i 1_{\{k_i>0\}} Q^{-1}, & (\bar{k}'=\bar{k}-e_i),\ (i\le N) \\ \overset{(-)}{\alpha_i}[k_i-1]1_{\{k_i\ge 1\}}1_{\{k_{i+1}<N_{i+1}\}}\;Q^{-1}, & (\bar{k}'=\bar{k}-e_i+e_{i+1}),\ (i\le N) \\ 1-[\lambda_i 1_{\{k_i<N_i\}} - \mu_i 1_{\{k_i>0\}} - \alpha_i[k_i-1]1_{\{k_i\ge 1\}}1_{\{k_{i+1}<N_{i+1}\}}]Q^{-1}, & (\bar{k}'=\bar{k}) \end{cases}$$

Then, by virtue of the uniformization technique and standard Markov reward arguments, the throughputs $\bar{L}$ and L for systems 1 and 2 respectively, can be obtained by

$$(5)\qquad \overset{(-)}{L} = \lim_{N\to\infty}\;[Q/N]\;\overset{(-)}{V_N}\;(\bar{0})$$

where $\bar{0} = (0,\ldots,0)$ denotes the empty state and where $\overset{(-)}{V_N}(.)$ is a function at S recursively determined by $\overset{(-)}{V_0}(.)=0$ and

(6) $\quad \overset{(-)}{V}_{n+1}(\bar{k}) = \Sigma_i \mu_i 1_{\{k_i>0\}} Q^{-1} + \Sigma_{\bar{k}'} \overset{(-)}{p}(\bar{k},\bar{k}') \overset{(-)}{V}_n(\bar{k}')$

The following lemma will be crucial for establishing an error bound. It provides estimates for so-called bias-terms $V_n(\bar{k}')-V_n(\bar{k})$ uniformly in n.

Lemma 1 For all $n \geq 0$ and $i \leq M$:

(7) $\quad 0 \leq V_n(\bar{k}+e_i) - V_n(\bar{k}) \leq 1$

Proof We will apply induction to n. Clearly, (7) holds for n=0. Suppose it holds for all i=1,...,M and $n \leq m$. Then, for n=m+1 and arbitrary i we can substitute (6) for $\bar{k}+e_i$ and $\bar{k}$. Before doing so it is noted in advance, however, that some terms will be artificially added and subtracted or split in order to obtain compatible terms that can be compared pairswise. Also in the final result the sixth term is indeed 0 but kept in for use of an argument later on. Further, let $h=Q^{-1}$ and read k-1=0 for k=0.

(8) $\quad V_{m+1}(\bar{k}+e_i) - V_m(\bar{k}) =$

$$\Big\{\Sigma_{j\neq i} h\mu_j\ 1_{\{k_j>0\}} + h\mu_i +$$

$$\Sigma_{j\neq i} h\lambda_j\ 1_{\{k_j<N_j\}} V_m(\bar{k}+e_j+e_i) +$$

$$h\lambda_i\ 1_{\{k_i+1<N_i\}} V_m(\bar{k}+e_i+e_i) + h\lambda_i\ 1_{\{k_i+1=N_i\}} V_m(\bar{k}+e_i) +$$

$$\Sigma_{j\neq i} h\mu_j\ 1_{\{k_j>0\}} V_m(\bar{k}-e_j+e_i) + h\mu_i\ V_m(\bar{k}) +$$

$$\Sigma_{j\neq i-1,i}\ h\alpha_j [k_j-1]\ 1_{\{k_{j+1}<N_{j+1}\}} V_m(\bar{k}-e_j+e_{j+1}+e_i) +$$

$$h\alpha_{i-1}[k_{i-1}-1]\ 1_{\{k_i+1<N_i\}} V_m(\bar{k}-e_{i-1}+e_i+e_i) +$$

$$h\alpha_{i-1}[k_{i-1}-1]\ 1_{\{k_i+1=N_i\}} V_m(\bar{k}+e_i)$$

$$h\alpha_i k_i\ 1_{\{k_{i+1}<N_{i+1}\}} V_m(\bar{k}+e_i-e_i+e_{i+1}) +$$

$$\{1 - [\Sigma_{j\neq i} h\lambda_j\ 1_{\{k_j<N_j\}} + h\lambda_i\ 1_{\{k_i<N_i\}} + \Sigma_{j\neq i}\ h\mu_j\ 1_{\{k_j>0\}} +$$

$$h\mu_i + \Sigma_{j\neq i-1,i}\ h\alpha_j [k_j-1]\ 1_{\{k_{j+1}<N_{j+1}\}} + h\alpha_{i-1}\ [k_{i-1}-1]\ 1_{\{k_i<N_i\}} +$$

$$h\alpha_i\ k_i\ 1_{\{k_{i+1}<N_{i+1}\}}]\}\ V_m(\bar{k}+e_i)\Big\}$$

$$
\Big\{ \Sigma_{j\neq i} h\mu_j \; 1_{\{k_j>0\}} + h\mu_i \; 1_{\{k_i>0\}} +
$$

$$
\Sigma_{j\neq i} h\lambda_j \; 1_{\{k_j<N_j\}} V_m(\bar{k}+e_j) + h\lambda_i \; 1_{\{k_i<N_i\}} V_m(\bar{k}+e_i) +
$$

$$
\Sigma_{j\neq i} h\mu_j \; 1_{\{k_j>0\}} V_m(\bar{k}-e_j) + h\mu_i 1_{\{k_i>0\}} V_m(\bar{k}-e_i) + h\mu_i \; 1_{\{k_i=0\}} V_m(\bar{k}) +
$$

$$
\Sigma_{j\neq i-1,i} \; h\alpha_j [k_j-1] \; 1_{\{k_{j+1}<N_{j+1}\}} V_m(\bar{k}-e_j+e_{j+1}) +
$$

$$
h\alpha_{i-1} \; [k_{i-1}-1] \; 1_{\{k_i+1<N_i\}} V_m(\bar{k}-e_{i-1}+e_i) +
$$

$$
h\alpha_{i-1} \; [k_{i-1}-1] \; 1_{\{k_i+1=N_i\}} V_m(\bar{k}-e_{i-1}+e_i) +
$$

$$
h\alpha_i \; [k_i-1] \; 1_{\{k_{i+1}<N_{i+1}\}} V_m(\bar{k}-e_i+e_{i+1}) + h\alpha_i \; 1_{\{k_{i+1}<N_{i+1}\}} V_m(\bar{k}) +
$$

$$
\{1 - [\Sigma_{j\neq i} h\lambda_j \; 1_{\{k_j<N_j\}} + h\lambda_i \; 1_{\{k_i<N_i\}} + \Sigma_{j\neq i} h\mu_j \; 1_{\{k_j>0\}} + h\mu_i +
$$

$$
\Sigma_{j\neq i-1,i} h\alpha_j \; [k_j-1] \; 1_{\{k_j+1<N_j\}} + h\alpha_{i-1} \; [k_{i-1}-1] \; 1_{\{k_i<N_i\}} +
$$

$$
h\alpha_i \; [k_i-1] \; 1_{\{k_{i+1}<N_{i+1}\}} + h\alpha_i \; 1_{\{k_{i+1}<N_{i+1}\}}]\} \; V_m(\bar{k}) \Big\}
$$

$$
=
$$

$$
h\mu_i \; 1_{\{k_i=0\}} +
$$

$$
\Sigma_{j\neq i} h \; \lambda_j \; 1_{\{k_j<N_j\}} [V_m(\bar{k}+e_j+e_i) - V_m(\bar{k}+e_j)] +
$$

$$
h\lambda_i \; 1_{\{k_i+1<N_i\}} [V_m(\bar{k}+e_i+e_i) - V_m(\bar{k}+e_i)] +
$$

$$
\Sigma_{j\neq i} h\mu_j \; 1_{\{k_j>0\}} [V_m(\bar{k}-e_j+e_i) - V_m(\bar{k}-e_j)] +
$$

$$
h\mu_i \; 1_{\{k_i>0\}} [V_m(\bar{k}) - V_m(\bar{k}-e_i)] + h\mu_i \; 1_{\{k_i=0\}} [V_m(\bar{k})-V_m(\bar{k})] +
$$

$$
\Sigma_{j\neq i-1,i} h\alpha_j \; [k_j-1] 1_{\{k_{j+1}<N_{j+1}\}} [V_m(\bar{k}-e_j+e_{j+1}+e_i)-V_m(\bar{k}-e_j+e_{j+1})] +
$$

$$
h\alpha_{i-1} \; [k_{i-1}-1] 1_{\{k_i+1<N_i\}} [V_m(\bar{k}-e_{i-1}+e_i+e_i)-V_m(\bar{k}-e_{i-1}+e_i)] +
$$

$$
h\alpha_{i-1} [k_{i-1}-1] 1_{\{k_i+1=N_i\}} [V_m(\bar{k}+e_i)-V_m(\bar{k}-e_{i-1}+e_i)] +
$$

$$h\alpha_i [k_i - 1]\ 1_{\{k_{i+1} < N_{i+1}\}} [V_m(\bar{k}+e_{i+1}) - V_m(\bar{k}-e_i+e_{i+1})] +$$

$$h\alpha_i\ 1_{\{k_{i+1} < N_{i+1}\}} [V_m(\bar{k}+e_{i+1}) - V_m(\bar{k})] +$$

$$\{1 - [\Sigma_{j\neq i} h\lambda_j\ 1_{\{k_j < N_j\}} + h\lambda_i\ 1_{\{k_i < N_i\}} + \Sigma_{j\neq i} h\mu_j\ 1_{\{k_j > 0\}} + h\mu_i +$$

$$\Sigma_{j\neq i-1,i}\ h\alpha_j\ [k_j - 1]\ 1_{\{k_{j+1} < N_{j+1}\}} + h\alpha_{i-1}\ [k_{i-1} - 1]\ 1_{\{k_i < N_i\}} +$$

$$h\alpha_i\ [k_i]\ 1_{\{k_{i+1} < N_{i+1}\}}]\}\ [V_m(\bar{k}+e_i) - V_m(\bar{k})].$$

We can now apply our induction hypothesis (7) for n=m and all i. The right hand side of expression (8) is then directly estimated from below after observing that in its tenth term we can write: $V_m(\bar{k}+e_i) - V_m(\bar{k}-e_{i-1}+e_i) = V_m(\bar{\ell}+e_{i-1}) - V_m(\bar{\ell})$ for $\bar{\ell}=\bar{k}-e_{i-1}+e_i$, and in its eleventh term: $V_m(\bar{k}+e_{i+1}) - V_m(\bar{k}-e_i+e_{i+1}) = V_m(\bar{\ell}+e_i) - V_m(\bar{\ell})$ for $\bar{\ell}=\bar{k}-e_i+e_{i+1}$.

To estimate this right hand side from above by 1, now recall that its sixth term is equal to 0 while its probability coefficient $h\mu_i 1_{\{k_i=0\}}$ is exactly equal to the first additional term. By also recalling (3) and h=1/Q and substituting the upper estimate 1 from (7) as per induction hypothesis, summing all terms yields an upper estimate 1. Inequality (7) has thus been proven for n=m+1, which completes the proof.

□

We are now able to present the main result.

Theorem 1. With $\bar{L}$ and L the throughputs of arbitrary $\{\bar{\alpha}_i\}$ and $\{\alpha_i\}$ models, we have:

(9) $$|\bar{L} - L| \leq 2M\ \max_i |\bar{\alpha}_i - \alpha_i| N_i$$

Proof. From (6) we conclude:

(10) $$\bar{V}_{n+1}(\bar{k}) - V_{n+1}(\bar{k}) =$$

$$\Sigma_{\bar{k}}\ \bar{p}(\bar{k},\bar{k}')[\bar{V}_n(\bar{k}) - V_n(\bar{k}')] +$$

$$\Sigma_{\bar{k}'}\ [\bar{p}(\bar{k},\bar{k}') - p(\bar{k},\bar{k}')]V_n(\bar{k}')$$

while from (4):

$$(11) \quad \Sigma_{\bar{k}'} \, [\tilde{p}(\bar{k},\bar{k}')-p(\bar{k},\bar{k}')]V_n(\bar{k}') =$$

$$\Sigma_i \; [\tilde{\alpha}_i - \alpha_i][k_i - 1]Q^{-1} \; 1_{\{k_i>0\}} \; 1_{\{k_{i+1}<N_{i+1}\}} \; [V_n(\bar{k}-e_i+e_{i+1})-V_n(\bar{k})]$$

By writing

$$(12) \quad V_n(\bar{k}-e_i+e_{i+1})-V_n(\bar{k})=[V_n(\bar{k}-e_i+e_{i+1})-V_n(\bar{k}-e_i)]+[V_n(\bar{k}-e_i)-V_n(\bar{k})],$$

taking absolute values and applying (7), we conclude from (10)-(12):

$$(13) \quad \max_{\bar{k}} |\tilde{V}_{n+1}(\bar{k})-V_{n+1}(\bar{k})| \le \max_{\bar{k}} |\tilde{V}_n(\bar{k})-V_n(\bar{k})| + 2MQ^{-1} \max_i |\tilde{\alpha}_i - \alpha_i| N_i .$$

As $\tilde{V}_0(.)=V_0(.)=0$, iterating (13) for $n=0,1,\ldots,N-1$ yields:

$$(14) \quad \tilde{V}_N(\bar{k}) - V_N(\bar{k}) \le 2NMQ^{-1} \max_i |\tilde{\alpha}_i - \alpha_i| N_i$$

for arbitrary $\bar{k}$ and N. Substituting $\bar{k}=\bar{0}$ and applying (5) completes the proof.

□

As a practical corollary, substituting $\tilde{\alpha}_i = 0$ for all i leads to:

Corollary With L the throughput of a given $\{\alpha_i\}$-system and $\tilde{L}$ the value as per (1) by Erlang's loss formula:

$$(15) \quad |\tilde{L}-L| \le 2M \max_i \alpha_i N_i$$

Remark The factors M and N_i in (9) or (15) are simple robust bounds but can in fact be sharpened by employing (11) and (12) more technically. For example, N_i can be reduced to the expected queue length at station i.

Remark Alternatively, assuming a jockeying rate $\alpha_i(k_i)$ at station i when k_i jobs are present, where $\alpha_i(k_i)$ is nondecreasing in k_i, $i=1,\ldots,M$, the proofs of lemma 1 and theorem 1 can most easily be modified to show that

$$|\tilde{L}-L| \le \max_{\bar{k}} \Sigma_i \{\tilde{\alpha}(\bar{k})-\alpha_i(\bar{k})\}$$

Acknowledgement I am indebted to professor Kleinrock for bringing the jockeying problem to my attention by his talk at the ITC-special seminar meeting in Adelaide, Australia, September 25-29. I am grateful to Australia Telecom and The University of Adelaide for the financial support to let me visit this meeting and university.

References.

[1] Kleinrock, L., On distributed systems, (1989), Proceedings: ITC Specialist Seminar, Adelaide, 25-29 Sept. 1989, Paper no. 32.
[2] Kleinrock, L., Queueing Systems, Volume I: Theory (1975), Wiley Interscience, New York.
[3] Livny, M. & Melman, M., "Load Balancing in Homogenous Broadcast Distributed Systems", Proceedings ACM Computer Network Performance Symposium, pp. 47-55, 1982.
[4] Tijms, H.C. (1986), "Stochastic Modelling and Analysis; A computational approach, Wiley, New York.