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A M S T E R D A M

A SIMPLE THROUGHPUT ESTIMATE AND ERROR BOUND FOR A DISCRETE-TIME SLOTTED ALOHA SYSTEM

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Abstract A simple throughput estimate is proposed for a discrete-time slotted ALOHA-system with an explicit error bound of order Δ the length of the time-slot.

1 Introduction

Since its introduction in the early seventies (cf. [1]) the famous ALOHA protocol has been widely implemented in computer performance evaluation, radio packet switching, satellite communication and data processing (cf. [2], [6], [10]). Particularly, with the present-day fast developing technology of digitized communication, slotted ALOHA has become a standard. The feature of time-slotting, however, introduces the complication of possible multiple transmission requests at the same time which in turn leads to collisions. Source interdependencies are hereby created and a standard birth-death type analysis so as to conclude a closed product form expression for the steady state distribution of busy sources is no longer applicable. Approximate results have therefore been widely investigated (cf. [3], [4], [5]). Generally, however, these can still be computationally expensive and do not guarantee an a priori error bound of their order of accuracy.

This paper aims to show that the throughput for a class of slotted ALOHA systems can be roughly evaluated by a simple and secure (continuous time) product form estimate. Particularly, an explicit error bound will hereby be provided which is of order Δ, the length of a time-slot, provided the intensities per unit of time are of finite order. Though robust this estimate may be practical for quick engineering purposes so as to obtain:

(i) A first indication of the order of magnitude.
(ii) Quick qualitative or quantitative insights.

The prooftechnique, based on inductive Markov reward arguments, is of interest in itself and seems promising for further extension to more complicated communication architectures such as to evaluate carrier-sense-multiple-access schemes (e.g. [7], [11]), or to study the effect of propagation delays (cf. [8]).

2 Model

Consider a communication system consisting of Z sources that generate packets (messages) to be transmitted as follows. The time is slotted in time-slots (intervals) of length Δ. A source is either idle or busy. At the end of a time slot an idle source h generates a packet for transmission with probability p_h, in which case it becomes busy, while a busy source h will attempt to transmit its packet with probability q_h. However, as there is only one channel for transmission, only one source can transmit at a time. With $H=\{h_1,\ldots,h_n\}$ representing the busy sources a source, an attempt of source $h\in H$ will thus be successful with probability

$$M(h|H) = q_h \prod_{\{\alpha\in H-h\}} [1-q_\alpha] \tag{1}$$

When more than one busy source attempt to transmit at the same time, a collision arises and all these sources remain idle. More than one source, however, can generate at the same time. More precisely, when the system is in state H, the probability that at the end of a time slot a set of source $G \subset H^c$ will generate a packet simultaneously is given by

$$\prod_{h \in G} p_h \prod_{h \in H^c/G} [1-p_h] \tag{2}$$

Due to the multiple transition and collision feature, the present system does not exhibit a simple closed form expression. We therefore aim to propose and determine the order of accuracy of a simple estimate for the throughput. Hereby the values p_h and q_h must be thought of as being of order Δ, as naturally corresponding to discrete-time analogs of continuous-time models with bounded intensities.

2.2 Estimate

Throughout let

$$H = \{h_1, \ldots, h_n\}$$

denote the state in which sources $h_1, \ldots, h_n$ are busy. Let $p(H,H')$ be the one-step transition probability for a change from state H into state H' of the Markov chain at time points $0, \Delta, 2\Delta, \ldots$. Clearly, this chain is irreducible at the finite set of states $S = \{H | H = \{h_1, \ldots, h_n\}, n \le M\}$ so that a unique steady state distribution $\pi(.)$ exists (e.g. [12]). Now let Q such that

$$Q \ge \sum_{h=1}^{Z} [p_h + q_h] \tag{3}$$

Then as the steady state distribution $\pi(.)$ is uniquely determined (up to normalization) by the global balance equations

$$\pi(H) \sum_{H' \ne H} p(H,H') = \sum_{H' \ne H} \pi(H')\, p(H',H), \tag{4}$$

where the transition from H into itself is deleted as it would contribute equally to both the left and right hand side, it also represent the steady state distribution of the Markov chain at S with one-step transition probabilities defined by

$$\begin{aligned} p_1(H,H) &= 1 - \textstyle\sum_{H' \ne H} p(H,H')/Q \\ p_1(H,H') &= p(H,H')/Q \qquad (H' \ne H) \end{aligned} \tag{5}$$

More precisely, with $\pi_1(.)$ its steady-state distribution we thus have $\pi(.) = \pi_1(.)$ at S. Now consider a Markov chain at S with one-step transition probabilities

$$\begin{aligned} p_2(H,H) &= 1 - \textstyle\sum_{h=1}^{Z} [p_h + q_h]/Q \\ p_2(H,H+h) &= p_h/Q \qquad (h \notin H) \\ p_2(H,H-h) &= q_h/Q \qquad (h \in H) \\ p_2(H,H') &= 0 \quad \text{otherwise} \end{aligned} \tag{6}$$

Then by standard birth-death arguments one directly verifies the steady state distribution $\pi_2(.)$, with c a normalizing constant, given by

$$\pi_2(H) = c \prod_{h \in H} [p_h/q_h] \tag{7}$$

The value

$$\tilde{g} = g_2 = \sum_H \pi_2(H) \; [\sum_{h \in H} q_h] \; \Delta^{-1} \tag{8}$$

can thus be easily computed and proposed as an estimate for

$$g = g_1 = \sum_H \pi_1(H) \; [\sum_{h \in H} M(h|H)] \; \Delta^{-1} \tag{9}$$

which represents the throughput, that is the mean number of successful transmissions per unit of time, of the original system. Clearly this estimate can be very robust but in what follows we will show that its accuracy is of order Δ provided the probabilities p_h and q_h are of order Δ. Roughly speaking that is, provided these probabilities correspond to finite intensities per unit of time.

Remark 1 The scaling by a factor Q corresponds to the standard so-called uniformization technique (cf. [12], p.110) for continuous-time Markov chains. Here it is used to make the original and approximate model compatible while the approximate model is to be introduced as a Markov chain, i.e. with $\sum_{H'} p(H,H')=1$.

2.3 Error Bound

Let

$$\begin{aligned} \lambda_h &= p_h/\Delta \\ \mu_h &= q_h/\Delta \\ L &= \max_h \lambda_h \\ M &= \max_h \mu_h \end{aligned} \tag{10}$$

so that p_h and q_h can be seen as approximate probabilities in time Δ for a continuous-time model with intensities λ_h and μ_h. Then, by standard Markov reward theory (e.g. [9], [12]) and letting $\emptyset$ represent the state without busy sources, we have:

$$\begin{aligned} g &= \Delta^{-1} \lim_{N\to\infty} \frac{Q}{N} V_N \; (\emptyset) \\ \tilde{g} &= \Delta^{-1} \lim_{N\to\infty} \frac{Q}{N} \tilde{V}_N \; (\emptyset) \end{aligned} \tag{11}$$

where the functions $V_N(.)$ and $\tilde{V}_N(.)$ at S for all N≥0 are defined by:

$$\begin{aligned} V_{n+1}(H) &= \sum_{h \in H} p_1(H,H-h) + \sum_{H'} p_1(H,H') \; V_n(H') \\ \tilde{V}_{n+1}(H) &= \sum_{h \in H} p_2(H,H-h) + \sum_{H'} p_2(H,H') \; \tilde{V}_n(H') \end{aligned} \tag{12}$$

In order to compare the simple estimate $\tilde{g}$ and the measure of interest g, the following lemma will be crucial. It provides bounds for so-called bias-terms uniformly in n.

Lemma 1 For all $n\geq 0$ and $H+h\in S$:

$$0 \leq \tilde{V}_n(H+h) - \tilde{V}_n(H) \leq 1 \tag{13}$$

Proof The proof will follow by induction to n. Clearly, it holds for $n=0$ as $\tilde{V}_0(.)=0$. Now assume that it holds for $n\leq m$. Then by (12):

$$\tilde{V}_{m+1}(H+h) - \tilde{V}_{m+1}(H)$$

$$= ([\textstyle\sum_{\alpha\in H} q_\alpha + q_h]\, Q^{-1}$$

$$\textstyle\sum_{\alpha\in H} q_\alpha\, Q^{-1}\, \tilde{V}_m(H-\alpha+h) + \sum_{\alpha\notin H+h} p_\alpha\, Q^{-1}\, \tilde{V}_m(H+\alpha+h) +$$

$$q_h\, Q^{-1}\, \tilde{V}_m(H) + [1 - \textstyle\sum_{\alpha\in H+h} q_\alpha\, Q^{-1} - \sum_{\alpha\notin H+h} p_\alpha\, Q^{-1}]\, \tilde{V}_m(H+h))$$

$$-$$

$$(\Sigma_{\alpha\in H}\, q_\alpha Q^{-1} + \Sigma_{\alpha\in H}\, q_\alpha Q^{-1}\, \tilde{V}_m(H-\alpha) + q_h Q^{-1}\, \tilde{V}_m(H) +$$

$$\Sigma_{\alpha\notin H+h}\, p_\alpha Q^{-1}\, \tilde{V}_m(H+\alpha) + p_h Q^{-1}\, \tilde{V}_m(H+h) +$$

$$[1 - \Sigma_{\alpha\in H+h} q_\alpha Q^{-1} + \Sigma_{\alpha\notin H+h}\, p_\alpha Q^{-1}]\tilde{V}_m(H))$$

$$=$$

$$q_h Q^{-1} +$$

$$\Sigma_{\alpha\in H}\, q_\alpha Q^{-1}[\tilde{V}_m(H-\alpha+h) - \tilde{V}_m(H-\alpha)] +$$

$$\Sigma_{\alpha\notin H+h}\, p_\alpha Q^{-1}[\tilde{V}_m(H+\alpha+h) - \tilde{V}_m(H+\alpha)] +$$

$$q_h Q^{-1}[\tilde{V}_m(H) - \tilde{V}_m(H)] + p_h Q^{-1}[\tilde{V}_m(H+h) - \tilde{V}_m(H+h)] +$$

$$[1 - \Sigma_{\alpha\in H+h}\, q_\alpha Q^{-1} - \Sigma_{\alpha\notin H+h}\, p_\alpha Q^{-1}][\tilde{V}_m(H+h) - \tilde{V}_m(H)] \tag{14}$$

where it is noted that the fourth and fifth term in the right hand side are equal to 0 but kept in for arguing below. By substituting the induction hypothesis $\tilde{V}_m(H+h)-\tilde{V}_m(H)\geq 0$ for all H and h, we directly conclude: $\tilde{V}_{m+1}(H+h)-\tilde{V}_{m+1}(H)\geq 0$. To estimate the right-hand side of (14) from above, now note that its fourth term is equal to 0 while its coefficient is exactly equal to the first additional nonnegative term $q_h Q^{-1}$. By also recalling (3) and substituting the induction hypothesis $\tilde{V}_m(H+h)-\tilde{V}_m(H)\leq 1$ for all H and h, we condlude: $\tilde{V}_{m+1}(H+h)-\tilde{V}_{m+1}(H)\leq 1$. The induction completes the proof. □

The following theorem can now be proven. It provides an explicit error bound for the accuracy of the simple estimate g for the throughput $\tilde{g}$ of the original system.

Theorem With

$$B = \Sigma_{k=1}^{Z} \binom{Z}{k} \Delta^k [\max(L,M)]^k = O(\Delta)$$

we have (15)

$$|g-\tilde{g}| \leq [2ZM + \Sigma_{\alpha=1}^{Z} \alpha \Delta^\alpha L^\alpha \binom{Z}{\alpha}]B$$

Proof From (12) we conclude for all $n\geq0$:

$$\begin{aligned}
&(V_{n+1} - \tilde{V}_{n+1})(H) = \\
&\textstyle\sum_{h\in H}[p_1(H,H-h) - p_2(H,H-h)] + \\
&\textstyle\sum_{H'}[p_1(H,H') - p_2(H,H')]\,\tilde{V}_n(H') + \sum_{H'} p_1(H,H')[V_n(H')-\tilde{V}_n(H')]
\end{aligned} \tag{16}$$

By (1), (3), (4), (8) and standard calculus we obtain:

$$\begin{aligned}
&\left|\textstyle\sum_{h\in H} [p_1(H,H-h)-p_2(H,H-h)]\right| = \\
&\left|\textstyle\sum_{h\in H} \mu_h\Delta\left(\prod_{\alpha\in H-h} [1-\mu_\alpha\Delta] - 1\right)Q^{-1}\right| \leq \\
&\Delta Q^{-1}\left(\textstyle\sum_{h\in H}\mu_h\right) \sum_{k=1}^{n} \binom{n}{k} \Delta^k M^k \leq \Delta ZMQ^{-1} \sum_{k=1}^{Z}\binom{Z}{k}\Delta^k M^k
\end{aligned} \tag{17}$$

Using that both $p_1(.,H')$ and $p_2(.,H')$ for all H' sum up to 1, and recalling the multiple transitions as per (2), we similarly conclude:

$$\begin{aligned}
&\left|\textstyle\sum_{H'} [p_1(H,H') - p_2(H,H')]\,\tilde{V}_n(H')\right| = \\
&\left|\textstyle\sum_{H'} [p_1(H,H') - p_2(H,H')]\,[\tilde{V}_n(H') - \tilde{V}_n(H)]\right| \leq \\
&\Delta ZMQ^{-1} \left[\textstyle\sum_{k=1}^{Z} \binom{Z}{k} \Delta^k M^k\right] \{\max_{h,H}|\tilde{V}_n(H-h)-\tilde{V}_n(H)|\} + \\
&\textstyle\sum_{\alpha=1}^{Z} \Delta^\alpha \binom{Z}{\alpha} L^\alpha \left[\sum_{k=1}^{Z} \binom{Z}{k} \Delta^k L^k\right]Q^{-1} \{\max_{h_1,\ldots,h_\alpha} \\
&\qquad |\tilde{V}_n(H+h_1+\ldots+h_\alpha) - \tilde{V}_n(H)|\}
\end{aligned} \tag{18}$$

From the lemma above and with

$$\delta = 2\Delta \ ZM \ B + \sum_{\alpha=1}^{Z} \alpha \ \Delta^{\alpha} \ L^{\alpha} \ \binom{Z}{\alpha} B$$

we thus conclude from (16), (17) and (18) that for all H and N≥0:

$$\max_H \left|(V_N - \tilde{V}_N)(H)\right| \le \delta Q^{-1} + \max_H \left|(V_{N-1} - \tilde{V}_{N-1})(H)\right| \le \dots \le \delta N Q^{-1} \qquad (19)$$

where the latter relation follows by iteration for n = N-1,...,0 and the fact that $\tilde{V}_0(.)=0$. Application of (11) now completes the proof. □

Remark 2 Note that the scaling factor Q does not appear in (15). It has merely been used to compare the original model with an approximate model in a convenient way, i.e. by Markov reward arguments.

Remark 3 Clearly, the error bound (15) primarily relies upon the order of magnitude of the values L and M, or more precisely λ_h and μ_h. Roughly speaking, these values represent the packet scheduling and transmission intensities normalized per unit of time. These values seem realistic to be of finite order in various typical present day applications.

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