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SIMPLE PERFORMANCE ESTIMATES AND ERROR BOUNDS
FOR SLOTTED ALOHA LOSS SYSTEMS

Nico M. van Dijk

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Simple performance estimates and error bounds for slotted ALOHA loss systems

Nico M. van Dijk
Free University, Amsterdam, The Netherlands

Abstract
Simple analytic estimates and corresponding error bounds are provided for communication or broadcast systems with state dependent message loss probabilities, such as slotted ALOHA loss systems.

Keywords
ALOHA system * throughput * product form * performance estimate * error bound.

1. Introduction

Since the introduction of the famous ALOHA communication network in the early 1970's (cf. [1]), communication or broadcast protocols such as slotted ALOHA, CSMA, BTMA and CDMA have obtained considerable attention with typical applications in computer performance evaluation, radio packet switching, satellite communication and data processing (cf. [3], [4], [11], [14], [17], [21], [22]).

These protocols involve practical features such as access limitations (e.g., a finite number of links or time slots), technical restrictions (e.g., a node cannot hear and transmit at the same time) and collisions (e.g., resulting from time slotting or propagation delays). As a result closed product form expressions have been reported for some special architectures, (cf. [4], [12], [15], [27]), but generally cannot be provided (e.g., [14]). Most of the associated literature, therefore, for example on ALOHA-systems, deals with modeling and stability issues and employs approximate analyses (e.g., [5]-[9], [22]). Particularly, approximate "averaging" assumptions such as aggregate attempt rates and/or aggregate state-independent success probabilities, are most common (e.g., [3], pp. 213-215, [11], pp. 166-169, [17], pp. 429-433, [20], [21]).

This paper concerns random access schemes with state dependent loss probabilities, such as a slotted ALOHA-loss system, and makes no averaging assumptions. In contrast, it allows the random access or success probabilities to depend on the detailed information of which other sources are busy. The main results developed are:

(i) Simple robust bounds for performance measures.
(ii) Analytic error bounds of their accuracy.

The performance bounds are based upon a product form simplification and are typically developed for quick engineering purposes such as to obtain:

(i) A first indication of order of magnitude.
(ii) Qualitative or quantitative insights.

The error bounds follow from a Markov reward comparison technique. This technique can be seen as a partial extension of monotonicity prooftechniques such as applied in [2], [18], [19] and [28] and has already been successful in various queueing network problems (cf. [24], [25], [26]). In contrast, however, in none of these references explicit error bounds are provided. Further, the application of this technique to random access protocols appears to be new.

Particularizing to slotted ALOHA-loss systems a simple throughput estimate is suggested and proven to be an upper bound within an accuracy of order d, the length of a time slot. The results seem promising for further application of the technique to more complex random access schemes, such as carrier sense multiple access protocols with collision detection (CSMA-CD).

The primary motivation for developing these results was to investigate the effect of "time slotting or relatedly "propagation delays" (cf. [13]) in ALOHA-systems. In principle, this would require a discrete-time analysis. For convenience of presentation, however, a continuous time modeling will be employed (e.g., similarly to [11], p. 168) without excluding the essential feature of interferences such as reflecting collisions.

2. Model, performance and error bounds

2.1 Model

Consider a communication system consisting of M transmitters (nodes), numbered 1,..,M. When idle, (i.e. not transmitting), node h wants to transmit a message after an exponential time with parameter γ_h. Its message length is exponential with parameter μ_h. Throughout, let $H = \{h_1, \dots, h_n\}$ denote the currently busy (i.e. transmitting) nodes and write $H + h = H \cup \{h\}$ and $H - h = H/h$. When nodes $H = \{h_1, \dots, h_n\}$ are busy (i.e. currently transmitting), a transmission that node h requests is

$$(1) \quad \begin{cases} \text{accepted and initiated with probability: } \beta(h|H) \\ \text{rejected and lost with probability: } 1-\beta(h|H). \end{cases}$$

Example 2.1 (Slotted ALOHA) Let d be the length of some fixed time slot and assume that a transmission can be started only at time integer values of d. Further, when two or more transmissions are requested in one and the same time slot they have to be aborted and to be considered as lost, or equivalently, they have to be rescheduled at the original scheduling rates. Then,

$$(2) \qquad \beta(h|H) = \prod_{s \notin H+h} e^{-d\gamma_s}.$$

Example 2.2 (Memory accessing) A transmission is to be intiated by first storing or retrieving some address at a memory disk. Each idle node h, however, regularly "communicates" with this memory disk, on the average during a fraction p_h of its idle time. As the memory disk has access for only one node at a time, we have

$$(3) \qquad \beta(h|H) = \prod_{s \notin H+h} [1-p_s].$$

2.2 Simple performance estimates

Let $\{\pi(H)\}_{H \in S}$ denote the steady state distribution of the system described above, assuming that this distribution is unique for the set of reachable states S.

An explicit product form expression for this distribution can be given only in special situations, such as with all nodes being indistinguishable or with different node-classes satisfying a "coordinate convexity" condition (cf. [4], [12], [15], [27]). Generally, however, with distinguishable nodes a closed-form expression cannot be provided.

Let $\{\bar{\pi}(H)\}_{H \in \bar{S}}$ be the corresponding steady state distribution for the system in which transmission requests are never rejected, i.e., assuming that for all h,H such that $H+h \in S$:

$$(4) \qquad \beta(h|H) = 1.$$

Then one easily checks or concludes from literature (e.g., [4], [27],) that with $\bar{c}$ a normalizing constant:

$$(5) \qquad \bar{\pi}(H) = \bar{c} \prod_{h \in H} [\gamma_h/\mu_h].$$

Now assume that for some performance function r(.) we are interested in the performance measure

$$(6) \qquad g = \sum_{H \in S} \pi(H) r(H).$$

A simple and computationally attractive estimate is then suggested by

$$(7) \qquad \bar{g} = \sum_{H \in \bar{S}} \bar{\pi}(H) r(H).$$

2.3 Error Bounds

To compare g and $\bar{g}$ let Q be such that

$$(8) \qquad Q \geq \sum_h [\gamma_h + \mu_h]$$

and define Markov chains X and $\bar{X}$ with one-step transition probabilities $p(H, H \pm h)$ and $\bar{p}(H, H \pm h)$ given by:

$$(9) \qquad \begin{cases} \overset{(-)}{p}(H,H) = 1 - \overset{(-)}{p}(H,H+h) - \overset{(-)}{p}(H,H-h) \\ \overset{(-)}{p}(H,H+h) = \gamma_h \overset{(-)}{\beta}(h|H)/Q \\ \overset{(-)}{p}(H,H-h) = \mu_h/Q \end{cases}$$

where the symbol (-) denotes throughout that the expression is to be read both with and without upper bar "-" symbol and where $\bar{\beta}(.|.) = 1$.

Further, define functions V_N and $\bar{V}_N$ for N=0,1,2,... by $\overset{(-)}{V}_0(.)=0$ and

$$(10) \qquad \overset{(-)}{V}_{n+1}(H) = r(H)/Q + \sum_{\tilde{H}} \overset{(-)}{p}(H,\tilde{H}) \overset{(-)}{V}_n(\tilde{H}).$$

Then by standard Markov reward arguments (e.g., [10], [16]) and the uniformization technique (e.g., [23], p. 110), we conclude

(11) $\overset{(\bar{\ })}{g} = \lim_{N\to\infty} \frac{Q}{N} \overset{(\bar{\ })}{V}_N(H)$

for arbitrary $H \in \overset{(\bar{\ })}{S}$. The following key-result can now be proven. It enables one to conclude that $\bar{g}$ is an upper or lower bound of g as well as to compute an error bound on its accuracy.

Theorem 2.1

(i) We have

(12) $\bar{g} \geq (\leq)\ g$

if for all h, H and n:

(13) $\bar{V}_n(H+h) \geq (\leq)\ \bar{V}_n(H)$.

(ii) We have

(14) $|\bar{g}-g| \leq \varepsilon\ C$

if for all h, H and n:

(15) $[1-\beta(h|H)] \leq \varepsilon$

(16) $|\bar{V}_n(H+h)-\bar{V}_n(H)| \leq C$.

Proof First note that for arbitrary $H \in S$: $p(H,H')$ remains restricted to $H' \in S$ while also $S \subset \bar{S}$. As a result, from (10) we derive for $H \in S$:

(17) $(\bar{V}_n-\bar{V}_n)(H) =$

$\sum_{H'} [\bar{p}(H,H')-p(H,H')]\bar{V}_{n-1}(H') +$

$\sum_{H'} p(H,H')[\bar{V}_{n-1}(H')-V_{n-1}(H')]$.

Further, from (9) we find:

(18) $\sum_{H'} [\tilde{p}(H,H')-p(H,H')]\tilde{V}_{n-1}(H') =$

$$\sum_{h \notin H} \gamma_h [1-\beta(h|H)][\tilde{V}_{n-1}(H+h)-\tilde{V}_{n-1}(H)]/Q.$$

As $p(H,H') \geq 0$ and $[1-\beta(h|H)] \geq 0$ for all H,H' and h, we obtain from substituting (18) in (17) that $\tilde{V}_n(H) \geq (\leq)\ V_n(H)$ provided $\tilde{V}_{n-1}(H') \geq (\leq)$ $V_{n-1}(H')$ for all H'. Induction to n, as $\tilde{V}_0(.) = V_0(.) = 0$, and applying (11) proves (i).

By substituting (18) in (17) again but now taking absolute values we obtain from (15) and (16) that for any $H \in S$:

(19) $$|\tilde{V}_n(H)-V_n(H)| \leq \varepsilon\ C/Q + \max_{H' \in S} |\tilde{V}_{n-1}(H')-V_{n-1}(H')| \leq \varepsilon\ n\ C/Q$$

where the latter inequality follows by iteration and noting that $\tilde{V}_0(.) =$ $V_0(.)$. Relation (11) hereby also proves (ii). □

Remarks 2.2

(i) (Bounds 12) Inequalities as (12) may seem trivial. For example, with $\tilde{g}$ and g representing the succesful number of transmission requests one directly expects the $\geq$ sign. However, one can give counterintuitive examples (cf. [2], [24], [26]) in which the throughput of service systems can be increased by rejecting specific arrivals.

(ii) (Condition 15) With $[1-\beta(.|.)]$ modeling some sort of collision due to time slotting or propagation delays, one should typically think of ε being small. For example, for $\beta(.|.)$ given by (2) the value ε is of order d, the length of the time slots.

(iii) (Condition 16) From standard Markov reward theory differences of the form $V_n(H)-V_n(H')$ are generally known to be uniformly bounded in n as based upon mean first passage times (e.g., [16]). These times, however, are generally just as difficult to estimate as the steady state distribution itself when a multi-dimensional state space is involved. In the next section therefore we apply a direct method to verify (16).

3. Application: A simple throughput bound, e.g. for slotted ALOHA-loss systems

As an application and illustration of the preceding results, in this section we will establish a simple upper bound $\bar{g}$ as well as an error bound of its accuracy on the system throughput g as determined by (5), (6), (7) and

(20) $\quad r(H) = \sum_{h \in H} \mu_h .$

First, a key-lemma is given.

Lemma 3.1 For all n,h and H:

(21) $\quad 0 \leq \bar{V}_n(H+h) - \bar{V}_n(H) \leq 1.$

Proof This will follow by induction to n. As $\bar{V}_0(.)=0$, (21) holds for n=0. Suppose that (21) holds for all $n \leq m$, h and H. Then by (10) and (20) we conclude:

(22) $\quad \bar{V}_{m+1}(H+h) - \bar{V}_m(H) =$

$$\Big\{\sum_{\ell \in H+h} [\mu_\ell/Q] + \sum_{\ell \notin H+h} [\gamma_\ell/Q]\bar{V}_m(H+h+\ell) +$$

$$\sum_{\ell \in H+h} [\mu_\ell/Q]\bar{V}_m(H+h-\ell) + (1-\sum_{\ell \notin H+h} [\gamma_\ell/Q] + \sum_{\ell \in H+h} [\mu_\ell/Q])\bar{V}_m(H+h)\Big\}$$

$$-$$

$$\Big\{\sum_{\ell \in H} [\mu_\ell/Q] + \sum_{\ell \notin H} [\gamma_\ell/Q]\bar{V}_m(H+\ell) +$$

$$\sum_{\ell \in H} [\mu_\ell/Q]\bar{V}_m(H-\ell) + (1-\sum_{\ell \notin H} [\gamma_\ell/Q] + \sum_{\ell \in H} [\mu_\ell/Q])\bar{V}_m(H)\Big\}$$

$$=$$

$$[\mu_h/Q] + \sum_{\ell \in H} [\mu_\ell/Q][\bar{V}_m(H+h-\ell) - \bar{V}_m(H-\ell)] +$$

$$\sum_{\ell \notin H+h} [\gamma_\ell/Q][\bar{V}_m(H+h+\ell) - V_m(H+\ell)] +$$

$$(1-\sum_{\ell \in H+h} [\mu_\ell/Q] - \sum_{\ell \notin H+h} [\gamma_\ell/Q]) \cdot$$

$$[\bar{V}_m(H+h) - \bar{V}_m(H)]$$

By substituting the induction hypothesis (21) for n=m and recalling (8), one immediately verifies (21) also for n=m+1. □

Now let $\bar{g}$ be computed by (5), (7) and (20). Then, by combining theorem 2.1 and lemma 3.1, we immediately obtain:

Corollary 3.2 (Throughput bounds) With ε given by (15):

(23) $\bar{g} \le g \le \hat{g} + \varepsilon$

Example (Slotted ALOHA). With $\beta(.|.)$ given by (2):

(24) $\bar{g} \le g \le \bar{g} + d[\gamma_1 + \ldots + \gamma_M]$.

Evaluation A technique is introduced so as to provide simple performance bounds or study the effect of interferences and collisions in ALOHA-type communication or broadcast structures. Extensions such as to CSMA-schemes seem promising.

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