SERIE RESEARCH MEMORANDA

TRUNCATION OF MARKOV DECISION PROBLEMS
WITH A
QUEUEING NETWORK OVERFLOW CONTROL APPLICATION

by

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Research Memorandum 1989-65
September 1989

VRIJE UNIVERSITEIT
FACULTEIT DER ECONOMISCHE WETENSCHAPPEN
EN ECONOMETRIE
AMSTERDAM

TRUNCATION OF MARKOV DECISION PROBLEMS WITH A QUEUEING NETWORK OVERFLOW CONTROL APPLICATION

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Abstract Conditions are provided to conclude an error bound for truncations and perturbations of Markov decision problems. Both the average and finite horizon case are covered. The results are illustrated by a truncation of a Jacksonian queueing network with overflow control. An explicit error bound for this example is obtained.

Key-words Markov Decision Problems * Truncation * Perturbation * Error Bound * Bias-Terms * Queueing Network * Overflow.

1 Introduction

Markov decision theory is known as a succesful modeling tool for dynamic sequential decision problems. Classical applications are found in fields as inventory management, maintenance and reliability while more presently a growing interest arises in manufacturing and, most notably, performance evaluation of computer or communication systems (e.g. Tijms [24]). One may think, for example, of dynamic or alternate routing of messages in communication systems or processor and resource allocation in computer networks (cf. Ott and Krishnan [12], Tijms [24]). Particulary in these latter present-day applications, however, the major practical drawback of computational complexity comes in as the number of states is often astronomic or even as in open systems, infinite. In addition, multi-dimensional structures are hereby frequently involved. As MDP's can only most rarely be solved analytically, state space truncation methods are thus of significant practical relevance.

For uncontrolled Markov chains the technique of state space truncation is a common feature in practice. However, even in this case theoretical support in terms of orders of accuracy or rates of convergence seems hardly available. Convergence proofs as the truncation size tends to infinity have already been investigated in the early fifties by Savymsakov and were cristallized most notably by Seneta [21], [22] with reference to private communications with Kendall. A detailed study of these convergence results as well as an extensive list of related literature can be found in Seneta [23]. In this latter reference, simple error bounds were provided (cf. theorem 6.4 and its corollary, p. 215), but these are just robust bounds which do not secure an order of accuracy. Recently, in Van Dijk [29] therefore, a condition has been provided from which error bound results for Markov chain truncation can be concluded. The crucial part of the verification of this condition comes down to the estimation (bounding) of so-called bias terms for appropriate reward structures. Its verification was illustrated for two specific two-dimensional queueing examples.

A somewhat related issue is that of computing an error bound for approximate or perturbed Markov (reward) chains. Error bound results to this end have been established. In Schweitzer [18] and Meyer [10] the effect of perturbations is studied for finite steady state distributions. In Whitt [33] and Hinderer [4] approximate dynamic programs are studied for the finite horizon and infinite horizon discounted reward case. In Van Dijk and Puterman [31] these results were generalized to the average reward case for uncontrolled Markov reward structures. None of these results, however, can directly be adopted for truncation purposes as they essentially all require one and the same or at most minorly perturbated state space (cf. Whitt [33]). Furthermore, only Whitt [33] and Hinderer [4] are concerned with MDP. Their error bounds though do not allow a limiting argument for the average reward case (see remark 2.1).

The present paper concerns both truncation, as it primary focus, and perturbation results in a unifying manner and extends the results from [29] and [31] to MDP's. This latter extension is briefly mentioned as rather direct in [31] for the perturbation part, but is less obvious and in fact as per the proof of theorem 2.2, turns out to be technically more complicated when also truncations and/or unbounded reward structures are included as total and possibly optimal reward structures are hereby cut off while also different states and optimal actions are to be compared. Furthermore, the essential step of estimating bias terms now has to be investigated under various policies. This in turn is briefly argued in [31] as similar to the uncontrolled case as based upon bounding mean first passage times but only for simple one-dimensional situations. For more complex structures such as queueing networks, however, it remained open.

The essential part of this paper, therefore, is its illustration of how estimates for the bias terms can be obtained also under different policies and for multi-dimensional applications such as a queueing network. To this end the truncation of a Jacksonian network with overflow control will be studied as example. An explicit error bound for this example will be derived.

2 General results

2.1 Model and notation

The reader is assumed to be familiar with the concept of a Markov decision problem (MDP) or otherwise referred to excellent standard books such as Bertsekas [1], Gihman and Skorohod [2], Heyman and Sobel [3], Hordijk [4], Howard [8], Ross [17] or Tijms [24] for a precise description. Below the essential ingredients are briefly reviewed.

Consider an original discrete-time MDP with state space $S\subset N$, action sets $A(i)$ in state i, one-step reward $r^a(i)$ under action a in state i and one-step transmission probabilities $p^a(i,j)$ for a transition from state i into state j under action a. A decision rule δ is a mapping from S into the set of actions $A=\bigcup_i A(i)$ such that $\delta(i)\in A(i)$ for all i. Let Γ be the set of all possible decision rules δ and let $\Delta\subset\Gamma$ be a particular subset.

Now, also consider a related MDP, referred to as modified model hereafter, with state space $\bar{S}\subset S$, action sets $A(i)$ as above but one-step rewards $\bar{r}^a(i)$ and transition probabilities $\bar{p}^a(i,j)$. The following assumption will be essential for comparing the original and the modified MDP.

Assumption 2.1 Under any $\delta\in\Delta$, the original and modified MDP are irreducible at some set S^δ and $\bar{S}^\delta$ respectively, where $S^\delta\supset\bar{S}^\delta$.

From now on, we always use an upper bar "$-$" symbol for an expression concerning the modified MDP, in contrast with no extra symbol for the original, while the symbol "$(-)$" is used when an expression is to be read for both the original and modified MDP. Further, for notational convenience we introduce the notation $r^\delta(.)$, $\bar{r}^\delta(.)$, $p^\delta(.,.)$ and $\bar{p}^\delta(.,.)$ by

$$\overset{(-)}{r}{}^\delta(i) = \overset{(-)}{r}{}^a(i) \qquad \text{for } a = \delta(i)$$

$$\overset{(-)}{p}{}^\delta(i,j) = \overset{(-)}{p}{}^a(i,j) \qquad \text{for } a = \delta(i)$$

and for arbitrary function $g^a(i)$ let $f = \sup_{\delta\in\Gamma} f^\delta$ be given by

$$f(i) = \sup_{\delta\in\Gamma} f^{\delta(i)}(i) = \sup_{a\in A(i)} f^{a}(i)$$

2.2 Average case

Define operators ${}^{(-)}T_t^{\delta}$, $t=0,1,\ldots$ on functions $f:{}^{(-)}S\to R$ by:

$${}^{(-)}T_t^{\delta} f(i) = \Sigma_j\ {}^{(-)}p^{\delta}(i,j)\ f(j)$$

$${}^{(-)}T_{t+1}^{\delta} = {}^{(-)}T^{\delta}\ {}^{(-)}T_t^{\delta} \tag{2.1}$$

$${}^{(-)}T_0^{\delta} = I$$

and introduce functions ${}^{(-)}V_N^{\delta}$, $N=0,1,2,\ldots$ by

$${}^{(-)}V_N^{\delta} = \Sigma_{t=0}^{N-1}\ {}^{(-)}T_t^{\delta}\,{}^{(-)}r^{\delta} \tag{2.2}$$

In words that is, ${}^{(-)}V_N^{\delta}(i)$ is the expected total reward over N periods when starting in state i at time $t=0$ and applying the stationary policy $\delta^{\infty} = (\delta,\delta,\delta,\ldots)$ which prescribes one and the same decision rule δ for each period. Then under the assumption that for some $\ell\in{}^{(-)}S^{\delta}$:

$${}^{(-)}g^{\delta} = \lim_{N\to\infty} \frac{1}{N}\ {}^{(-)}V_N^{\delta}(\ell) \tag{2.3}$$

is well-defined, as naturally guaranteed when r is bounded and the MDP irreducible at ${}^{(-)}S^{\delta}$ for all stationary policies δ^{∞}, our quantity of interest is given by

$${}^{(-)}g = \sup_{\delta\in\Delta}\ {}^{(-)}g^{\delta}, \tag{2.4}$$

which represents the optimal expected average reward under all stationary policies $\{\delta^{\infty} | \delta\in\Delta\}$. The following theorem provides conditions to conclude an error bound for $\bar{g}$-g.

Theorem 2.1 *(Average case) Suppose that for some nonnegative function Φ, some initial state ℓ, with $\ell \in \bar{S}^{\delta} \subset S^{\delta}$ for all $\delta \in \Delta$, some constants $\varepsilon, \gamma, \beta > 0$, and all $\delta \in \Delta$, $i \in \bar{S}$, $t \geq 0$:*

$$\left|\Sigma_j [\bar{p}^{\delta}(i,j) - p^{\delta}(i,j)][V_n^{\delta}(j) - V_n^{\delta}(i)]\right| \leq \varepsilon\ \Phi(i) \tag{2.5}$$

$$\left|\bar{r}^{\delta}(i) - r^{\delta}(i)\right| \leq \gamma\ \Phi \tag{2.6}$$

$$\bar{T}_t^{\delta}\ \Phi(\ell) \leq \beta. \tag{2.7}$$

Then

$$|\bar{g} - g| \leq [\varepsilon + \gamma]\beta \tag{2.8}$$

Proof As for all t:

$$^{(\bar{})}V_{t+1}^{\delta} = {}^{(\bar{})}r^{\delta} + {}^{(\bar{})}T^{\delta} V_t^{\delta} \tag{2.9}$$

by virtue of (2.2), while the transition probabilities $\bar{p}(.,.)$ remain restricted to $\bar{S}^{\delta} \subset S^{\delta}$, for arbitrary $i \in \bar{S}^{\delta}$ we can write:

$$\begin{aligned}(\bar{V}_N^{\delta} - V_N^{\delta})(i) &= (\bar{r}^{\delta} - r^{\delta})(i) + (\bar{T}^{\delta}\bar{V}_{N-1}^{\delta} - T^{\delta}V_{N-1}^{\delta})(i) \\ &= (\bar{r}^{\delta} - r^{\delta})(i) + (\bar{T}^{\delta} - T^{\delta})V_{N-1}^{\delta}(i) + \bar{T}^{\delta}(\bar{V}_{N-1}^{\delta} - V_{N-1}^{\delta})(i) \\ &= \Sigma_{t=0}^{N-1}\ \bar{T}^{\delta}([\bar{r}^{\delta} - r^{\delta}] + [(\bar{T}^{\delta} - T^{\delta})V_{N-t-1}^{\delta}])(i) + \bar{T}^{\delta}(\bar{V}_0^{\delta} - V_0^{\delta})(i),\end{aligned} \tag{2.10}$$

where the latter equality follows by iteration. Now note that the last term in the last right hand side is equal to 0 as $\bar{V}_0^{\delta}(.) - V_0^{\delta}(.) = 0$ by definition. Further, as both $\bar{p}^{\delta}(.,.)$ and $p^{\delta}(.,)$ have row sums equal to one, we obtain for any s and i:

$$\begin{aligned}(\bar{T}^{\delta} - T^{\delta})V_s(i) &= \Sigma_j [\bar{p}^{\delta}(i,j) - p^{\delta}(i,j)]V_s^{\delta}(j) \\ &= \Sigma_j [\bar{p}^{\delta}(i,j) - p^{\delta}(i,j)][V_s^{\delta}(j) - V_s^{\delta}(i)]\end{aligned} \tag{2.11}$$

By substituting (2.11) in (2.10), taking absolute values and noting that $\bar{T}_t$ is a monotone operator for all $t\geq 0$, we obtain from (2.5), (2.6), (2.7) and (2.10):

$$|(\bar{V}_N^\delta - V_N^\delta)(\ell)| \leq [\gamma+\varepsilon]\ \Sigma_{t=0}^{N-1}\ \bar{T}_t^\delta \Phi(\ell) \leq [\gamma+\varepsilon]\beta N \tag{2.12}$$

Applying (2.4) completes the proof. □

Remark 2.1 (Literature)
(i) In Hinderer [4] and Whitt [33] the average case has not been dealt with. In the present setting their results would essentially lead to an error bound in (2.12) of order $O(N^2)$ (see [ ]) or $O(1/(1-\alpha)^2)$ (see [ ]), where α is a discount factor per step, so that the average case by using $\lim_{N\to\infty} 1/N$ or $\lim_{\alpha\to 1}(1-\alpha)$ cannot be concluded.

(ii) The uncontrolled results from Van Dijk and Puterman [31] are closely related but do not incorporate the special function Φ and initial state ℓ. In contrast they essentially require $\bar{S}=S$, which excludes truncations, and $r(.)$ to be uniformly bounded. The inclusion of this special function Φ and state ℓ will be crucial for truncations and unbounded rewards. (See remark 2.3).

Remark 2.2 (Importance of bias-terms) The crucial step for the above theorem is the simple relation (2.11). This step enables one to transform conditions upon $V_t^\delta(.)$ in so-called bias-terms: $V_t^\delta(j)-V_t^\delta(i)$. While $V_t^\delta(.)$ can grow linearly in t, bias terms for given i and j are generally bounded uniformly in t. More precisely, when $r^\delta(.)$ is bounded, say $|r^\delta(i)|\leq B$ for all i,δ, then by simple Markov reward arguments (cf. [31]) one proves:

$$|V_t^\delta(j)-V_t^\delta(i)| \leq 2B \min\ [R_{ij}^\delta, R_{ij}^\delta]$$

where R_{ij}^δ is the expected number of steps (mean first passage-time, e.g. [17]) to reach state j out of state i under decision rule δ. A similar though more technical result in terms of such times can be given also for unbounded rewards (cf. [27]). Most essentially, however, closed form expressions or even simple bounds for such times seem to be limited to simple one-dimensional random walks (cf. [31]). In the next section therefore, we will illustrate how estimates for these bias-terms can be derived in a different analytic manner. Most notably, this applies also to multi-dimensional applications such as queueing networks.

Remark 2.3 (Use of conditions) Roughly speaking, theorem 2.1 can be applicable in the following twofold manners given that the bias-terms can be uniformly bounded from above:

(i) By showing that the expected value of the scaling function Φ or the probability of being in states where differences in reward and transition structure are significant, is sufficiently small. This is typically the case for truncations or other types of transition modifications.

(ii) By showing that these differences themselves are sufficiently small, possibly up to a scaling funcation Φ. This typically applies to perturbations.

Both situations need to be regarded conditional to only one particular initial state ℓ at time t=0.

Remark 2.4 (Unbounded rewards) Note that no conditions are imposed upon the one-step reward function r(.) other than that we implicitly assume the average rewards g and $\bar{g}$ to be well-defined. Particularly, unbounded rewards are allowed as will be used in section 3.

Remark 2.5 (Combination) Clearly, the conditions (2.5), (2.6) and (2.7) could have been combined in one bounding condition that can be applied directly to (2.10). The present slightly more restrictive conditions are preferred as they are naturally verified.

Remark 2.6 (Subset Δ) Note that the value $\overset{(-)}{g}$ as per (2.4) and dealt with in theorem 2.1 concerns an optimum over a possibly restricted subset $\Delta \subset \Gamma$.

2.3 Finite horizon case

Assume $\Delta=\Gamma$ and for arbitrary finite integer N let the finite horizon optimal reward functions $\bar{V}_N(.)$ be recursively determined by $\bar{V}(.)=0$ and

$$\bar{V}_{n+1} = \sup_{\delta\in\Gamma}[\bar{r}^{\delta} + \bar{T}^{\delta}\bar{V}_n] \qquad (n=0,1,\ldots,N-1) \tag{2.13}$$

The following theorem provides conditions, similarly to theorem 2.1, to conclude an error bound for $\bar{V}_N - V_N$ linearly in N. To this end, for arbitrary $\pi = (\delta_0,\delta_1,\delta_2,\ldots,\delta_{N-1})$ with $\delta_j \in \Delta$, $j=0,\ldots,N-1$, define for $t=0,1,\ldots,N-1$:

$$\bar{T}^{\pi}_{t+1} = \bar{T}^{\delta_0}\,\bar{T}^{\delta_1}\ldots\bar{T}^{\delta_t}.$$

Theorem 2.2 *Suppose that for nonnegative function Φ, some initial state $\ell\in\bar{S}$, some constants $\varepsilon,\gamma>0$, all $i\in\bar{S}$ and $n\le N$:*

$$\sup_{a\in A(i)} \left|\Sigma_j[\bar{p}^a(i,j)-p^a(i,j)][V_n(j)-V_n(i)]\right| \le \varepsilon\,\Phi(i) \tag{2.14}$$

$$\sup_{a\in A(i)} \left|\bar{r}^a(i)-r^a(i)\right| \le \gamma\,\Phi(i) \tag{2.15}$$

$$\sup_{\pi}\ \bar{T}^{\pi}_n\,\Phi(\ell) \le \beta \tag{2.16}$$

Then

$$\left|(\bar{V}_N - V_N)(\ell)\right| \le [\varepsilon+\gamma]\,\beta\,N. \tag{2.17}$$

Proof First note that for arbitrary functions $g_1(.,.), g_2(.,.)$: $\bar{S}\times A \to R$

$$\left|\sup_{\delta\in\Gamma} g_1^{\delta}(i) - \sup_{\delta\in\Gamma} g_2^{\delta}(i)\right| =$$

$$\left|\sup_{a\in A(i)} g_1(i,a) - \sup_{a\in A(i)} g_2(i,a)\right| \le$$

$$\sup_{a\in A(i)}\left|g_1(i,a) - g_2(i,a)\right| = \sup_{\delta\in\Gamma}\left|g_1^{\delta}(i) - g_2^{\delta}(i)\right| \tag{2.18}$$

As a result, from (2.13), (2.18) and the fact that $\bar{T}^{\delta}$ remains restricted to $\bar{S}\subset S$, we obtain similarly to (2.10) for $i\in\bar{S}$:

$$|(\bar{V}_{n+1} - V_{n+1})(i)| \le$$

$$|\sup_{\delta\in\Gamma}[\bar{r}^{\delta}+\bar{T}^{\delta}\bar{V}_n](i) - \sup_{\delta\in\Gamma}[\bar{r}^{\delta}+\bar{T}^{\delta}V_n](i)| +$$

$$|\sup_{\delta\in\Gamma}[\bar{r}^{\delta}+\bar{T}^{\delta}V_n](i) - \sup_{\delta\in\Gamma}[r^{\delta}+T^{\delta}V_n](i)| \le$$

$$\sup_{\delta\in\Gamma}|\bar{T}^{\delta}(\bar{V}_n-V_n)(i)| + \sup_{\delta\in\Gamma}|(\bar{r}^{\delta}-r^{\delta})(i)| + \sup_{\delta\in\Gamma}|(\bar{T}^{\delta}-T^{\delta})V_n(i)| \tag{2.19}$$

Now note that for arbitrary $\tau>0$ and any $n\le N$ there exist a decision rule δ_n such that for all $i\in\bar{S}$:

$$\sup_{\delta\in\Gamma}|\bar{T}^{\delta}(\bar{V}_n-V_n)(i)| \le |\bar{T}^{\delta_n}(\bar{V}_n-V_n)(i)| + [\tau/N] \tag{2.20}$$

Repeating (2.19) for $n=N,\dots,0$ and using (2.14), (2.15), (2.16) and (2.20) gives:

$$|(\bar{V}_N-V_N)(\ell)| \le N[\tau/N] + [\varepsilon+\gamma]\sum_{t=0}^{N-1} \bar{T}^{\delta_N}\bar{T}^{\delta_{N-1}}\dots T^{\delta_{t+1}}\Phi(\ell)$$

$$\le \tau + [\varepsilon+\gamma]\,\beta\, N \tag{2.21}$$

Choosing τ arbitrarily yields (2.17). □

Remark 2.7 (Average result) Under standard regularity conditions (e.g. Ross [17] we have

$$\overset{(-)}{g} = \lim_{N\to\infty}\frac{1}{N}\overset{(-)}{V}_N(\ell)\ , \tag{2.22}$$

so that (2.8) might also be concluded by applying theorem 2.2 for arbitrarily large N. Clearly, however, the conditions are more difficult to verify, of which particularly the estimation of the bias-terms $V_n(j)-V_n(i)$. Moreover, note that in this subsection we imposed the restrictive condition $\Delta=\Gamma$, as used in the crucial inequality (2.18).

2.4 Special situations

(i) Pure perturbation: The case of merely perturbed one-step rewards and transition probabilities as in [27] and [31] is obtained by assuming:

$$\bar{S} = S$$

(ii) Pure truncation : To illustrate how truncations are covered assume for instance that for some $L < \infty$:

$$\begin{cases} \bar{p}^{\delta}(i,j) = 0 & j>L,\ i\le L \\ \bar{p}^{\delta}(i,j) = p^{\delta}(i,j) & j\ne t^{\delta}[i],\ j\le L,\ i\le L \\ \bar{p}^{\delta}(i,t^{\delta}[i]) = p^{\delta}(i,t^{\delta}[i]) + \sum_{j>L} p^{\delta}(i,j) & i\le L \end{cases} \tag{2.23}$$

where $t^{\delta}[i]\le L$ is some given "state of truncation" for any $i\le L$. In words that is, all transitions of the original matrix $p^{\delta}(i,j)$ out of state i beyond a certain threshold L are reflected to one and the same state $t^{\delta}[i]$. Condition (2.5) then simply reduces to:

$$\left|\sum_{j>L} p^{\delta}(i,j)[V_t^{\delta}(j)-V_t^{\delta}(t^{\delta}[i])]\right| \le \varepsilon\, \Phi(i) \tag{2.24}$$

The fact that different absorption states $t^{\delta}[i]$ for different states i can be chosen will naturally come up when multi-dimensional applications are transformed in a one-dimensional description given.

Remark 2.8 (Other truncations) The truncation (2.23) is a natural one as it corresponds to the original model as long as the truncation limit L is not exceeded. Clearly, similar conditions can be provided for other types of truncations. For example, rather than letting a transition $i\to j$ for all $j>L$ transform into one and the same state $t^{\delta}[i]$, we can also let it transform into different states in a randomized manner.

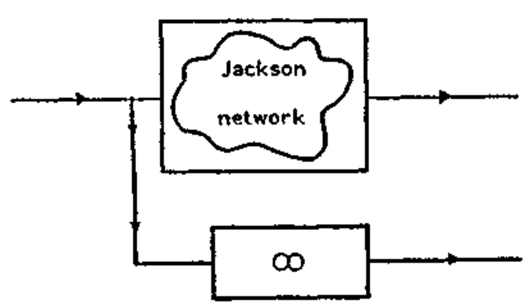
3. Application

In this section we wish to illustrate the preceding results and most of all the verification of the necessary conditions in a concrete situation. To this end, a queueing network with overflow control will be investigated. Even in the uncontrolled case or under a fixed stationary policy a closed form expression for this example is not available. The verification of the conditions, such as the estimation of bias-terms, is of particular interest as a multi-dimensional state space is involved (see remark 2.2).

As per our prime motivation, we restrict the application to the average case (section 2.2) and a state-space trucation (section 2.4 ii). The essential difficulties of having to estimate bias-terms under different decisions and to verify condition (2.7) with β small for an appropriate scaling function Φ are hereby illustrated. With more complex notation small perturbations in the transmission and reward structure can easily be included.

3.1 Model.

Consider an open Jacksonian queueing network of N service stations with jobs routing from one station to another to receive certain amounts of service before leaving the network again. Jobs arriving from outside at the network however can be rejected and rerouted to an additional "overflow" service station N+1 when the network is "overloaded".



More precisely, when accepted by the network an arriving job is assigned station j with probability $\alpha_j = p_{0j}$, j=1,...,N. Upon service completion at a station i, i=1,...,N a job routes to another station j, j=1,...,N with probability p_{ij} or leaves the system with probability $p_{i0} = [p_{i1}+\ldots+p_{iN}]$. The service rate at station i is $\mu_i(n_i)$ when n_i jobs are present, where $\mu_i(.)$ is assumed to be bounded and nondecreasing. With $\bar{n} = (n_1,\ldots,n_N)$ denoting the number of jobs n_i at stations $i=1,\ldots,N$, the arrival rate at the network is state-dependent as denoted by $\lambda(\bar{n})$, where $\lambda(\bar{n})$ is assumed to be bounded by some number λ.

Overflow control. Upon arrival a job can be rejected and rerouted to the overflow station N+1 depending upon the current state $\bar{n}$ of the Jackson network only. A control policy will determine in which states jobs are accepted/rejected by the network under the condition that the network is assumed to have a large but limited capacity of no more than C jobs in total. The overflow station is a multi-server station, say with M exponential servers at a rate μ each, and an infinite capacity. Hence, $\mu_{N+1}(m) = m\mu$ for $m < M$ and $M\mu$ for $m \geq M$, denotes its service rate when m jobs are present. (Here one may typically think of M to be very large). Assume $\lambda < \mu M$.

Objective. The control objective is to minimize the work to be offered by the system, i.e. the total amount of service provided at any of the stations $i=1,\ldots,N+1$ per unit of time when the system is in equilibrium.

Remark. Under fixed control policies the system is not of product form due to the dynamic (state-dependent) routing feature. For example, even with N=1 (only one regular station), $\lambda(.)=\lambda$ for some constant λ and under the simple decision rule that jobs are routed to the overflow station only when $n_1 \geq C$ (in that case the capacity of station 1), an explicit product form expression for the steady state distribution does not hold.

A computational procedure, such as successive approximation or policy improvement, is therefore required. To this end, in turn, as the state space is infinite, a state space truncation is needed.

3.2 Parametrization

In advance it is noted that it is more natural and convenient to use a multi-dimensional rather than one-dimensional description. Clearly, in order to apply the results of section 2 directly we could label the states one-dimensionally. It is more convenient though, to simply reread all results of section 2 with multi-dimensional states by identifying a state with a symbol i or j. This will be assumed hereafter without further mentioning.

Denote by $[\bar{n},m]$ the state with the network configuration $\bar{n}=(n_1,\ldots,n_N)$ with n_j jobs at station $j=1,\ldots,N$ and m jobs at the overflow station N+1, and consider the set of possible actions $A = A(\bar{n},m)$ for all $(\bar{n},m)$ given by

$$A = \{1,2\}$$

where

1: "accept an arriving job"
2: "reroute an arriving job"

Let $S = \{(\bar{n},m) \mid n_1+\ldots+n_N \le C,\ m \ge 0\}$, and

$$\Delta = \{\delta\colon S \to A \mid \delta(\bar{n},m)=\gamma(\bar{n}) \text{ for some } \gamma(.) \text{ and } \gamma(\bar{n})=2 \text{ for } n_1+\ldots+n_N = C\}$$

the set of possible decision rules δ describing action $\delta(\bar{n},m)=1$ (accept) or $\delta(\bar{n},m)=2$ (reroute) for all states $(\bar{n},m)\in S$. Further, let $Q<\infty$ be a finite number such that

$$Q \ge \lambda(\bar{n}) + [\mu_1(n_1)+\ldots+\mu_N(n_N)] + \mu(m) \quad \text{for all } (\bar{n},m)\in S. \tag{3.1}$$

By $\bar{n}+e_i$, $\bar{n}-e_i$ and $\bar{n}+e_i-e_j$ we denote the network configuration equal to $\bar{n}$ up to one job more at station i, one job less at station i or one job moved from station i to station j respectively.

The underlying continuous-time control problem can then be transformed in a discrete-time MDP by virtue of the data-transformation or uniformization technique (e.g. [24], p.110) as follows:

$$r^a([\bar{n},m]) = [\Sigma_{j=1}^{N} \mu_j(n_j) + \mu(m)]/Q \qquad (a=1,2)$$

and

$$p^a([\bar{n},m],[\bar{n}+e_i,m]) = \lambda(\bar{n})\alpha_i/Q \qquad (a=1) \quad (i=1,\dots,N)$$

$$p^a([\bar{n},m],[\bar{n},m+1]) = \lambda(\bar{n})/Q \qquad (a=2)$$

$$p^a([\bar{n},m],[\bar{n}-e_i,m]) = \mu_i(n_i)p_{io}/Q \quad (a=1,2) \quad (i=1,\dots,N) \qquad (3.2)$$

$$p^a([\bar{n},m],[\bar{n},m-1]) = \mu(m)/Q \qquad (a=1,2)$$

$$p^a([\bar{n},m],[\bar{n}-e_i+e_j,m]) = \mu_i(n_i)p_{ij}/Q \quad (a=1,2) \quad (i,j=1,\dots,N)$$

$$p^a([\bar{n},m],[\bar{n},m]) = 1 - [\lambda(\bar{n}) + \mu_1(n_1)+\dots+\mu_N(n_N) + \mu(m)]/Q \quad (a=1,2)$$

Assumption 3.1 For each $\delta\in\Delta$ the above MDP is irreducible at some set S^δ with $[\bar{0},0]\in S$, where $\bar{0}=(0,\dots,0)$.

The value g^δ as defined by (2.3) is then well-defined and represents the work W^δ offered by the system per unit of time in equilibrium under the stationary control policy δ.

3.3 Truncation

We will truncate the queue length of the overflow station, say at some level Z. More precisely, let

$$\bar{S} = \{(\bar{n},m) \mid n_1+\dots+n_N \le C, \ m \le Z\}$$

and define the modified MDP as according to (2.23) with $t([\bar{n},m])=[\bar{n},m]$ by

$\bar{r}^a(.) = r^a(.)$

$\bar{p}^a([\bar{n},Z],[\bar{n},Z+1]) = 0 \quad (a=2)$

$\bar{p}^a([\bar{n},Z],[\bar{n},Z]) = p^a([\bar{n},Z],[\bar{n},Z]) + p^a([\bar{n},Z],[\bar{n},Z+1])$

$$\bar{p}^a([\bar{n},m],[\bar{n}',m']) = p^a([\bar{n},m],[\bar{n}',m']) \quad \text{otherwise.} \tag{3.3}$$

The following assumption is a natural consequence of the assumption above for the original MDP.

Assumption 3.2 For each $\delta\in\Delta$ the truncated MDP is irreducible at some set $\bar{S}^\delta \subset (S^\delta \cap \bar{S})$ with $[\bar{0},0]\in\bar{S}^\delta$.

Then also the value $\bar{g}^\delta$ as defined by (2.3) is well-defined.

We are now ready to apply the results of section 2.2 or more precisely theorem 2.1 with $\ell=[\bar{0},0]$.

3.4 Error bound

To apply theorem 2.1 the following key-lemma will be proven first.

Lemma 3.1. *For any $\delta\in\Delta$, all $t\geq 0$ and $[\bar{n},m+1],[\bar{n},m]\in\bar{S}$:*

$$0 \leq V_t^\delta([\bar{n},m+1]) - V_t^\delta([\bar{n},m]) \leq 1. \tag{3.4}$$

Proof. This will be given by induction to t. Clearly, (3.4) holds for $t=0$ as $V_0^\delta(.)=0$. Suppose that (3.4) holds for $t\leq z$. Then we will verify (3.4) for $t=z+1$.
To this end, let $1_{\{A\}}$ denote an indicator of an event A, i.e. $1_{\{A\}}=1$ if A is satisfied and $1_{\{A\}}=0$ otherwise, and for convenience substitute $Q^{-1}=h$.

$$V^\delta_{z+1}(\bar{n},m+1) - V^\delta_{z+1}(\bar{n},m)$$

$$=$$

$$\Big\{\Sigma^N_{j=1}\ h\ \mu_j(n_j) + h\mu(m+1) +$$

$$h\ \lambda(\bar{n})\ 1_{\{\gamma(\bar{n})=2\}}\ V^\delta_z(\bar{n},m+2) +$$

$$h\ \lambda(\bar{n})\ 1_{\{\gamma(\bar{n})=1\}}\ \Sigma^N_{j=1}\ \alpha_j\ V_z(\bar{n}+e_j,m+1) +$$

$$h\ \Sigma^N_{j=1}\ \Sigma^N_{i=1}\ \mu_j(n_j)\ p_{ji}\ V_z(\bar{n}-e_j+e_i,m+1) +$$

$$h\ \Sigma^N_{j=1}\ \mu_j(n_j)\ p_{j0}\ V_z(\bar{n}-e_j,m+1) + h\mu(m+1)\ V_z(\bar{n},m) +$$

$$[1 - h\lambda(\bar{n}) - h\ \Sigma^N_{j=1}\mu_j(n_j) - h\mu(m+1)]\ V_z(\bar{n},m+1)\Big\}$$

$$-$$

$$\Big\{\Sigma^N_{j=1}\ h\mu_j(n_j) + h\mu(m) +$$

$$h\ \lambda(\bar{n})\ 1_{\{\gamma(\bar{n})=2\}}\ V^\delta_z(\bar{n},m+1) +$$

$$h\ \lambda(\bar{n})\ 1_{\{\gamma(\bar{n})=1\}}\ \Sigma^N_{j=1}\alpha_j\ V^\delta_z(\bar{n}+e_j,m) +$$

$$h\ \Sigma^N_{j=1}\ \Sigma^N_{i=1}\ \mu_j(n_j)\ p_{ji}\ V_Z(\bar{n}-e_j+e_i,m) +$$

$$h\ \Sigma^N_{j=1}\ \mu_j(n_j)\ p_{j0}\ V_z(\bar{n}-e_j,m) + h\mu(m)\ V^\delta_z(\bar{n},m-1) +$$

$$[1 - h\ \lambda(\bar{n}) - h\ \Sigma^N_{j=1}\ \mu_j(n_j) - h\mu(m)]\ V^\delta_z(\bar{n},m)\Big\}$$

$$=$$

$$h[\mu(m+1) - \mu(m)] +$$

$$h\ \lambda(\bar{n})\ 1_{\{\gamma(\bar{n})=2\}}\ [V^\delta_z(\bar{n},m+1) - V^\delta_z(\bar{n},m)] +$$

$$h\ \lambda(\bar{n})\ 1_{\{\gamma(\bar{n})=1\}}\ \Sigma^N_{j=1}\ \alpha_j\ [V^\delta_z(\bar{n}+e_j,m+1) - V^\delta_z(\bar{n}+e_j,m)] +$$

$$h\ \Sigma^N_{j=1}\ \Sigma^N_{i=1}\ \mu_j(n_j)\ p_{ji}\ [V_z(\bar{n}-e_j+e_i,m+1)-V_z(\bar{n}-e_j+e_i,m)] +$$

$$h\ \Sigma^N_{j=1}\ \mu_j(n_j)\ p_{j0}\ [V_z(\bar{n}-e_j,m+1) - V^\delta_z(\bar{n}-e_j,m)] + h\mu(m)\ [V^\delta_z(\bar{n},m)-V^\delta_z(\bar{n},m-1)] +$$

$$h\ [\mu(m+1) - \mu(m)]\ [V^\delta_z(\bar{n},m) - V^\delta_z(\bar{n},m)] +$$

$$[1 - h\lambda(\bar{n}) - h\ \Sigma^N_{j=1}\mu_j(n_j) - h\mu(m+1)]\ [V^\delta_z(\bar{n},m+1) - V^\delta_z(\bar{n},m)] \qquad (3.5)$$

Here it is noted that the one but last term is equal to 0 but kept in for clarity of the arguments below. As $\mu(m)$ is nondecreasing in m, the right hand side of (3.5) is directly estimated from below by 0 by substituting the lower estimate 0 from (3.4) for t=z as per induction hypothesis. To estimate the right hand side of (3.5) from above by 1, now recall that the one but last term is equal to 0 while its coefficient is exactly equal to the first term in this right hand side: $h[\mu(m+1)-\mu(m)]$. By substituting the upper estimates 1 from (3.4) for t=z as per induction hypothesis again, summing all terms and recalling (3.1) with $Q=h^{-1}$, the upper estimate 1 is then concluded, that is (3.4) with t=z+1.

We are now able to verify condition (2.5). By combining (3.2), (3.3) and (3.4), similarly to (2.24) we find for any $\delta\in\Delta$ and $[\bar{n},m]\in\bar{S}^{\delta}$:

$$|\Sigma_{[\bar{n}',m']}\ [\bar{p}^{\delta}([\bar{n},m],[\bar{n}',m']) - p^{\delta}([\bar{n},m],[n',m'])][V_t^{\delta}([\bar{n}',m']) - V_t^{\delta}([\bar{n},m])]$$

$$=$$

$$|\lambda(\bar{n})Q^{-1}1_{\{\delta(\bar{n},m)=Z\}}\ 1_{\{m=Z\}}[V_t^{\delta}([\bar{n},Z+1]) - V_t^{\delta}\ ([\bar{n},Z])]| \le 1_{\{m=Z\}} \quad (3.6)$$

With $\varepsilon=1$ and $\gamma=0$ (since $\tilde{r}(.)=r(.)$ as per (3.3)), for applying theorem 2.1 it thus remains to verify (2.7) with

$$\Phi([\bar{n},m]) = 1_{\{m\ge Z\}} \quad (3.7)$$

and $\bar{T}_t^{\delta}$ as defined by (2.1) with (3.3) substituted. This will be established by the following technical lemma.

Lemma 3.2 *For any $\delta\in\Delta$ and all $t\ge 0$:*

$$\bar{T}_t^{\delta}\ \Phi([\bar{0},0]) \le T_t^{\delta}\ \Phi([\bar{0},0]) \quad (3.8)$$

Proof. Let $f: S\to \mathbb{R}$ be an arbitrary function such that for all $[\bar{n},m]\in S$:

$$f([\bar{n},m+1]) - f([\bar{n},m]) \ge 0 \quad (3.9)$$

Then, from (2.1) and the fact that $\bar{S}^{\delta} \subset S^{\delta}$ we obtain similarly to (2.10) or by direct telescoping:

$$(\bar{T}_t^{\delta}-T_t^{\delta})\ f([\bar{0},0]) = \Sigma_{s=0}^{t-1}\ \bar{T}_s^{\delta}(\bar{T}^{\delta}-T^{\delta})T_{t-s-1}^{\delta}f([\bar{0},0]) \qquad (3.10)$$

Substituting (2.1), (3.2) and (3.3) we also obtain as in (3.6) for any $[\bar{n},m]\in\bar{S}^{\delta}$:

$$(\bar{T}^{\delta}-T^{\delta})\ f([\bar{n},m]) = \lambda(\bar{n})Q^{-1}1_{\{\delta(\bar{n},m)=2\}}1_{\{m=Z\}} \times$$

$$[f([\bar{n},Z+1]) - f([\bar{n},Z])] \le 0 \qquad (3.11)$$

for any f(.) satisfying (3.9). As the operators (or transition matrices) $\bar{T}_s^{\delta}$ remain restricted to $\bar{S}^{\delta}$ and are monotone (i.e. $\bar{T}_s^{\delta}\psi \le 0$ if $\psi\le 0$), from (3.10) and (3.11) we thus conclude

$$(\bar{T}_t^{\delta} - T_t^{\delta})\ f([\bar{0},0]) \le 0 \qquad (3.12)$$

provided (3.9) holds also with f replaced by $T_s^{\delta}f$ for all s and any f which itself satisfies (3.9). This in turn will be proven by induction to s as follows.
Clearly, for s=0 it holds as $T_0^{\delta}f=f$ by definition. Suppose that $T_s^{\delta}f$, with f satisfying (3.9), satisfies (3.9) for s≤z. Then, similarly to (3.5) and with $h=Q^{-1}$ we derive

$$T_{z+1}^{\delta}f(\bar{n},m+1) - T_{z+1}^{\delta}f(\bar{n},m) =$$

$$h\ \lambda(\bar{n})\ 1_{\{\gamma(\bar{n})=2\}}\ [T_z^{\delta}f(\bar{n},m+2)-T_z^{\delta}(\bar{n},m+1)] +$$

$$h\ \lambda(\bar{n})\ 1_{\{\gamma(\bar{n})=1\}}\ \Sigma_{j=1}^{N}\ \alpha_j\,[T_z^{\delta}f(\bar{n}+e_j,m+1)-T_z^{\delta}f(\bar{n}+e_j,m)] +$$

$$h\ \Sigma_{j=1}^{N}\ \Sigma_{i=1}^{N}\ \mu_j(n_j)\ p_{ji}\ [T_z^{\delta}f(\bar{n}-e_j+e_i,m+1) - T_z^{\delta}f(\bar{n}-e_j+e_i,m)] +$$

$$h\ \Sigma_{j=1}^{N}\ \mu_j(n_j)\ p_{j0}\ [T_z^{\delta}f(\bar{n}-e_j,m+1) - T_z^{\delta}f(\bar{n}-e_j,m)] +$$

$$h\ \mu(m)\ [T_z^{\delta}f(\bar{n},m) - T_z^{\delta}f(\bar{n},m-1)] +$$

$$h[\mu(m+1) - \mu(m)]\ [T_z^{\delta}f(\bar{n},m) - T_z^{\delta}f(\bar{n},m)] +$$

$$[1 - h\lambda(\bar{n}) - h\ \Sigma_{j=1}^{N}\mu_j(n_j)-h\mu(m+1)]\ [T_z^{\delta}f(\bar{n},m+1)-T_z^{\delta}f(\bar{n},m)] \qquad (3.13)$$

Substitution of the induction hypothesis $T_z^\delta f(\bar{n},m+1)-T_z^\delta f(\bar{n},m)\geq 0$ for all $(\bar{n},m)$ then directly shows that the right hand side is estimated from below by 0, i.e. (3.9) holds also with f replaced by $T_{z+1}^\delta f$. Inequality (3.12) is thus proven for arbitrary $t\geq 0$. □

Lemma 3.3 *Let $\pi(.)$ be the steady state distribution of an infinite single queue with Poisson arrivals with parameter λ, where $\lambda \geq \lambda(\bar{n})$ for all $\bar{n}$, and M exponential servers at a rate μ each. Then*

$$T_t^\delta \ \Phi \ ([\bar{0},0]) \leq \pi(m \geq Z) \tag{3.14}$$

Proof. Consider the Markov chain at N defined by transition probabilities $\bar{\bar{p}}(i,j)$:

$$\bar{\bar{p}}(i,j) = \begin{cases} h\lambda & j=i+1 \\ h\mu(i), & j=i-1 \\ [1-h\lambda-h\mu(i)], & j=i \end{cases}$$

and let $\bar{\bar{T}}_t$ be the corresponding operators as defined, similarly to (2.1), by

$$\bar{\bar{T}}_0=I,\ \bar{\bar{T}}_{t+1} = \bar{\bar{T}}\ (\bar{\bar{T}}_t), \qquad (t\geq 0),\ \text{and}$$

$$\bar{\bar{T}}f(i) = \Sigma_j \bar{\bar{p}}(i,j)f(j).$$

Then, as in [28] ór similarly to the proof of lemma 3.2 ór by using the standard sample path arguments such as in [32], one can show that

$$T_t\Phi(\bar{0},0) \leq \bar{\bar{T}}_t\ \bar{\bar{\Phi}}(0) \qquad (t\geq 0)$$

where $\bar{\bar{\Phi}}(m)=1_{\{m\geq Z\}}$.

Roughly speaking that is, at any time t, the probability of at least Z jobs at the overflow station is bounded from above by the corresponding probability when this station is considered in isolation with a constant dominating Poisson arrival rate λ and starting with an empty queue at time t=0. By virtue of dominated monotone convergence and the fact that $\bar{\bar{\Phi}}$ is nondecreasing, the proof is hereby completed by showing

$$\bar{\bar{T}}_{t+1}f(0) \geq \bar{\bar{T}}_t f(0) \tag{3.15}$$

for all t and nondecreasing functions f(.). This will be proven by induction. For t=0:

$$\bar{\bar{T}}_1 f(0) = \bar{T}f(0) = h\lambda f(1) + [1-h\lambda]f(0) \geq f(0) = T_0 f(0) \tag{3.16}$$

Suppose that (3.15) holds for $t \leq z$. Then

$$(\bar{\bar{T}}_{z+2} - \bar{\bar{T}}_{z+1})f(0) = (T_{z+1} - T_z)(Tf)(0) \geq 0 \tag{3.17}$$

by induction assumption provided Tf is nondecreasing for any nondecreasing f. This in turns follows similarly to (3.13), by

$$(Tf)(m+1) - (Tf)(m) =$$

$$h\lambda[f(m+2) - f(m+1)] + h\mu(m)[f(m) - f(m-1)] +$$

$$h[\mu(m+1) - \mu(m)][f(m) - f(m)] + [1-h\lambda-h\mu(m+1)][f(m+1)-f(m)] \geq 0 \tag{3.18}$$

□

As the value

$$\pi(m \geq Z) = \begin{cases} c[\Sigma_{k=Z}^{M-1} (\lambda/\mu)^k/k! + (\lambda/\mu M)^M \mu M/(\mu M-\lambda)], & (Z < M) \\ \\ c(\lambda/\mu M)^Z \mu M/(\mu M-\lambda), & (Z \geq M), \end{cases} \tag{3.19}$$

with

$$c^{-1} = \Sigma_{k=0}^{M-1} (\lambda/\mu)^k/k! + (\lambda/\mu M)^M \mu M/(\mu M-\lambda),$$

is rather simple to estimate and of small order for Z sufficiently large, all ingredients of theorem 2.1 have hereby been established. More precisely, by combining (3.6) (as based upon lemma 3.1), (3.8) and (3.9), and applying theorem 2.1 with $\varepsilon=1$, $\gamma=0$, $\ell=[\bar{0},0]$, $\Phi(\bar{n},m)=1_{\{m\geq Z\}}$ and $\beta=\pi(m\geq Z)$, we have proven:

Result 3.4 (Error bound) *With $\pi(m\geq Z)$ given by (3.19), and g and $\bar{g}$ the optimal (i.e. minimal) amounts of work offered by the system in equilibrium for the original (infinite) and the truncated (finite) Z-model respectively, (precisely: $\overset{(-)}{g} = \inf_{\delta\in\Delta} \overset{(-)\delta}{g}$), we have:*

$$|\bar{g}-g| \leq \pi(m\leq Z) \tag{3.20}$$

Remark 3.5 Note that theorem 2.1 is here applied for minimization rather than maximization. This however is directly justified by standardly adding a minus sign $-$, that is considering costs as a negative reward.

Conclusion 3.6 (Finite MDP) For arbitrary system parametrizations $\lambda(\bar{n})$ and $\mu_j(.)$ for $j\leq N$, the service optimization (minimization) problem of section 3.1 can be solved up to an error bound $\pi(m\geq Z)$ (given by (3.19)) by solving a finite MDP. To this end, standard computational procedures can be employed such as most notably:

(i) Successive approximation along with Odoni is error bounds (cf. [11], [24]) for its accuracy at each step.
(ii) Modified policy iteration (improvement) methods as developed in [13], [14], [15].
(iii) Linear programming codes based on LP-formulations (cf. [6], [7]).

For a detailed description of these methods the reader is referred to these references among various others.

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