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PSEUDO MAXIMUM LIKELIHOOD TECHNIQUES IN A SIMPLE

RATIONING MODEL OF THE DUTCH LABOUR MARKET

Harry Bierings

Koos Sneek

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ABSTRACT: Using pseudo maximum likelihood methods (combined with a simulated (Monte Carlo) objective function we estimate several variants of an aggregated (fix-price) rationing model for the Dutch economy. Our findings lend support to the following conclusions: (i) The theoretically most efficient PML method considered in the paper is not robust with respect to the existence of micro markets; (ii) The simulated objective function variants of the model yield parameter estimates that roughly converge to the analytic PML estimates when a sufficiently large number of replications is used. However, consistency for a finite number of replications is not approved. To fulfill the consistency requirement we introduce a bias correction, resulting in substantial gains in computer efficiency; (iii) a useful extension of the basic model both from a theoretical and empirical perspective allows for disturbances that are heteroskedastic.

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*) Central Bureau of Statistics P.O. Box 4481, 6401 CZ Heerlen (tel. 045-736666 tst 3138).

**) Free University of Amsterdam, Faculty of Economics, Department of Econometrics, P.O. Box 7161, 1007 MC Amsterdam (tel. 020-5487061).

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1. INTRODUCTION

There has been a growing interest in the estimation of non-linear econometric models by Pseudo Maximum Likelihood (PML) methods [see eg. Gourieroux et al. (1984)]. A very promising extension of PML estimation is provided by Laroque and Salanié (1988), further referred to as L&S, who combine this technique with a simulated objective function. Their application lies in the field of micro based rationing econometrics using aggregate data. Within this field their major contribution lies in the fact that the aggregation matter can be solved more easily by "aggregation by simulation" over fix-price micro markets instead of "aggregation by integration" [Kooiman and Kloek (1979), Lambert (1988)]. This is still accentuated when micro labour markets and goods markets are assumed to be related in the sense of Malinvaud (1977) [compare Kooiman (1985) where highly intractable functions are derived using the "aggregation by integration" technique]. The alternative approach of estimating an aggregate discrete rationing model without micro considerations, at least theoretically becomes obsolete, not to mention the usual but troublesome (Full Information Maximum Likelihood) estimation involved [see Kooiman and Kloek (1985)].[1] In this paper we will elaborate on the work of L&S in several ways.

Our main objective is to test the performance of a simple aggregate (fix-price) rationing model for the Dutch labour market using Pseudo Maximum Likelihood estimation techniques. Related purposes are (i) to test the robustness of the estimators with respect to the introduction of micro markets and (ii) to examine the compatibility of the various PML estimation results.

As far as the estimation method is concerned, the paper builds on L&S. We consider two versions of PML estimation. In the terminology of the latter authors we distinguish PML_1, which is a first order based method and PML_3, which also takes into account second order effects. Each application of the PML method defines an aggregate and a disaggregate variant and each variant is subdivided into an analytic version and a Monte Carlo version; in the latter the objective function is computed through simulation. For PML_1 the aggregate and the disaggregate variant are observationally equivalent.

Our results constitute an extension of the ongoing literature in several ways. The econometric contributions of the paper are of both a practical and

[1]See Quandt (1988) and the references cited there for the applications in this respect.

theoretical nature. The practical side is embedded in the two points mentioned earlier. Theoretical results relate to asymptotic theory. L&S prove the almost sure convergence of the simulated PML estimates to the analytical PML estimates if the number of replications G in the Monte Carlo experiment to compute the moments of the endogenous variables goes to infinity. They further prove that the asymptotic distribution of the simulated PML estimators when sample size $T\to\infty$ coincides with that of the PML estimators if $G/\sqrt{(T)}\to\infty$. A major finding in this paper is the convergence of the simulated PML_1 estimates to the PML_1 estimates if $T\to\infty$ for fixed G if a bias correction is applied. Furthermore, the asymptotic distribution of the corrected simulated PML_1 estimators if $T\to\infty$ coincides with that of the PML_1 estimators if $G\to\infty$. As to be expected, substantial gains in computer efficiency can be obtained. From an economic point of view, the paper gives some new results in the field of (micro based) macroeconomic rationing modelling as we show how an estimator of the number of micro labour markets can be obtained.

The remainder of the paper is organized as follows. In Section 2 we present the basic model and describe the two PML estimation methods. Section 3 lays out the analytic results for the aggregate and disaggregate rationing model. Section 4 is attributed to the consistency matter for a finite number of replications in the Monte Carlo version of PML_1. In Section 5 the empirical analysis is presented. An extension of the basic model together with its estimation results is considered in Section 6. The final section presents the conclusions and some suggestions for future research. Details on the calculation of the covariance matrix are described in the Appendices.

2. BASIC MODEL AND ESTIMATION METHODS

In this section we present the (fix-price) rationing model for the labour market and discuss the (pseudo maximum likelihood) estimation techniques. Since our focus lies on the performance of the pseudo maximum likelihood techniques and not on a rigorous description of economic behaviour, the model is kept intentionally simple. However, it still has attractive properties which will become clear later.

Suppose that the typical labour market j can be represented by the following equations of demand D_j, supply S_j, and employment Q_j:[2]

$$D_j = \frac{E(D)}{N} + \frac{\sigma_1}{\sqrt{N}}\eta_{1j}, \qquad (1)$$

$$S_j = \frac{E(S)}{N} + \frac{\sigma_2}{\sqrt{N}}\eta_{2j}, \qquad (2)$$

$$Q_j = \text{Min}(D_j, S_j), \qquad (3)$$

where E(D) and E(S) relate to the expectations of demand and supply over j (j=1,...,N), respectively; N (treated as a real number) is the number of labour markets and $\eta_j = (\eta_{1j}, \eta_{2j})$ is normal with zero mean and unit covariance matrix

$$\eta_j \sim N(0,I). \qquad (4)$$

The variables η_j are assumed to be independent over time. The parameter $\sigma = (\sigma_1, \sigma_2)$ is a vector of scales of the disturbances in (1) and (2). It is clear that the model can only be a simple representation of reality. All markets have identical expectations of D and S. Heterogeneity across markets comes up stochastically in actual labour demand, supply and employment, that is D_j, S_j and Q_j, respectively. It seems that the specification of the model implies that the value of N should be interpreted as a lower limit. To develop some intuition, if one multiplies E(D) and E(S) with some random variable in order to bring about some variation over j in the size of markets, N will probably be estimated higher. This is because the expected value of Q_t must be well below E(D) and E(S), but splitting a large market contributes more to a decrease in $E(Q_t)$ than splitting a small market. Further note that model (1)-(4) is only defined for a value of N which is sufficiently small to ensure $D_j, S_j > 0$. Too large a value of N breaks down the normality assumption. We indicate the above model specification as the disaggregate variant. The aggregate variant, which we also consider to test the robustness of the disaggregate model with respect to the introduction of

[2]Time subscripts are deleted throughout the text unless ambiguity arises.

micro labour markets, is but a special case defined for N=1.[3]

To estimate model (1)-(4), we use the above mentioned two PML methods. We assume that only E(D), E(S) and $\sum_{j=1}^{N} Q_j$ (=Q) are observed from the data. To be more explicit:

$$E(D) = L^d = Q + V, \tag{5}$$

$$E(S) = L^s = Q + U, \tag{6}$$

where Q, V, U represent aggregate employment, aggregate vacancies and aggregate unemployment, respectively.

To describe the PML procedure, let $Q = (Q_1,\ldots,Q_T)'$, and E(Q) and V(Q) be the expectation of Q and the variance of Q, respectively where V(Q) is assumed to be diagonal. Then, the PML methods provide consistent estimators of σ_1 and σ_2 by minimizing the PML function ψ_i:

$$\psi_{iT} = (Q - E(Q))'(V_i(Q))^{-1}(Q - E(Q)) + \log(\det(V_i(Q))), \tag{7}$$

where i=1 and i=3 refer to PML_1 and PML_3, respectively and $V_1(Q)=I$ and $V_3(Q)=V(Q)$.

In addition to the aggregate and disaggregate variant, an analytic, and a Monte Carlo (MC) variant of PML estimation is considered. The distinction between the analytic and MC variants is made to monitor the extra variability in the parameter estimates induced by the simulation of the objective function. In the analytic variants, explicit expressions for $E(Q_t)$ and $V(Q_t)$ are used, whereas in the MC variants these are assumed to be untractable.

In the MC variants, the procedure comprises the minimization of an approximate PML function, ψ_{iT}^{G}, as defined in (7), except that the compu-

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[3]For N=1 the aggregate (discrete rationing) model of L&S emerges. In particular, their model shows:

$D = E(D) + \sigma_1\eta_1$; $S = E(S) + \sigma_2\eta_2$; $Q = \min(D,S)$ with $Var(\eta) = I$.

One easily verifies that aggregation over markets j yields the aggregate model with the desired variance property. Note in this respect that:

$$\eta_1 = \frac{\sum_{j=1}^{N} \eta_{1j}}{\sqrt{N}} \text{ and } \eta_2 = \frac{\sum_{j=1}^{N} \eta_{2j}}{\sqrt{N}}.$$

tation of E(Q) and V(Q) is performed by using Monte Carlo methods.[4] As an example we describe the aggregate MC PML_3 variant more precisely, i.e. the model implied by (1)-(4) for N=1. The remaining variants can be easily understood from this perspective.

The objective is to minimize the function ψ defined in (7) for i=3. It is assumed that the quantities $E(Q_t)$ and $V(Q_t)$ can not be obtained explicitly from the data, so for each period t and for given σ we generated sequences $\{D_{jg}\}$, $\{S_{jg}\}$ and $\{Q_{jg}\}$, g=1,...G (G is fixed in advance and j=1 in this case) by

$$D_{jg} = E(D) + \sigma_1 a_{1jg},$$

$$S_{jg} = E(S) + \sigma_2 a_{2jg},$$

$$Q_{jg} = \mathrm{Min}(D_{jg}, S_{jg}),$$

using observed quantities on E(D) and E(S). The sequences $\{a_{1jg}\}$ and $\{a_{2jg}\}$ are random but once generated fixed throughout the minimization. The next step is to approximate $E(Q_t)$ and $V(Q_t)$ from

$$E(Q_t) = \frac{1}{G}\sum_{g=1}^{G} Q_{jg} = \bar{Q}_{jg}, \tag{8}$$

$$V(Q_t) = \frac{1}{G-1}\{\sum_{g=1}^{G} Q_{jg}^2 - G\,\bar{Q}_{jg}^2\}. \tag{9}$$

Finally, the function ψ in (7) is evaluated using the previous approximations and can then be minimized with respect to σ.[5] Assuming some regularity conditions (satisfied in the model), L&S prove that for any T, any converging sequence of approximate estimators, in our case σ_{1T}^G, σ_{2T}^G and N_T^G, converge to the corresponding PML estimators σ_{1T}, σ_{2T}, N_T when G goes to

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[4]Under PML_1, only $E(Q_t)$ is computed.

[5]Note that numerical errors in the use of explicit expressions for $E(Q_t)$ and $V(Q_t)$ can be seen as special cases of the procedure if G is large, apart perhaps from the normal distribution we used.

infinity (compare their Theorem 1).[6] The magnitude of G is mainly constrained by computer budgets. Furthermore, the asymptotic properties of the PML estimator itself are taken from Gourieroux et al. (1984). We will return to this subject in Section 4 where we dispose of the requirement that $G \to \infty$ to obtain consistency in case of PML_1.

3. ROBUSTNESS WITH RESPECT TO THE INTRODUCTION OF MICRO LABOUR MARKETS

The conventional approach to the estimation of aggregate rationing models pictures employment as the minimum of demand for labour and supply of labour.[7] Its fundamental weakness lies in the inability to describe the simultaneous existence of unemployment and vacancies in the aggregate. In the more modern approach this limitation is obviated through the introduction of fix-price labour markets at the micro level. In its formal development this approach can explain the aggregate and simultaneous occurrence of unemployment and vacancies [see e.g. Kooiman and Kloek (1979) and Lambert (1988)].[8]

Despite the shortcoming of the discrete switching model it is still being applied.[9] We will try to settle this controversy empirically for our application of the model to the Dutch labour market. For that purpose we use PML_3 estimation using analytical expressions for the moments of Q_t implied by (1)-(4).

First consider the expectation of Q, E(Q), and the variance of Q, V(Q). By definition:

$$E(Q_t) = E(\sum_{j=1}^{N} Q_{tj}). \tag{10}$$

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[6]In this respect also note that for our model, N must be bounded away from zero because of the Lipschitz continuity and must be sufficiently small for other reasons (see text).

[7]Compare eq. (1)-(4) for N=1. We will refer to this model as the aggregate model.

[8]Compare eq. (1)-(4) for N>1. We will refer to this model as the disaggregate model.

[9]See e.g. Quandt and Rosen (1978 and 1987) in the setting of one aggregate market model and e.g. Sneessens (1983), Artus et al. (1985), Kooiman and Kloek (1985) in an environment of two (related) aggregate markets.

After some manipulation, using the formulas in Kooiman and Kloek (1979, Appendix A), this can be rewritten as:

$$E(Q_t) = L_t^d\Phi(-Y_t) + L_t^s\Phi(Y_t) - (\zeta\sqrt{N})\psi(Y_t), \tag{11}$$

where $Y_t = \dfrac{Z_t}{\zeta\sqrt{N}}$, $L_t^d = E(D_t)$, $L_t^s = E(S_t)$, $Z = L_t^d - L_t^s$ and $\zeta = \sqrt{(\sigma_1^2+\sigma_2^2)}$.

Equation (11) satisfies the following properties: $\partial E(Q_t)/\partial L_t^d > 0$, $\partial E(Q_t)/\partial L_t^s > 0$ and $\partial E(Q_t)/\partial N < 0$.[10] Obviously σ_1, σ_2 and N are not individually identified.

For the computation of $V(Q_t)$ we use

$$\sum_{j=1}^{N}(E(Q_{tj}))^2 = \frac{1}{N}\{L_t^d\Phi(-Y_t) + L_t^s\Phi(Y_t) - \zeta\sqrt{N}.\psi(Y_t)\}^2 , \tag{12}$$

and

$$\sum_{j=1}^{N}E(Q_{tj}^2) = (\frac{(L_t^d)^2}{N} + \sigma_1^2)\Phi(-Y_t) + (\frac{(L_t^s)^2}{N} + \sigma_2^2)\Phi(Y_t) - \frac{\zeta\sqrt{N}}{N}(L_t^d + L_t^s)\psi(Y_t). \tag{13}$$

Finally, the expression for $V(Q_t)$ can be easily determined using independence over j:

$$V(Q_t) = \sum_{j=1}^{N}V(Q_{tj}) = \sum_{j=1}^{N}E(Q_{tj}^2) - \sum_{j=1}^{N}(E(Q_{tj}))^2. \tag{14}$$

V(Q) is not monotonous in any of the parameters σ_1, σ_2 and N. Note that σ_1, σ_2 and N are identified at the second order. Further note that for N=1 the expressions coincide with the moments of the aggregate model (compare L&S). Here we wish to stress the very interesting feature of our PML_3 version that

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[10]Note that model (1)-(4) is defined for sufficient small values of N as mentioned before. Expressions like $\lim_{N\to\infty} E(Q) \to -\infty$ therefore have no meaning.

a lower bound of the number of micro markets is identified. It is clear that, when it comes to estimation PML_1 is robust with respect to the introduction of micro markets, whereas PML_3 is not.

4. THE MODIFIED SIMULATED PML_1 ESTIMATOR

In this section we propose a modification to the procedure in L&S who prove consistency of the simulated PML estimators if T and G go to infinity, where G can be considered to be the number of replications in the Monte Carlo part. For the simulated PML_1 case we suggest a bias correction such that consistency results even for finite G.

Let $\psi_{1T} = \frac{1}{T}\sum_{t=1}^{T} (Q_t-E(Q_t))'(Q_t-E(Q_t))$, where $E(Q_t)$ is a function of the parameter θ of interest and where (in this section only) Q_t may be a vector. The PML method is based on the fact that ψ_{1T} converges to some function $\psi_1(\theta)$; if ψ_1 has a unique minimum at θ^0 and θ^0 is the "true" parameter we then hope that $\hat{\theta}^T$ converges to θ^0, where $\hat{\theta}^T$ minimizes ψ_{1T}.

For the simulated PML method $E(Q_t)$ is replaced by a random variable that converges to $E(Q_t)$ if some index G goes to infinity. We write the corresponding function as

$$\psi_{2T} = \frac{1}{T}\sum_{t=1}^{T} (Q_t-E(Q_t)+a_{tG})'(Q_t-E(Q_t)+a_{tG}), \tag{15}$$

where a_{tG} is a random variable that is independent of Q_t; we choose a_{tG} to have zero mean and finite second order moments. We note that the variables are independent for different values of t. Then using the results of Gourieroux <u>et al.</u> (1984) and L&S the following theorem applies:

THEOREM 1: $\psi_{2T} - (\psi_{1T} + \frac{1}{T}\sum_{t=1}^{T} a'_{tG}a_{tG})$ *converges almost surely to zero if* $T\to\infty$.

Note that the expression in the theorem equals $\frac{2}{T}\sum_{t=1}^{T} a'_{tG}(Q_t-E(Q_t))$.

We leave it to the reader to impose conditions such that some version of a law of large numbers applies. In this respect the results obtained by L&S are more general, but in their context a bias correction is not easily implemented.

In our study a_{tG} is a simple average of simulated $(Q_t - E(Q_t))$ values according to so called 'crude' Monte Carlo [Hammersley and Handscomb (1964)]. Of course variance reducing techniques like importance sampling, control variates or antithetic variables may in general alleviate the computational burden of the simulated PML method substantially, but is not applied here as it is rather specific to each application. For the same reason we did not consider replacing simple averages of simulated values by more robust and under certain conditions more efficient functions of simulated values.

From the theorem we conclude that there are two possibilities with respect to consistency:

(i) if $Var(a_{tG})$ does not depend on θ, then $\frac{1}{T}\sum_{t=1}^{T} a_{tG}' a_{tG}$ will become a 'flat' function if $T \to \infty$. In this case the sequence $\{a_{tG}\}$ will not have any influence on the location of the minimum of $\psi(\theta)$ and if T is large enough, then $var(a_{tG})$ does not necessarily have to be very small for all t. Consistency, in this case is approved for finite G.

(ii) if $var(a_{tG})$ is a function of θ then there is a problem, because the shape of $\psi(\theta)$ will be different from $\psi(\theta) + \frac{1}{T}\sum_{t=1}^{T} a_{tG}' a_{tG}$. In this case the number of replications G must be large enough to have a small $\frac{1}{T}\sum_{t=1}^{T} a_{tG}' a_{tG}$ relative to $\psi(\theta)$. Consistency is only obtained if $G \to \infty$.

It is precisely point (ii) of the conclusion above which applies to the simulated PML_1 method: $var(a_{tG})$ depends on θ, since $var(Q_t)$ depends on σ_1, σ_2 and N. Thus, the method is not consistent for any finite G.

We will show how consistency can be obtained for finite G, replacing ψ_{2T} in (15) by

$$\psi_{3T} = \frac{1}{T}\sum_{t=1}^{T}\{(Q_t - E(Q_t)+a_{tG})'(Q_t - E(Q_t)+a_{tG}) - tr(Var(a_{tG}))\}. \qquad (16)$$

Then we have the following Theorem:

THEOREM 2: *Let the modified simulated PML_1 estimator be given by the parameter value $\hat{\theta}^T$ that minimizes ψ_{3T}, then this estimator is consistent if $T\to\infty$, even for finite G.*

Note from the definitions of ψ_{1T} and ψ_{3T} that

$$\psi_{3T} - \psi_{1T} = \frac{1}{T}\sum_{t=1}^{T}\{a_{tG}'a_{tG} - tr(Var(a_{tG})) + 2a_{tG}'(Q_t-E(Q_t))\}. \tag{17}$$

Under appropriate moment conditions this expression converges almost surely to zero if $T\to\infty$ for any G. In our setup convergence is uniform over θ (note that $var(a_{tG}) = \frac{1}{G}\, var(Q_t)$).

As for the asymptotic distribution of the modified simulated PML_1 estimator it is unlikely that it coincides with that of the PML_1 estimator itself for any finite G. Developing ψ_{1T} and ψ_{3T} into the familiar Taylor expansions we get:

$$\sqrt{T}\,\frac{\partial\psi_3(\hat{\theta}_3^T)}{\partial\theta} = 0 = \sqrt{T}\frac{\partial\psi_3(\theta_0)}{\partial\theta} + \frac{\partial^2\psi_3(\theta_0)}{\partial\theta\,\partial\theta'}\sqrt{T}(\hat{\theta}_3^T-\theta_0),$$

$$\sqrt{T}\,\frac{\partial\psi_1(\hat{\theta}^T)}{\partial\theta} = 0 = \sqrt{T}\frac{\partial\psi_1(\theta_0)}{\partial\theta} + \frac{\partial^2\psi_1(\theta_0)}{\partial\theta\,\partial\theta'}\sqrt{T}(\hat{\theta}^T-\theta_0).$$

The two matrices of second derivatives will converge to the same (constant) matrix if $T\to\infty$ (compare L&S). What we need is the same asymptotic distribution for $\sqrt{T}\frac{\partial\psi_3(\theta_0)}{\partial\theta}$ and $\sqrt{T}\frac{\partial\psi_1(\theta_0)}{\partial\theta}$ but from (17) we would obtain

$$\sqrt{T}\frac{\partial\psi_3(\theta_0)}{\partial\theta} = \sqrt{T}\frac{\partial\psi_1(\theta_0)}{\partial\theta} + K, \tag{18}$$

where K is a non-degenerate finite stochastic variable. The variance of K would approach zero if $G\to\infty$.

The bias correction can have considerable practical advantages. Assume that we want to approximate the PML_1 estimator as closely as possible by the adjusted simulated PML_1 estimator. The first option obviously is to increase G, but computationally this is not very attractive. Moreover, we would still have no idea about the variability introduced by simulating the objective function for that particular value of G (we just get one realization of the variable K). Another option now can be to do several independent estimations for a fixed G; the estimates can be averaged, the variability introduced by simulation goes down and one will have an idea about the magnitude of this variability (we sample over K). We stress the point, however, that for fixed G and T one still may have a small sample bias.

A bias correction in the PML_3 case cannot be made as straightforwardly as for the PML_1 case. For expository reasons we assume again that Q_t is a scalar. Suppose $E(Q_t)$ and $V(Q_t)$ are obtained through simulation as discussed before and consider

$$\frac{1}{T}\sum_{t=1}^{T}\{\frac{(Q_t-E(Q_t)+a_{tG})^2}{V(Q_t)+b_{tG}}+\log(V(Q_t)+b_{tG})\},$$

where for simplicity we assume a_{tG} and b_{tG} to be independent (one can make them independent!), and $Var(a_{tG})$ and $Var(b_{tG})$ approach zero if $G\to\infty$. Through expansion of denominator and logarithm we obtain

$$\frac{1}{T}\sum_{t=1}^{T}[\{\frac{(Q_t-E(Q_t))^2}{V(Q_t)}+\frac{a_{tG}^2}{V(Q_t)}+\frac{2a_{tG}(Q_t-E(Q_t))}{V(Q_t)}\}\{1-\frac{b_{tG}}{V(Q_t)}+\frac{b_{tG}^2}{(V(Q_t))^2}\ldots\}+$$

$$+\log(VQ_t)+\{\frac{b_{tG}}{V(Q_t)}-\frac{b_{tG}^2}{(V(Q_t))^2}\ldots\}].$$

Taking the expectation of this expression one can use the independence between a_{tG} and b_{tG}. If b_{tG} is constructed from an unbiased estimator of $V(Q_t)$, then $\frac{1}{T}\sum_{t=1}^{T}\frac{b_{tG}}{V(Q_t)}$ will not cause the problem (apart from the existence of sufficiently high order moments) and also $\frac{1}{T}\sum_{t=1}^{T}\frac{a_{tG}^2}{V(Q_t)}$ may be dealt with. Both series within the curly brackets, however, are in general non-converging asymptotic series in G [see Sneek (1983) for a definition of asymptotic series], so the optimal cut-off point depends on G. Although for finite but large enough G it is possible to reduce the asymptotic bias if $T\to\infty$ of the MC PML_3 procedure we were not able to reduce the bias completely. The subject, remains high for future research.

5. EMPIRICAL ANALYSIS

A. general remarks

The data used are taken from Kooiman and Kloek (1979). They contain information on employment, unemployment and unfilled vacancies over the period 1948-1975. We used a Davidon-Fletcher-Powell optimizing algorithm with a unimodal line search routine to obtain the estimates, using analytical first derivatives. In case of PML_1 we furthermore compared the analytical second derivatives with the (inverse) updated Hessian from the optimizing routine.[11]

From our previous discussion it follows that in principle eight variants can be distinguished. These are displayed in Diagram I. In the actual application only I, III, VII and VIII are implemented. Variants I and II are observationally equivalent (note that N and ζ are not individually identified from equation (11)). Furthermore we rely on the fact that comparison of the estimation results of I and V provides sufficient insight into the extra variability induced by the MC variant compared with its analytic counterpart to make the estimation of V, VI and VII redundant.

N / ψ	N=1	N>1	N=1	N>1
ψ_1	Analytic (I)	Analytic (II)	Monte Carlo (III)	Monte Carlo (IV)
ψ_3	Monte Carlo (V)	Monte Carlo (VI)	Analytic (VII)	Analytic (VIII)

Diagram I--Alternative PML variants

[11]The first derivatives for the analytical variants and the second derivatives for the MC variants are presented in Appendix I. Appendix II derives the covariance matrix in a more general setting.

Before discussing the estimation results in some detail, several points need to be stressed. First, the computation of E(Q) and V(Q) in the MC variants is done by using different numbers of replications of η_t with G varying from 25 to 400 and sample of T=28. Second, in generating the random functions, in the MC PML_1 case, we imposed $\sigma_1=\sigma_2$ for convenience.[12]

Third, the differences in identifiability of the parameters between the two PML cases complicate the comparison of the parameter estimates. From PML_1 estimation we get an estimate of the standard error of the function $h(\sigma) = \sqrt{(\sigma_1^2+\sigma_2^2)}$, whereas PML_3 gives the standard error of both σ_1 and σ_2. To facilitate comparison we also need an estimate of the standard error of $h(\sigma)$ under PML_3 estimation. For that purpose a linearization of the function $h(\sigma)$ with respect to σ_0 is used, i.e.

$$h(\sigma) = h(\sigma_0) + \frac{\partial h}{\partial \sigma'}(\sigma - \sigma_0),$$

so that the variance of $h(\sigma)$ can easily be calculated as

$$\text{var}(h(\sigma)) = \frac{\partial h}{\partial \sigma'}\,\text{var}(\sigma)\,\frac{\partial h}{\partial \sigma}. \tag{19}$$

The optimizing routine produces an estimate of var(σ) and the first derivatives of h with respect to σ can be calculated straightforwardly.

Last, the estimations allow for the computation of the level of structural unemployment, U* (that is the level of unemployment for which applies $L^d = L^s$[13]). This provides us with an additional check on the plausibility of the estimates. In the disaggregate PML_3 variant the following explicit form for U* in terms of total labour supply (denoted by u*) can be obtained[14]

$$u^* = \zeta\sqrt{(N)}\,\frac{\psi(0)}{L^s}. \tag{20}$$

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[12]The parameters σ_1 and σ_2 are not individually identified (see eq. 11) in the PML_1 case so we imposed this "restriction". Note that this "restriction" does not imply restricted optimization. The only property that we use is that $E(Q_t)$ is estimated consistently if $G\to\infty$.

[13]This is the same as demanding that the number of unfilled vacancies equals the number of unemployed persons [compare equations (5) and (6)].

[14]In the following we misleadingly use the "level of structural unemployment" to indicate u^*.

B. Aggregate PML Estimation

Table IA presents the estimation results from our experiments with the various aggregate (= disaggregate) PML_1 variants and aggregate PML_3 variants. Let us consider the PML_1 case. Reported values on H^{-1} correspond to the updated (inverse) Hessian matrix from the DFP algorithm; H_{an}^{-1} is the analytic (inverse) Hessian computed according to equation (A17) of Appendix I.

TABLE IA

ESTIMATES FOR PML_1 AND PML_3 -AGGREGATE MC AND AGGREGATE ANALYTIC VARIANTS[1)]

	G	$\hat{\zeta}$	$SD(\hat{\zeta})$	H^{-1}	S.D.	H_{an}^{-1}	S.D.	$SE(\hat{\zeta})$[2)]	S.D.	$\hat{u}*$[3)]
- PML_1 -										
	25	171.5	9.7	0.1330	0.0128	0.1170	0.0117	2.557	0.126	
SE_{mean}			1.9							
	50	178.2	8.6	0.1223	0.0101	0.1215	0.0091	2.607	0.098	
SE_{mean}			1.7							
	100	181.3	5.2	0.1244	0.0161	0.1234	0.0056	2.628	0.059	
SE_{mean}			1.0							
	200	183.4	3.8	0.1262	0.0048	0.1248	0.0044	2.644	0.046	
SE_{mean}			0.8							
	400	184.2	2.0	0.1268	0.0031	0.1254	0.0028	2.650	0.029	
SE_{mean}			0.4							
										2.00%
	∞	185.6		0.1330				2.730		2.01%

- PML_3 -	$\hat{\sigma}_1$	$SE(\hat{\sigma}_1)$	$\hat{\sigma}_2$	$SE(\hat{\sigma}_2)$	$\hat{\zeta}$	$SE(\hat{\zeta})$	
ANALYTIC	98.1	44.9	0.0016	408.9	98.1	44.9	1.06%

Notes: 1) The reported figures are means computed for 25 independent runs and associated standard deviations.
Standard errors of the means can be computed by dividing the reported S.D. values by $\sqrt{25}$. In the table only the standard errors of the means of ζ are explicitly shown to facilitate interpretation of the results.

2) Estimated from H^{-1} in each run.

3) Structural unemployment, u^* is related to L^s. In the table we took the sample mean of L^s (3600) to compute it.

The reported statistics are based on 25 independent runs for each situation. G, the number of replications required for the computation of the random function of the MC variant is reported in the first column; $G=\infty$ is the analytical variant.

It is obvious from the table that with the number of replications increasing, the estimate of ζ converges to its value obtained from the analytic variant. It is also clear that the standard error of the estimates as computed from the Hessian is in all cases very close to the one obtained from the analytic version. Now each estimate coincides with one realization of a stochastic variable K similar to (18), and it seems that although K evidently induces a bias it does not influence the estimate of the variance very much. From one single MC PML estimation it is not possible to obtain an estimate of the variance induced by the simulation, and from several estimations for a single value of G it is not possible to estimate the extra bias. We find it encouraging that broadly the values H^{-1} and H_{an}^{-1} are alike; this implies that second derivatives as approximated by the optimization routine are reasonable, at least in the one parameter case.

Theoretically our estimates based on the analytic PML_1 method should coincide with those of Kooiman and Kloek.[15] However, we find a slight difference for which we do not have an explanation. Their estimate of ζ is 181.7 (our estimate is 185.6) with standard error 5.7 (our estimate is 2.73); corresponding levels of structural unemployment are equally close, around 2%.

Summarizing, the above results of the MC variant seem promising for applications to more complex models for which no analytic expression of $E(Q_t)$ is available. However, a sufficient large value of G for a given T is required. Note for instance that in our example, a standard deviation of 2.0 for G=400 in the simulation has to be combined with the standard error 2.65 as estimated from the Hessian. So to require a minimum of 400 replications does not seem overdone.

In Table IB we present some results for the bias corrected MC PML_1 estimation procedure. It is very obvious that at a sample size of T=28 the bias is reduced considerably for all reported values of G; in all cases the averages of the parameter estimates are within one standard error of the

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[15]First, recall that we used the same data. Second, it can easily be verified that our analytic PML_1 method is essentially the same as the method applied by Kooiman and Kloek who minimise the sum of squared residuals in (11) with a disturbance term added.

analytical PML_1 estimate, i.e. the bias is essentially removed. In Table IA on the other hand even for G=400 we report a bias that is statistically different from zero. We also experimented with a larger number of runs. Using 200 independent runs for G=25 we find a smaller value of ζ, i.c. 187.93 with associated value $SD(\hat{\zeta})$ = 12.4 and thus with standard error 0.875.

To illustrate the importance of the bias correction graphically Figure I is presented. On the left side one finds the estimation results for the ordinary MC PML_1 estimations based on the 25 independent runs. On the right hand side one finds the bias corrected results. Note that we graphed 1.96 times standard errors and furthermore we subtracted the standard errors from the heights of the bars for estimation. We see that the bias correction is at the cost of only a slight increase in the variation due to the Monte Carlo part.

TABLE IB

ESTIMATES FOR PML_1 -AGGREGATE MC WITH BIAS CORRECTION[1)]

	G	$\hat{\zeta}$	$SD(\hat{\zeta})$	H^{-1}	S.D.	S.E.[2)]	S.D.
	25	188.7	12.0	0.1787	0.0342	3.150	0.287
SE_{mean}			2.4				
	50	187.0	9.5	0.1787	0.0138	3.161	0.123
SE_{mean}			1.9				
	100	185.7	5.5	0.1785	0.0136	3.160	0.120
SE_{mean}			1.1				
	200	185.7	3.1	0.1787	0.0063	3.163	0.056
SE_{mean}			0.8				
	400	185.4	2.0	0.1789	0.0045	3.165	0.039
SE_{mean}			0.4				
	∞	185.6		0.1330		2.730	

Notes: 1) see note 1 of Table IA
2) see note 2 of Table IA.

For G=25 we found 14 negative values for ψ [see eq. (16)], for G=50 there were only 3 negative ones and for G≥100 none; we do not know whether a negative value of ψ indicates that G is too small, though it is tempting to draw this conclusion.

From the results we conclude that the bias corrected version is highly superior to the uncorrected one and that it is preferable to average several independent estimations for a moderate value of G instead of doing one estimation for a large value of G; the bias is negligible for moderate values of G, the standard deviation due to the simulation goes to zero anyway through averaging and at the same time one can estimate the latter standard deviation.

The results from PML_3 estimation are troublesome. We find negative $Q_t-E(Q_t)$ values for each t both for the analytical and the MC variant. This is inacceptable. On the one hand we note that the estimate of ζ is much lower in the PML_3 case (compare 98.1 with 185.6). On the other hand, the corresponding standard error of h(o) in the PML_3 case, which equals 44.9 is substantially larger than the PML_1 value of 2.7 (see Table IA).

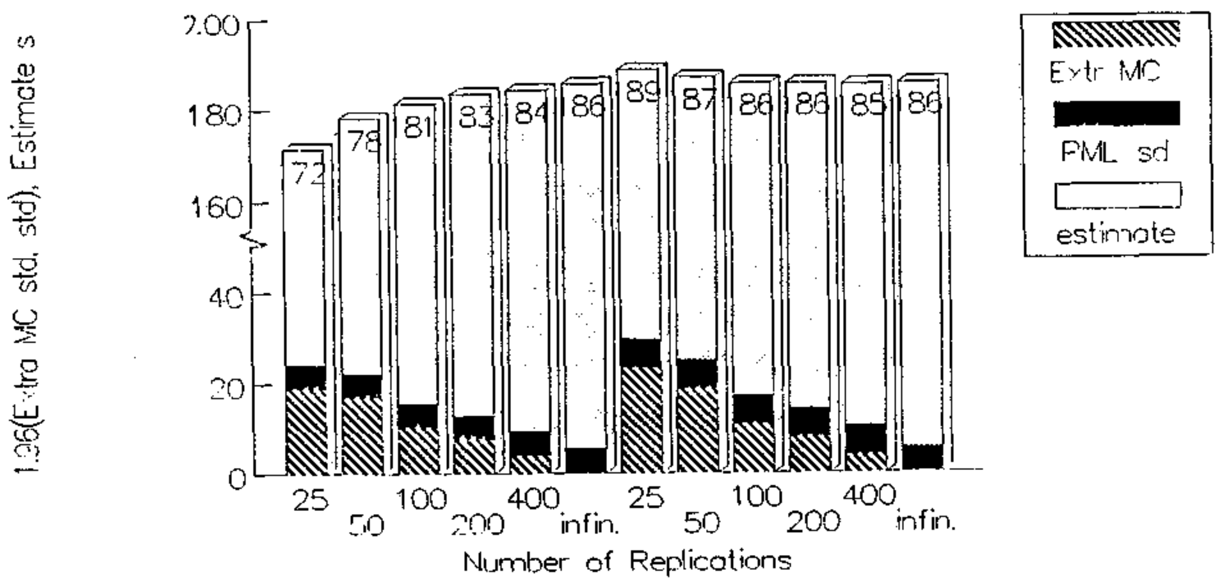

Figure I—Monte Carlo PML estimation/Extra Monte Carlo variation

From asymptotic theory PML_3 should be more efficient than PML_1, though in this case the sample size T is only 28 and under PML_3 one more parameter is estimated. Besides efficiency considerations the explanation may also come from the fact that the model is inadequate. We have estimated the aggregate model, under PML_1 we essentially have estimated the disaggregate model as well (they are equivalent), but PML_3 does distinguish between the aggregate and the disaggregate model. Recall that $E(Q_t)$ goes down if N gets larger; if N=1 is imposed as a restriction, then under PML_3 indeed all values $(Q_t-E(Q_t))$ could be expected to be negative. We concluded that we are dealing with a trade-off between efficiency and robustness: PML_1 is less efficient, but robust with respect to the existence of micro markets, whereas PML_3 is more efficient but sensitive to the existence of micro markets.

C. Monte Carlo Evidence

To substantiate the claim above we undertook a limited Monte Carlo study by generating artificial data according to the aggregate model implied by equations (1)-(4) for N=1. We tried four different pairs of pre-set values of σ_1 and σ_2. For each of the four sets of parameter values, the following was done 50 times. Two vectors of 28 (= actual sample size for which observations on E(D) and E(S) are available) independent N(0,1) variables were drawn. These were the values of η_{1t} and η_{2t}. D_t, S_t and Q_t were generated by (1), (2) and (3), using the given values of the explanatory variables. Subsequently we used PML_1 and PML_3 to obtain the parameter estimates.

In Table II the results from the Monte Carlo simulation are presented. Note that we reported the standard deviations as emerging from 50 estimates (the line entries) and the estimated standard deviations as computed from H^{-1}, denoted by $SD(\hat{\cdot})$ where the dot indicates the parameter (the column entries). From the table it is clear that both PML_1 and PML_3 estimates are noticeably biased downwards in all cases. From the table we conclude that PML_3 performs superior to PML_1 as the downward bias is considerably less (compare e.g in the first line entry 26.6 and 22.0 with 28.3, respectively) and the standard deviations of $\hat{\zeta}$ are far less (compare e.g. in the second line entry 3.82 and 15.5). Examining a couple of sets of residuals for both models revealed 'random' sequences. However, $SD(\hat{\cdot})$ under PML_1 severely underestimates the true standard deviation (compare, e.g. for $\hat{\zeta}$=28.3, the value 6.00 in the first line entry with 15.5 in the second line entry). For PML_3 we see that the bias is upwards (compare, e.g. for $\hat{\zeta}$=28.3, the value 20.30 in the first line entry with 3.82 in the second line entry). Test statistics based on H^{-1} therefore are likely to yield incorrect significance levels; under PML_1 parameter values would be severely underestimated, under PML_3 they would be overestimated. The standard deviations for $\hat{\zeta}$ were usually smaller than those of $\hat{\sigma}_1$ and $\hat{\sigma}_2$, because of positive correlation between $\hat{\sigma}_1$ and $\hat{\sigma}_2$. Note that under PML_3 usually none of the estimated parameters would appear to be statistically different from zero.

TABLE II

ESTIMATES FOR ζ UNDER PML_1 AND PML_3

(ARTIFICIAL DATA FOR DIFFERENT PAIRS OF σ_1 and σ_2)

		→ PML_1		↔	PML_3					←
(σ_1, σ_2)		ζ	$\hat{\zeta}$	$SD(\hat{\zeta})$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\zeta}$	$SD(\hat{\sigma}_1)$	$SD(\hat{\sigma}_2)$	$SD(\hat{\zeta})$
(20, 20)		28.3	22.0	6.00	18.6	18.5	26.6	20.6	19.6	20.30
	S.D.		15.5	2.955	3.86	4.10	3.82	4.53	4.51	3.34
	S.E.		2.19	0.418	0.546	0.579	0.540	0.641	0.638	0.472
(50,100)		111.8	99.5	3.27	48.0	88.3	103.5	112.3	99.5	72.6
	S.D.		32.9	0.649	22.1	19.5	16.3	51.6	33.7	11.0
	S.E.		4.65	0.091	3.124	2.754	2.307	7.305	4.765	1.557
(100, 50)		111.8	94.3	3.17	91.7	40.4	103.9	106.4	194.4	76.3
	S.D.		37.4	0.520	21.9	25.8	19.5	44.3	239.6	17.1
	S.E.		5.29	0.074	3.104	3.651	2.753	6.259	33.889	2.423
(100,100)		141.8	122.1	2.96	95.3	87.7	133.2	160.8	163.3	90.8
	S.D.		44.2	0.419	24.8	27.1	19.4	62.0	70.7	16.9
	S.E.		6.25	0.059	3.508	3.828	2.749	8.765	9.997	2.385

Note: Means and standard deviations obtained from 50 independent runs. Standard errors of the means are computed by dividing the S.D. by $\sqrt{50}$.

Interpreting the results of this (limited) Monte Carlo study we conclude that PML_3 is more efficient than PML_1, but evidently less robust with respect to the existence of micro markets. Additionally we conclude that the aggregate model is not an adequate model for our data because PML_3 performs reasonably well on the simulated data sets (for which the true model is indeed as the 'aggregate' variant). This implies a warning towards an uncritical application of the aggregate discrete switching model in "disequilibrium" macroeconometrics as in Quandt and Rosen (1987).

D. Disaggregate PML_3 Estimation

To complete this section we present the analytic disaggregate PML_3 estimates. These are reported in Table III. Now the results are more in agreement with the previous estimates. The parameter estimate of $\zeta\sqrt{N}$ roughly corresponds to the estimated value of ζ in the PML_1 variants. Furthermore,

TABLE III

RESULTS FOR THE ANALYTIC DISAGGREGATE PML_3 VARIANT

	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{N}$	$\hat{\zeta}/N$	$\hat{u}^*$
parameter estimates	17.606	0.000	109.540	184.3	1.99%
(S.E. between brackets)	(12.4)		(158.0)	(25.5)	

Note: u^* is computed for L^S=3600.

note that none of the parameters appears to be statistically different from zero, similar perhaps to the MC results in Table II; fortunately the function $\hat{\zeta}/N$ is estimated somewhat better. We did not carry through the MC PML_3 estimation, as we regard the MC PML_1 variant has proved its significance.

6. EXTENSION AND EMPIRICAL RESULTS

In this section, we suggest an extension to the aggregate PML_1 model implied by equations (1)-(4) for N=1 by allowing for the introduction of heteroskedasticity in the variance of the disturbances.[16] We argue that the constraint imposed by the basic model stating that the variability of the variables D and S is independent of the magnitude of a market is likely to be violated in practice. A relatively large market, i.e. E(D) and E(S) are relatively large, is expected to have relatively large deviations around the respective expectations. Therefore we propose the following alternative specification:

$$D = E(D) + \sigma E(D)\eta_1, \tag{21}$$

$$S = E(S) + \sigma E(S)\eta_2, \tag{22}$$

[16]Ignoring heteroskedasticity of disturbances in econometric models does not in general prevent consistent point estimation, but it typically entails inefficient point estimators. However, we note in this respect that to obtain consistency in the MC PML variant a bias correction is necessary. Furthermore we note that the type of heteroskedasticity considered here should be distinguished from the heteroskedasticity aspect of PML_3 estimation.

$$Q = \mathrm{Min}(D, S), \tag{23}$$

where we have imposed the restriction $\sigma_1=\sigma_2=\sigma$.[17] Note that the variances of the disturbances are not fixed as in the basic model, but proportional to E(D) and E(S), respectively.

Estimation of the heteroskedastic model demands a slight adaptation of the PML procedure, but it remains essentially the same as for the basic model. For example, the expression for $E(Q_t)$ now becomes:

$$E(Q_t) = L_t^d.\Phi(-X_t) + L_t^s\Phi(X_t) - \sigma\sqrt{((L_t^d)^2 + (L_t^s)^2)}\psi(X_t), \tag{24}$$

where $X_t = \dfrac{Z_t}{\sigma\sqrt{((L_t^d)^2 + (L_t^s)^2)}}$.

We find the estimate of σ to be equal to 0.034 with a standard error of 4.8902E-04. The corresponding estimate of the level of structural unemployment is equal to 1.9%, which is almost the same as found for the basic model.[18] In Figure II the patterns of the estimated disturbances of the basic and heteroskedastic model are portrayed. As usual, of course, a comparison of the degree of fit between the heteroskedastic model and the basic model might be blurred due to serial correlation in the disturbances, which is evidently the case here. In this respect we know from Gourieroux _et al._ (1985) (p. 317) that for the usual rationing model (also our model), consistency of parameter estimates is guaranteed but efficiency is not.[19] However, we find it encouraging that broadly the heteroskedastic model produces a lower value of the sum of squared residuals over the sample as can be seen from Figure II.

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[17]It is worthwhile mentioning that in this case the number of parameters under PML_1 equals that under PML_3. Furthermore note that σ here is a scalar and not a vector as in Section 2.

[18]It can be shown that the expression for the level of structural unemployment is: $u^* = \lambda\psi(0)$, with $\lambda=\sqrt{(2)}\sigma^2$. Note that, contrary to the basic model, u^* does not relate to the level of L^s; u^*, however is defined in terms of L^s so that in absolute terms dependency of L^s is maintained.

[19]The authors also provide a score test to detect first order serial correlation.

It is exactly for the serial correlation that we decided not to carry through the PML_3 variants. The serial correlation problem asks for a distinct approach based on a consistent PML procedure taking into account correlation over time of the endogenous variable in rationing models at a disaggregate level. In another paper we hope to present some new results on this.[20]

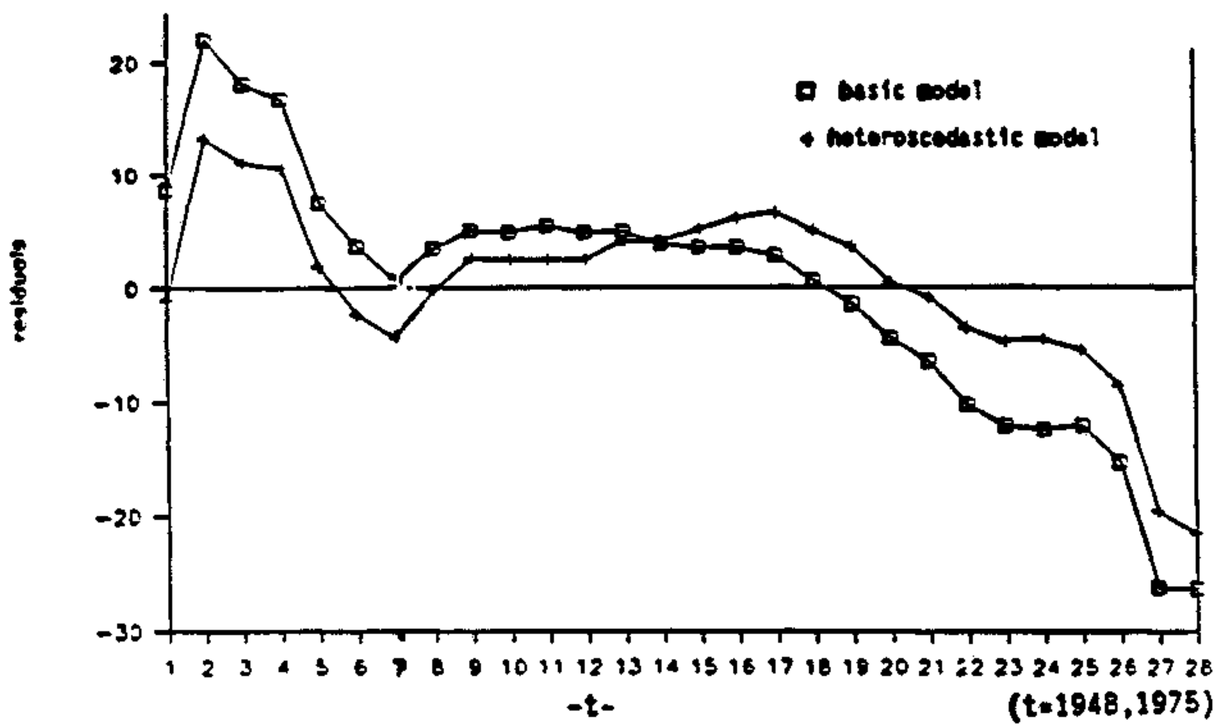

FIGURE II—The basic model and the heteroskedastic model (PML_1)

7. CONCLUSIONS

The results of this paper lend support to the following conclusions. First, the parameter estimates are not robust with respect to the introduction of micro markets. Our Monte Carlo experiments for PML_1 and PML_3 clearly show that micro markets exist. Second, the Monte Carlo PML estimation variants of the model yield parameter estimates that converge to the analytic PML

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[20] In a paper by Hajivassiliou (1989) it is found that a large part of the residual serial correlation can be explained by allowing for macro-economic shocks in the micro level rationing model.

variant when a sufficiently large number of replications is used, but a bias correction as indicated in Section 4 is highly recommended. Last, the heteroskedastic model is theoretically and empirically more appealing.

It is also obvious that PML_1 is the more robust estimation method, since it is observationally equivalent for both the aggregate and the disaggregate model. The penalty however is the extra loss in efficiency.

L&S proved the almost sure uniform convergence of the approximate pseudo-likelihood function, ψ_{GT} towards $E(Q-E(Q))^2$, when G and T simultaneously go to infinity. From our experiments with the MC PML_1 variants we concluded that G should be "sufficiently" large to decrease the simulation error to acceptable levels. As a result, this can put a large claim on computer time. Moreover, consistency is not approved with finite G. Introducing a bias correction in the pseudo-likelihood function established consistency for finite G. Consequently, a significant reduction on the computational burden can be reached, since relatively lower values for G are sufficient to obtain convergence to the analytical parameter estimates.

A useful extension of the present paper's environment includes allowing for systems with disturbances that are serially correlated. In another paper we will try to cope with this problem by developing a PML based method which takes into account correlations in time of the endogenous variable in rationing models at a disaggregate level.

APPENDIX I
COMPUTATION OF DERIVATIVES

We use E, E2, V to abbreviate E(Q), $E(Q^2)$ and V(Q), respectively.

- PML_3 ANALYTIC AGGREGATE MODEL -

Let $\zeta = \sqrt{(\sigma_1^2+\sigma_2^2)}$ and $f=f(\zeta,\sigma_1,\sigma_2)$ be differentiable, then

$$\frac{\partial f}{\partial \sigma_i} = \frac{\partial f}{\partial \sigma_i} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial \sigma_i},$$

where $\partial f/\partial \sigma_i$ on the left hand side means the 'total' derivative with respect to σ_i and on the right side it is strictly the partial derivative with respect to σ_i. From this chain rule one obtains for i=1,3

$$\frac{\partial E}{\partial \sigma_i} = \{(L^s-L^d)\psi(Z/\zeta)(-Z/\zeta^2)-\psi(Z/\zeta)[1+(Z^2/\zeta^2)]\}(\sigma_i/\zeta)), \tag{A1}$$

$$\frac{\partial E2}{\partial \sigma_i} = \{[(L^{s^2}-L^{d^2}) + (\sigma_1^2-\sigma_2^2)]\psi(Z/\zeta)(-Z/\zeta^2) - [(L^d+L^s)\psi(Z/\zeta)][1 + (Z^2/\zeta^2)]\}(\sigma_i/\zeta) - 2\sigma_i\Phi(Z/\zeta) + 2\sigma_i. \tag{A2}$$

From $V(Q) = E(Q^2) - (E(Q))^2$ it follows that

$$\frac{\partial V}{\partial \sigma_i} = \frac{\partial E2}{\partial \sigma_i} - 2E(Q)\frac{\partial E}{\partial \sigma_i}.$$

Computation of the derivatives of the PML functions in the text is straightforward using

$$\frac{\partial \psi}{\partial \sigma_i} = \frac{\partial \psi}{\partial E}\frac{\partial E}{\partial \sigma_i} + \frac{\partial \psi}{\partial V}\frac{\partial V}{\partial \sigma_i}, \tag{A3}$$

where $\psi=\psi(E,V)$ for PML_3 and $\psi=\psi(E)$ for PML_1.

Now let $\zeta = \surd(N)\surd(\sigma_1^2+\sigma_2^2)$, then

$$\frac{\partial E}{\partial \sigma_1} = \frac{\partial E}{\partial \zeta}\,\frac{\surd(N)\sigma_1}{\surd(\sigma_1^2+\sigma_2^2)} = \frac{\partial E}{\partial \zeta}\,\frac{N\sigma_1}{\zeta}, \tag{A4}$$

$$\frac{\partial E}{\partial N} = \frac{\partial E}{\partial \zeta}\,\frac{½\zeta}{N}. \tag{A5}$$

Letting now denote $E2= \sum_{j=1}^{N} E(Q_{tj}^2)$ we get

$$\frac{\partial E2}{\partial \zeta} = \left(\frac{(L^s)^2-(L^d)^2}{N} + \sigma_2^2-\sigma_1^2\right)\Phi\left(\frac{Z}{\zeta}\right)\left(\frac{-Z}{\zeta^2}\right) - \frac{L^d+L^s}{N}\,\psi\left(\frac{Z}{\zeta}\right) - \frac{L^d+L^s}{N}\,\psi\left(\frac{Z}{\zeta}\right)\frac{Z^2}{\zeta^2}. \tag{A6}$$

Because $\zeta=\zeta(\sigma_1,\sigma_2,N)$ one has, using the chain rule,

$$\frac{\partial E2}{\partial \sigma_1} = \frac{\partial E2}{\partial \sigma_1} + \frac{\partial E2}{\partial \zeta}\,\frac{\partial \zeta}{\partial \sigma_1}, \tag{A7}$$

$$\frac{\partial E2}{\partial N} = \frac{\partial E2}{\partial N} + \frac{\partial E2}{\partial \zeta}\,\frac{\partial \zeta}{\partial N}, \tag{A8}$$

where as before the symbols on the left and right hand side should be interpreted appropriately. It is straightforward to obtain for the ('strict') partial derivatives of E2:

$$\frac{\partial E_2}{\partial \sigma_1} = -\,2\sigma_1\Phi\left(\frac{Z}{\zeta}\right) + \sigma_1, \tag{A9}$$

$$\frac{\partial E_2}{\partial \sigma_2} = 2\sigma_2\Phi\left(\frac{Z}{\zeta}\right), \tag{A10}$$

$$\frac{\partial E_2}{\partial N} = \frac{((L^d)^2-(L^s)^2)}{N^2}\,\Phi\left(\frac{Z}{\zeta}\right) - \frac{(L^d)^2}{N^2} + \frac{\zeta}{N^2}\,(L^d+L^s)\psi\left(\frac{Z}{\zeta}\right). \tag{A11}$$

The derivatives in (A9), (A10) and (A11) have to be substituted into the right hand side of (A7) and (A8). The derivatives of V are finally obtained using

$$V(Q_t) = \sum_{j=1}^{N} E(Q_{tj}^2) - \sum_{j=1}^{N} E(Q_{tj})^2.$$

- PML_1 MONTE CARLO AGGREGATE MODEL -

In this case $E(Q_t)$ is replaced by $\bar{Q}_t = \frac{1}{G} \sum_{g=1}^{G} Q_{tg}$, where

$Q_{tg} = \text{Min } \{L_t^d + \sigma_1 \eta_{1tg}, L_t^s + \sigma_2 \eta_{2tg}\}$. Because only the function $\zeta = \sqrt{(\sigma_1^2+\sigma_2^2)}$ is identified we took $\sigma_1=\sigma_2=\zeta/\sqrt{2}$, i.e.

$$Q_{tg} = \text{Min } \{L_t^d + (\zeta/\sqrt{2})\, \eta_{1tg},\ L_t^s + (\zeta/\sqrt{2})\, \eta_{2tg}\}. \tag{A12}$$

One clearly has

$$\frac{dQ_{tg}}{d\zeta} = \begin{cases} (1/\sqrt{2})\eta_{1tg} \text{ if } E(D) + (\zeta/\sqrt{2})\eta_{1tg} < E(S)\ (\zeta/\sqrt{2})\eta_{2tg}, & \text{(A13)} \\ (1/\sqrt{2})\eta_{2tg} \quad \text{elsewhere} & \text{(A14)} \end{cases}$$

(note that the derivatives are discontinuous).

Defining $\delta_{tg} = 1$ if the condition in (A13) is satisfied and $\delta_{tg} = 0$ elsewhere, one obtains

$$\frac{d\bar{Q}_t}{d\zeta} = \frac{1}{G\sqrt{2}} \sum_{g=1}^{G} [\delta_{tg}(\eta_{1tg} - \eta_{2tg})+\eta_{2tg}], \tag{A15}$$

$$\frac{d\psi_{1T}}{d\zeta} = -2 \sum_{t=1}^{T} \{(Q_t-\bar{Q}_t)\, \frac{d\bar{Q}_t}{d\zeta}, \tag{A16}$$

$$\frac{d^2\psi_{1T}}{d\zeta^2} = 2 \sum_{t=1}^{T} (\frac{d\bar{Q}_t}{d\zeta})^2. \tag{A17}$$

Note that in (A17) the identity $\frac{d^2\bar{Q}_t}{d\zeta^2} = 0$ is used. In Table IA the inverse of the Hessian matrix using (A17) is compared with the one obtained from the DFP algorithm.

- *PML_3 MONTE CARLO AGGREGATE MODEL* -

In this case $E(Q_t)$ is replaced by $\bar{Q}_t = \frac{1}{G}\sum_{g=1}^{G} Q_{tg}$ and $V(Q_t)$ is replaced by $Var(Q_t) = \frac{1}{G-1}\{\sum_{g=1}^{G} Q_{tg}^2 - G(\bar{Q}_t)^2\}$; Q_{tg} is defined as in (A12) but without the restriction $\sigma_1 = \sigma_2 = \zeta/\sqrt{2}$. We therefore have

$$\left.\begin{aligned} \frac{dQ_{tg}}{d\sigma_1} &= \eta_{1tg} \\ \frac{dQ_{tg}}{d\sigma_2} &= 0 \end{aligned}\right\} \quad \text{if } L_t^d + \sigma_1 \eta_{1tg} < L_t^s + \sigma_2 \eta_{2tg},$$

$$\left.\begin{aligned} \frac{dQ_{tg}}{d\sigma_1} &= 0 \\ \frac{dQ_{tg}}{d\sigma_2} &= \eta_{2tg} \end{aligned}\right\} \quad \text{if } L_t^d + \sigma_1 \eta_{1tg} > L_t^s + \sigma_2 \eta_{2tg}. \tag{A18}$$

All derivatives are now obtained through simple substitution.

- *PML_3 MONTE CARLO AGGREGATE MODEL* -

All derivatives can now easily be obtained from the previous case using $\sigma_1 = \sigma_2 = \zeta/\sqrt{2}$

APPENDIX II

COMPUTATION OF COVARIANCE MATRIX FOR PML_1 AND PML_3

The asymptotic covariance matrix of θ is $J^{-1}IJ^{-1}$ [Gourieroux et al. (1984)], where

$$J_{ij} = E_x E\{-\frac{\partial^2\psi(x,\theta)}{\partial\theta_i\partial\theta_j}\} \text{ and } I_{ij} = E_x E\{\frac{\partial\psi(x,\theta)}{\partial\theta_i}\frac{\partial\psi(x,\theta)}{\partial\theta_j}\}, \tag{1}$$

where x is a vector of exogenous variables; θ is a vector of parameters.

-PML_1-

The pseudo function according to PML_1 is given by:

$$\Psi(x,\theta) = ½(y - f(x,\theta))'\Sigma^{-1}(y - f(x,\theta)) = \frac{1}{2\sigma^2}\sum_{t=1}^{T}\psi_t(x,\theta), \tag{2}$$

where Σ is the diagonal covariance matrix of the endogeneous variable y; f(.,.) represents the expectation of y as a function of x and θ. Note that for expository reasons we took y and f to be vectors instead of matrices, i.e. for each t we are dealing with a scalar. From standard differentiation rules one has

$$E\{\frac{\partial^2\psi(x,\theta)}{\partial\theta_i\partial\theta_j}\} = \frac{\partial f'}{\partial\theta_i}\Sigma^{-1}\frac{\partial f}{\partial\theta_j}, \tag{3}$$

$$E\{\frac{\partial\psi(x,\theta)}{\partial\theta_i}\frac{\partial\psi(x,\theta)}{\partial\theta_j}\} = E\{\frac{\partial f'}{\partial\theta_i}\Sigma^{-1}(y - f)(y - f)'\Sigma^{-1}\frac{\partial f}{\partial\theta_j}\} =$$

$$= \frac{\partial f'}{\partial\theta_i}\Sigma^{-1}\frac{\partial f}{\partial\theta_j}, \tag{4}$$

$$E_x\{\frac{\partial\psi(x,\theta)}{\partial\theta_i}\frac{\partial\psi(x,\theta)}{\partial\theta_j}\} = \frac{1}{T}\sum_{t=1}^{T}\frac{\partial\psi_t(x,\theta)}{\partial\theta_i}\frac{\partial\psi_t(x,\theta)}{\partial\theta_j}\,. \tag{5}$$

-PML_3-

The results presented here are from private correspondence with Bernard Salanié. The pseudo function according to PML_3 is given by:

$$\Psi(x,\theta) = \tfrac{1}{2}(y - f(x,\theta))' \, g^{-1}(x,\theta) \, (y - f(x,\theta)) + \tfrac{1}{2} \, \text{logdet}(g(x,\theta)), \qquad (6)$$

where g is the diagonal covariance matrix of y as a function of x and θ. From L&S we know that:

$$E\{\frac{\partial^2\psi(x,\theta)}{\partial\theta_i\partial\theta_j}\} = \frac{\partial f'}{\partial\theta_i} g^{-1} \frac{\partial f}{\partial\theta_j} + \tfrac{1}{2} \, \text{Tr}(g^{-1}\frac{\partial g}{\partial\theta_i} g^{-1} \frac{\partial g}{\partial\theta_j}), \qquad (7)$$

$$E\{\frac{\partial\psi}{\partial\theta_i} \frac{\partial\psi}{\partial\theta_j}\} = \frac{\partial f'}{\partial\theta_i} g^{-1} \frac{\partial f}{\partial\theta_j} + \tfrac{1}{2} \, \text{Tr}(g^{-1}\frac{\partial g}{\partial\theta_i} g^{-1} \frac{\partial g}{\partial\theta_j}). \qquad (8)$$

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