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ASYMPTOTIC ANALYSIS FOR BUFFER BEHAVIOUR
IN COMMUNICATION SYSTEMS

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ASYMPTOTIC ANALYSIS FOR BUFFER BEHAVIOUR IN COMMUNICATION SYSTEMS

by

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ABSTRACT. An important problem in computer communication systems is the design of finite buffers such that a small overflow probability is achieved. For the case of batch input and general service times, we derive an asymptotically exponential expansion for the overflow probability when the batch-arrival process is Poissonian. Next the results are extended to handle applications with periodic service opportunities and random interruptions in service. A convenient model to handle this type of applications is the versatile queueing model with exceptional first services.

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1. INTRODUCTION

Buffer overflow is an important problem in communication and production systems. This paper considers the practical situation of a small overflow probability and gives an easily computed asymptotic expansion of this probability. The finite-capacity $M^X/G/1$ queue is a useful model to analyse the behaviour of buffers in computer communication systems with batch arrivals. In section 2 we derive the overflow probability by relating the finite-capacity queue to the infinite-capacity queue through simple arguments from regenerative processes. Under mild assumptions on the batch-size and service-time distributions we give in section 3 an asymptotically exponential expansion for the overflow probability. Also, some numerical results are given showing the practical usefulness of the asymptotic expansion. In the final section 4 we extend the results to the case with periodic opportunities for service and random interruptions in service. This paper is related to earlier work of Bruneel (1983), Heines (1979), Tijms (1986) and Woodside and Ho (1987). The first two references consider the infinite buffer case and the latter two references deal with the case of a finite buffer but with single arrivals.

2. THE REGENERATIVE ANALYSIS

Suppose that at a single-server channel batches of packets arrive according to a Poisson process with rate λ. The batch size has a general probability distribution $\{\beta_j, j=1,2,..\}$ with finite mean β. The buffer has only capacity for K packets including the one in service. A batch whose size exceeds the remaining capacity in the buffer is partially lost due to overflow of the excess. The channel can handle only one packet at a time and the service time of the packets are independent random variables having a common probability distribution function F(x) with finite mean μ. Letting $\rho=\lambda\beta\mu$, it is assumed that the offered load ρ is less than 1.

Using simple probabilistic arguments, we derive in this section a formula for the long-run fraction of packets that overflow. To do so, denote by X_n the number of packets left behind at the n^{th} service completion of a packet. In the model the overflow probability is more easily obtained by analysing the embedded Markov chain $\{X_n\}$ rather than the continuous-time process describing the number of packets present at an arbitrary time. For the finite-capacity model let $\{q_j(K),\ j=0,\ldots,K-1\}$ be the limiting distribution of the embedded Markov chain $\{X_n\}$. Note the interpretation

$q_j(K)$ = the long-run fraction of service completions at which j packets are left behind.

Similarly, define $\{q_j(\infty),\ j=0,\ 1,\ldots\}$ as the equilibrium distribution of the Markov chain $\{X_n\}$ for the infinite-capacity model. Further, for the finite-capacity model, let

$\pi_{loss}(K)$ = the long-run fraction of packets that are lost.

The main result of this section is the following theorem.

<u>Theorem 1</u>. For any finite buffer size K,

$$\pi_{loss}(K) = \frac{\beta q_0(\infty) - (1-\rho)\sigma_K}{\beta q_0(\infty) + \rho\sigma_K}, \tag{1}$$

where

$$\sigma_K = \sum_{j=0}^{K-1} q_j(\infty) \quad \text{and } q_0(\infty) = \frac{1-\rho}{\beta}.$$

To prove this result, we first establish several lemmas. To do so, we need the following notation. Let us say that a cycle starts each time an arriving batch finds the system empty. For the finite-capacity model, define

$N_j(K)$ = the number of service completions during one cycle at which j packets are left behind ($j=0,\ 1,\ldots,\ K-1$),

and

$N(K)$ = the number of packets served during one cycle.

Similarly, we define $N_j(\infty)$ and $N(\infty)$ for the infinite-capacity model. Noting that only the batch size in excess of the remaining buffer capacity overflows and that accepted packets are served one at a time, the following lemma is a consequence of the lack of memory of the Poisson arrival process.

Lemma 2 For any finite K,

$$E[N_j(K)] = E[N_j(\infty)] \quad \text{for } j=0,1,\ldots,K-1.$$

Next we prove

Lemma 3 For any finite K,

$$q_j(K) = q_j(\infty)/\sum_{n=0}^{K-1} q_n(\infty) \qquad \text{for } j=0,1,\ldots,K-1.$$

Proof. By the theory of regenerative processes (cf. Ross (1983))

$$q_j(K) = \frac{E[N_j(K)]}{E[N(K)]} \quad \text{and} \quad q_j(\infty) = \frac{E[N_j(\infty)]}{E[N(\infty)]} \quad .$$

By Lemma 1, we next see that $q_j(K)/q_j(K-1) = q_j(\infty)/q_{j-1}(\infty)$ for $1 \le j \le K-1$. Hence, putting $\sigma_K = q_0(\infty)/q_0(K)$, we have $q_j(K) = q_j(\infty)/\sigma_K$ for $1 \le j \le K-1$. Since $\Sigma_{j=0}^{K-1}\, q_j(K) = 1$, it follows that $\sigma_K = \Sigma_{j=0}^{K-1}\, q_j(\infty)$. This ends the proof.

Lemma 4 For the infinite capacity model,

$$q_0(\infty) = \frac{1-\rho}{\beta} \quad .$$

Proof The long-run fraction of time the server is providing service equals ρ as can be intuitively seen by assuming a cost at rate 1 when the server is busy and then noting that the long-run average cost rate equals the average arrival rate of packets times the average service time of a packet, cf. also Ross (1983). Next, by the property Poisson arrivals see time averages, it follows that the fraction of batches finding upon arrival the system empty equals $1-\rho$. Hence the fraction of packets served as first one from their batch equals

$(1-\rho)/\beta$. Also, this fraction equals $1/E[N(\infty)]$. By $q_0(\infty) = 1/E[N(\infty)]$ we now get the desired result.

We are now in a position to prove Theorem 1.

Proof of Theorem 1 By the theory of regenerative processes, we have for the finite-capacity model

$$\pi_{loss}(K) = \frac{E[\text{number of packets lost during one cycle}]}{E[\text{number of packets arriving during one cycle}]} .$$

Obviously, the numerator of this ratio equals the the denominator minus $E[N(K)]$. Using $q_0(K) = 1/E[N(K)]$ and Lemma 3, we find

$$E[N(K)] = \frac{\sigma_K}{q_0(\infty)} .$$

By this relation and Wald's equation we have that the expected total service time during one cycle equals $\sigma_K\mu/q_0(\infty)$. Hence

$$E[\text{length of one cycle}] = \frac{1}{\lambda} + \frac{\sigma_K\mu}{q_0(\infty)} .$$

Next, by applying Wald's equation again,

$$E[\text{number of packets arriving during one cycle} = \lambda\beta \left[\frac{1}{\lambda} + \frac{\sigma_K\mu}{q_0(\infty)}\right].$$

and so we get $\pi_{loss}(K) = 1-[\sigma_K/q_0(\infty)]\,[\beta+\rho\sigma_k/q_0(\infty)]^{-1}$ yielding the desired result.

To conclude this section, we give a recursion scheme for the equilibrium probabilities $q_j(\infty)$. Therefore we use the following up- and downcrossings argument. The long-run fraction of services at whose completions n packets are left behind must be equal to the long-run fraction of services having the property that just prior to their beginning no more than n packets are waiting in queue and just prior to their completion more than n packets are in the system.

Hence, denoting by $\{a_j\}$ the probability distribution of the number of packets to arrive during the service time of one packet, we find for $n=0,1,\ldots$ the recursion equation

$$q_n(\infty) = q_0(\infty)\sum_{j=n+1}^{\infty} \beta_j + q_0(\infty)\sum_{j=1}^{n} \beta_j \sum_{h=n+1-j}^{\infty} a_h + \sum_{k=1}^{n} q_k(\infty)\sum_{h=n+1-k}^{\infty} a_h \qquad (2)$$

This recursion scheme involves no subtractions and is therefore numerically stable. In general the compound Poisson probabilities a_j are not easy to compute except for the following two cases. For the special case of a constant service time D, the a_j's can be recursively computed from

$$a_0 = e^{-\lambda D}, \quad a_j = \frac{\lambda D}{j}\sum_{k=1}^{j} k\beta_k a_{j-k} \quad \text{for } j\geq 1,$$

see section 1.6 in Tijms (1986). A simple recursion scheme for the a_j's can also be given when the service time S has a Coxian-2 distribution, that is, S equals S_1 with probability 1-b and equals S_1+S_2 with probability b, where S_1 and S_2 are independent exponentials with respective means $1/\mu_1$ and $1/\mu_2$. Then, denoting by $a_j^{(2)}$ the probability of j packet arrivals during a remaining service time distributed as S_2, we have for $j=1,2,\ldots$ the recursion equations

$$a_j^{(2)} = \frac{\lambda}{\lambda+\mu_2}\sum_{k=1}^{j} \beta_k a_{j-k}^{(2)} \quad \text{and} \quad a_j = \frac{\lambda}{\lambda+\mu_1}\sum_{k=1}^{j} \beta_k a_{j-k} + \frac{\mu_1 b}{\lambda+\mu_1} a_j^{(2)} ,$$

where $a_0^{(2)} = \mu_2/(\lambda+\mu_2)$ and $a_0 = \mu_1(\lambda+\mu_1)^{-1}\{1-b+b\mu_2/(\lambda+\mu_2)\}$.

In the next section it will be seen that for practical purposes the overflow probability $\pi_{loss}(K)$ can be much more easily calculated from an asymptotically exponential expansion than from (1) in conjunction with the recursion scheme (2).

3. THE ASYMPTOTIC EXPANSION

By Theorem 1, an asymptotically exponential expansion for the overflow probability $\pi_{loss}(K)$ of the finite-capacity model is obtained when the tail probabilities $q_j(\infty)$ for the infinite-capacity model exhibit a geometric behaviour for j large enough. These tail probabilities will decrease geometrically fast only when the batch-size and service-time distributions have no extremely long tails. Therefore we make the following assumption.

<u>Assumption</u>

(a) The convergence radius R of the power series $\beta(z) = \Sigma_{j=1}^{\infty} \beta_j z^j$ is larger than 1 and the integral $\int_0^\infty e^{st} dF(t)$ is finite for some s>0.

(b) $\lim_{s\to A} \int_0^\infty e^{st} dF(t) = \infty$ where $A = \sup\{s \mid \int_0^\infty e^{st} dF(t) < \infty\}$.

(c) A number $R_0 \in (1,R]$ exists such that $\lim_{x\to R_0} \beta(x) = 1+A/\lambda$.

This assumption is satisfied in cases of practical interest, e.g. when the batch-size distribution has finite support and the service-time distribution is of the phase-type. It is pointed out that assumption (b) excludes densities of the form $ce^{-\alpha t}/t^2$.

Under the above assumption there is a unique number $z_0 \in (1,R_0)$ such that

$$\int_0^\infty e^{\lambda t\{\beta(z_0)-1\}} dF(t) = z_0 \ . \tag{3}$$

To prove this, define the generating function $A(z) = \Sigma_{n=0}^{\infty} a_n z^n$, $|z| \le 1$, for the probability distribution $\{a_j\}$ of the number of packets to arrive during the service time of one packet. Using that

$$a_n = \int_0^\infty dF(t) \sum_{k=0}^{n} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \beta_n^{k*} \qquad \text{for } n=0,1,\ldots \ .$$

with $\{\beta_n^{k*}\}$ denoting the k-fold convolution of $\{\beta_n\}$ with itself, we obtain

$$A(z) = \int_0^\infty e^{\lambda t\{\beta(z)-1\}} dF(t). \tag{4}$$

By the above assumption the right side of (4) is analytic for $|z| < R_0$. Thus, by Taylor's theorem, the power serves representation $A(z) = \Sigma_{n=0}^{\infty} a_n z^n$ can be extended to the domain $|z| < R_0$. Consequently, $A''(x)>0$ for all $x \in (0,R_0)$ and so $A(x)$ is strictly convex on $(0,R_0)$. Therefore the graph of $A(x)$ can intersect the line $y=x$ in at most two points. We have $A(1)=1$. Moreover, by $A'(1)=\rho$ and the assumption $\rho < 1$, the function $A(x)-x$ is decreasing at $x=1$. Hence for some $a > 0$, $A(x)<x$ for $x \in (1, 1+a)$. Next, using that $\lim_{x \uparrow R_0} A(x)>R_0$ by the representation (4) and part (c) of the above assumption, it follows that $A(x)-x$ has a unique zero z_0 on the interval $(1,R_0)$. As a by-product of this proof we find that $A(x)<x$ for $1 < x < z_0$. This property will be needed in the proof of the following theorem.

<u>Theorem 5</u> For the infinite-capacity model,

$$q_j(\infty) \sim q_0(\infty) \frac{[\beta(z_0)-1]}{A'(z_0)-1} z_0^{-j} \qquad \text{for } j \text{ large enough},$$

where $\beta(z) = \Sigma_{j=1}^{\infty} \beta_j z^j$, $A'(z_0) = \lambda\beta'(z_0) \int_0^\infty t e^{\lambda t\{\beta(z_0)-1\}} dF(t)$, and z_0 is defined by (3).

<u>Proof</u>. The proof is based on partial fraction expansion of the generating function of the $q_j(\infty)$'s and requires some complex function analysis. The power series $Q(z) = \Sigma_{j=0}^{\infty} q_j(\infty) z^j$ is certainly convergent in the domain $|z| \le 1$ of the complex plane. Using the equilibrium equations

$$q_n(\infty) = \sum_{k=1}^{n+1} q_k(\infty) a_{n+1-k} + q_0(\infty) \sum_{k=1}^{n+1} \beta_k a_{n+1-k}, \quad n \ge 0,$$

it is a matter of simple algebra to obtain that

$$Q(z) = q_0(\infty)\ [1-\beta(z)]\ \frac{A(z)}{A(z)-z} \quad . \tag{5}$$

Since the power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\beta(z) = \sum_{n=0}^{\infty} \beta_n z^n$ are convergent for all $|z| < R_0$, it follows that the right side of (5) is analytic in the domain $|z| < R_0$ with the possible exception of the zeros of the denominator $A(z)-z$. Since the power series $Q(z)$ is convergent for all $|z| \le 1$, the right side of (5) has no pole on or within the circle $|z|=1$ although $z=1$ is a zero of $A(z)-z$. The dominant pole of $Q(z)$ in the domain $1<|z|<R_0$ determines the asymptotic behaviour of $q_j(\infty)$. We first show that $A(z)-z$ has no zeros in the domain $1<|z|<z_0$ and that the real number z_0 is the only zero of $A(z)-z$ on the circle $|z|=z_0$. The zero z_0 of $A(z)-z$ has multiplicity 1 since $A'(z_0)-1\neq 0$ by the strict convexity of $A(x)$. We have already proved above that $A(x)<x$ on the interval $(1,z_0)$. Hence, by the power series representation of $A(x)$, we get $|A(z)| \le A(|z|) < |z|$ for all z in the domain $1<|z|<z_0$. To prove that $A(z)-z$ has z_0 as the only zero on the circle $|z|=z_0$, it suffices to verify that $A(z_1)$ is real when $A(z_1)=z_1$ with $|z_1|=z_0$. To do so, denote by x_1 en y_1 the real part and imaginary part of $\beta(z_1)$. Since $|\beta(z_1)|=(x_1^2+y_1^2)^{\frac{1}{2}}$ and $|\beta(z_1)| \le \beta(|z_1|)=\beta(z_0)$, we have $(x_1^2+y_1^2)^{\frac{1}{2}} \le \beta(z_0)$ and so $x_1 \le \beta(z_0)$. Then, using that $|e^{iy}|=1$, we obtain from (4) and (3) that

$$z_0 = |A(z_1)| \le \int_0^{\infty} e^{\lambda(x_1-1)t} dF(t) \le \int_0^{\infty} e^{\lambda(\beta(z_0)-1)t} dF(t) = z_0 \ .$$

This implies that $x_1=\beta(z_0)$. Next, by $(x_1^2+y_1^2)^{\frac{1}{2}} \le \beta(z_0)$, we find $y_1=0$. Hence $\beta(z_1)$ is a real number and so, by (4), $A(z_1)$ is a real number.

By the above analysis there is a number R_1 with $z_0<R_1<R_0$ such that $A(z)-z$ has no zeros in the domain $|z| \le R_1$ except at the point $z=z_0$. Here we use that the zero $z=z_0$ has a neighbourhood containing no other zeros of $A(z)-z$, since $A(z)-z=(z-z_0)\varphi(z)$ for some analytic function $\varphi(z)$ with $\varphi(z_0)\neq 0$ by the Taylor expansion and the fact that the zero z_0 is of order 1. Consequently, we can write (5) as $Q(z)=H(z)/(z-z_0)$ for some analytic function $H(z)$ in $|z| \le R_1$ with

$H(z_0)\neq 0$. Using the Taylor expansion $H(z)=H(z_0)+(z-z_0)U(z)$, we next find that an equation of the form

$$Q(z) = \frac{a_{-1}}{z-z_0} + U(z) \tag{6}$$

is true for $|z|\leq R_1$, $z\neq z_0$, where $U(z)$ is some analytic function in the domain $|z|\leq R_1$ and the residue a_{-1} is given by

$$a_{-1} = \lim_{z\to z_0}(z-z_0)\ Q(z) = \frac{q_0(\infty)[1-\beta(z_0)]\ A(z_0)}{A'(z_0)-1}\ .$$

Note that $a_{-1}\neq 0$. Since $U(z)$ is analytic for $|z|\leq R_1$, a Taylor series expansion $U(z)=\Sigma_{j=0}^{\infty}u_j z^j$ is true for $|z|\leq R_1$. The power series $\Sigma_{j=0}^{\infty}u_j z^j$ is convergent for $z=R_1$ and so $u_j R_1^j$ is bounded in j. Since $Q(z)=\Sigma_{j=0}^{\infty}q_j(\infty)z^j$ for $|z|<z_0$, we obtain from the series expansion of the right side of (6) that

$$q_j(\infty) = \frac{q_0(\infty)[\beta(z_0)-1]\ A(z_0)}{A'(z_0)-1}\ z_0^{-j-1} + O(R_1^{-j}) \quad \text{for all } j\geq 0.$$

Using that $A(z_0)=z_0$ and $R_1>z_0$, we finally get the desired result.

As an immediate consequence of the Theorems 1 and 5, we have

<u>Theorem 6</u> For K large enough,

$$\pi_{loss}(K)\sim \frac{(1-\rho)\gamma_0 z_0^{-K}}{1-\rho\gamma_0 z_0^{-K}}\ , \tag{7}$$

where

$$\gamma_0 = \frac{(1-\rho)}{\beta}\ [\lambda\beta'(z_0)\int_0^{\infty} te^{\lambda t(\beta(z_0)-1)}dF(t)-1]^{-1}[\beta(z_0)-1]\ \frac{z_0}{z_0-1}\ . \tag{8}$$

Defining $K(\alpha)$ as the smallest integer K for which $\pi_{loss}(K)\leq\alpha$, it follows from Theorem 6 that $K(\alpha)$ can be approximated by

$$K(\alpha) \approx \ln(\gamma_0(1-\rho+\rho\alpha)/\alpha)/\ln(z_0). \tag{9}$$

when α is small enough. Typically in practical applications α will be small. It is an empirical finding that the asymptotic expansion for the $q_j(\infty)$'s applies already for relatively small values of j provided that ρ is not very small. Thus for practical purposes we can compute $K(\alpha)$ from (9). This is confirmed by the numerical results in table 1. For the cases of a constant batch size and a geometrically distributed batch size, we give in table 1 the asymptotic values of $K(\alpha)$ (rounded to above) for several service-time distributions and several values of ρ and α. In nearly all cases the asymptotic values (9) are equal to the exact values of $K(\alpha)$. The few cases in which they differ are marked by *. In each marked case the difference between the exact value and the approximate value is 1 except for the case marked with ** for which the difference is 2. The offered load ρ is varied as 0.2, 0.5, 0.8 and 0.9, while the service level α is varied as 10^{-1}, 10^{-3} and 10^{-5}. Denoting by $c_S^2 (=\sigma^2(S)/E^2(S))$ the squared coefficient of variation of the service time S, we vary c_S^2 as 0, ½, 1, 2 and 5. The values $c_S^2=0$, ½ and 1 correspond to the deterministic, Erlang-2 and exponential distributions, while the values $c_S^2=2$ and 5 correspond to hyperexponential distributions of order 2 (the H_2-distribution is a special case of a Coxian-2 distribution). For purposes of sensitivity analysis, we consider for the H_2-distribution both the normalization of balanced means (b) and the gamma normalization (g). For the first normalization the three parameters of the H_2-density $p_1\mu_1 e^{-\mu_1 t}+p_2\mu_2 e^{-\mu_2 t}$ are chosen such that $p_1/\mu_1=p_2/\mu_2$ whereas for the second normalization the parameters are chosen such that the first three moments of the distribution are the same as those of a gamma distribution. The numerical results in table 1 confirm that

$$K(\alpha) \approx (1-c_S^2)K_{det}(\alpha) + c_S^2 K_{exp}(\alpha) \tag{10}$$

is an excellent approximation provided that c_S^2 is not too large, where $K_{det}(\alpha)$ and $K_{exp}(\alpha)$ denote $K(\alpha)$ for the particular cases of constant and exponential services with the same means E(S). Approximations of this type were advocated in Tijms (1986) who gives many examples in which queue-size or waiting-time percentiles can be approximated as in (10).

Table 1. The minimal buffer sizes $K(\alpha)$

		const.(β=2)			geom.(β=2)			const.(β=5)			geom.(β=5)		
		10^{-1}	10^{-3}	10^{-5}	10^{-1}	10^{-3}	10^{-5}	10^{-1}	10^{-3}	10^{-5}	10^{-1}	10^{-3}	10^{-5}
ρ=0.2	c_s^2=0	3	7	10	5	13	22	6	15	24	13	38	64
	=1/2	3	7	11	5	13	22	6	16	25	13	39	65
	=1	3	8	12	5	14	23	6	16	26	13	39	65
	=2(b)	2*	9	15	4*	14	24	6	17	28	12*	39	66
	=2(g)	3	8	14	4*	14	24	6	17	28	12*	39	66
	=5(b)	2*	12	22	3**	16	30	5*	20	36	11**	40	70
	=5(g)	3	11	19	4*	16	28	6	20	34	12*	41	70
ρ=0.5	c_s^2=0	4	11	19	6	19	33	9	27	45	16	57	98
	=1/2	4	13	22	6	21	35	9	29	48	17	58	101
	=1	5	15	25	6	22	38	9	30	52	17	59	103
	=2(b)	5	19	33	7	25	45	10	34	59	17	62	109
	=2(g)	5	18	31	7	25	44	10	34	58	17	62	108
	=5(b)	6	30	55	8	36	66	11	45	81	18	72	127
	=5(g)	6	28	49	8	34	61	11	43	76	18	71	124
ρ=0.8	c_s^2=0	6	26	47	9	41	75	14	63	117	24	120	224
	=1/2	7	31	58	9	46	85	14	63	117	25	125	233
	=1	8	37	68	10	51	94	15	74	138	26	130	243
	=2(b)	9	48	91	12	62	116	17	86	159	27	141	263
	=2(g)	9	48	89	12	61	115	17	85	158	27	141	263
	=5(b)	14	84	159	17	93	183	21	120	226	32	174	327
	=5(g)	14	80	150	17	93	175	22	117	220	32	172	322
ρ=0.9	c_s^2=0	8	46	90	11	70	139	17	113	224	30	209	415
	=1/2	9	56	111	12	80	159	14	124	245	31	219	436
	=1	10	67	133	13	90	180	20	134	266	32	229	456
	=2(b)	12	88	177	15	112	223	22	155	310	35	250	498
	=2(g)	12	88	175	15	111	222	22	155	309	35	249	498
	=5(b)	19	154	310	22	176	355	29	220	442	41	313	627
	=5(g)	19	150	301	22	173	348	29	217	436	42	311	622

4. OVERFLOW PROBABILITY FOR INTERRUPTED SERVICE

In this section we first discuss an extension of the previous results to the case of a set-up time after an idle period and next apply the extended results to a computer communication model with periodic opportunities for service and random interruptions of service.

Suppose that a warming-up time W is required before the server can start the service of the first packet from a batch that finds upon arrival the system empty. The service times and the warming-up times are independent. Let $G_w(t)$ denote the probability distribution function of the warming-up time. Defining $q_0(K)$ and $\pi_{loss}(K)$ as before, an examination of the analysis of section 2 reveals that the main Theorem 1 remains valid provided we replace the formulae for $\pi_{loss}(K)$ and $q_0(\infty)$ by

$$\pi_{loss}(K) = \frac{\beta q_0(\infty)\{1+\lambda E(W)\}-(1-\rho)\sigma_K}{\beta q_0(\infty)\{1+\lambda E(W)\}+\rho\sigma_K} \tag{11}$$

and

$$q_0(\infty)= \frac{1-\rho}{\beta\{1+\lambda E(W)\}} \quad . \tag{12}$$

The modification (11) is explained by noting that the expected length of a cycle in the finite-capacity model is now given by $1/\lambda+E(W)+\sigma_K\mu/q_0(\infty)$. We find (12) by the following modification of the proof of Lemma 3. For the infinite-capacity model the long-run fraction of time the server is servicing packets remains equal to $\rho=\lambda\beta\mu$. Denote by f_0 and f_w the long-run fractions of time that the system is empty respectively a warming-up period is in progress. Then $f_0+f_w=1-\rho$. Imagine now that the system incurs a cost at rate 1 whenever a warming up time is in progress. Then the long-run average cost rate equals f_w and is given by the average arrival rate of batches finding the system empty times the average length of a warming-up period. Hence $f_w=\lambda f_0 E(W)$ yielding that $f_0=(1-\rho)/(1+\lambda E(W))$. The remaining part of the proof of (12) proceeds as in Lemma 3.

The only other modification required in section 2 concerns the recursion equation (2) in which the second term $q_0(\infty)\Sigma_j\beta_j\Sigma_h a_h$ should be replaced by $q_0(\infty)\Sigma_j\beta_j\Sigma_h a_h^*$, where the probability distribution $\{a_j^*\}$ is the convolution of the probability distributions $\{a_j\}$ and $\{w_j\}$. Here w_j is the probability that the total number of packets to arrive during a warming-up period equals j. Denoting by $W(z)=\Sigma_{j=0}^{\infty}w_j z^j$ the generating function of the w_j's, we obtain similarly to (4) that

$$W(z)=\int_0^{\infty} e^{\lambda t(\beta(z)-1)}dG_w(t).$$

Next we can easily modify the results of section 3. The equation (5) becomes

$$Q(z)=q_0(\infty)[1-\beta(z)]\ W(z)\ \frac{A(z)}{A(z)-z} \quad ,$$

and thus

$$q_j(\infty)\sim\ q_0(\infty)\ \frac{[\beta(z_0)-1]\ W(z_0)}{A'(z_0)-1}\ z_0^{-j} \quad \text{for } j \text{ large.} \tag{13}$$

Here we need the technical assumption that $\sup\{s|\int_0^{\infty}e^{st}dG_w(t)<\infty\}$ is larger than $\lambda(\beta(z_0)-1)$. The decay coefficient z_0 is again determined by (3). Defining the constant γ_w by

$$\gamma_w = \frac{\gamma_0 W(z_0)}{1+\lambda E(W)}$$

with γ_0 given by (8), it follows from (11)-(13) that the asymptotic formulae (7) and (9) remain valid for the model with set-up times provided we replace in these formulae γ_0 by γ_w.

We can go a step further and make the following useful extension. In addition to the warming-up times, suppose that the first service starting a busy period has a different distribution than the other services. Let $F_1(t)$ denote the probability distribution of the exceptional first services and let μ_1 denote its mean. Then the formulae (7) and (9) remain valid provided we replace γ_0 by

$$\gamma_{we}=\frac{\gamma_0\ W(z_0)\ A_1(z_0)/z_0}{1+\lambda E(W)+\lambda\mu_1-\lambda\mu} \quad ,$$

where $A_1(z_0)=\int_0^\infty e^{\lambda t\{\beta(z_0)-1\}}dF_1(t)$. This modification can be seen as follows. For the finite-capacity model the expected length of a cycle becomes $1/\lambda+E(W)+\sigma_K\mu/q_0(\infty)-\mu+\mu_1$, while for the infinite-capacity model $q_0(\infty)$ becomes $1-\rho$ divided by $1+\lambda E(W)+\lambda\mu_1-\lambda\mu$ and the term $A(z)$ in the numerator of (5) should be replaced by $A_1(z)W(z)$.

A communication system with interrupted services

Consider a communication channel at which batches of packets arrive according to a Poisson process with rate λ, where the batch size has a general distribution. The packets are temporarily stored in a finite buffer to await transmission. Overflow occurs for those packets of an arriving batch which are in excess of the remaining buffer capacity. The transmission time of each packet is a constant slot length of one time unit. The beginnings of the time slots provide the only opportunity to start the transmission of a packet. The transmission channel is subject to random service interruptions. It is assumed that at the beginning of each time slot the channel is available for transmission with a given probability f, independently of the state of the channel in the previous time slots. Equivalently, the transmission of a packet is successful with probability f, otherwise the transmission has to be retried in the next time slot.

To find the long-run fraction of packets that overflow, we convert this model with random service interruptions into a finite-capacity $M^X/G/1$ model with set-up times. The service time of a packet is defined as the number of time slots from the moment the packet is ready for transmission until the packet is succesfully received. Hence the service time S of a packet has the geometric distribution

$$P(S=j)=(1-f)^{j-1}f,\ j=1,2,\dots\ .$$

A batch arriving when the system is empty has to wait until the next periodic opportunity for service. Therefore the warming-up time W is defined as the time from the moment that a batch arrives when the system is empty until the beginning of the next time slot. Obviously, the probability distribution of W is given by

$$P(W \le t) = \frac{e^{-\lambda(1-t)} - e^{-\lambda}}{1 - e^{-\lambda}} \qquad \text{for } 0 \le t \le 1 .$$

Then we can apply the above results using the specification of the probability distributions of the service time and the set-up time. Our unifying analysis extends results in Woodside and Ho (1987). Using the queueing model with exceptional first services we can also solve the model with periodic service opportunities and service interruptions when the process describing the service interruptions is an exogenous two-state Markov chain. Then the probability distribution of the exceptional first services can be specified by using simple Markov chain analysis as given in Woodside and Ho (1987). Finally, the versatile queueing model with exceptional first services can also be used when the process describing the on-state and off-state for service is an alternating renewal process in which the on-times have a geometric distribution and the off-times have a general discrete distribution.

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