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STOCHASTIC MARKET EQUILIBRIA WITH RATIONING AND LIMITED PRICE FLEXIBILITY

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Abstract

In this paper we consider a market where a heterogeneous population of individual actors demands units of various types of a heterogeneous good (e.g., housing) and also have the possibility to withdraw from the market. It is assumed that allocation on the market does not (completely)take place by means of the price mechanism, either because prices are completely fixed, or because they can only vary within some limited range. Rationing is assumed takes place by preventing some actors to realize the alternative they have chosen.

We will proof three existence theorems. First of all we demonstrate the existence of a rationed equilibrium when all prices are completely fixed. Second, we show the existence of a mixed equilibrium, i.e., an equilibrium where demand and supply are matched partly by means of price adjustments and partly by means of quantity rationing. Third, we proof the existence of a mixed equilibrium in a situation where the rationing is allowed to vary over the different classes of demanders and over their present situation. This introduces the possibility to give a preferential treatment to some groups of demanders. The second theorem is a generalisation of the first ; the third a generalisation of the second.

1 Introduction.

In recent years there has been a lot of interest in the existence and uniqueness of price-equilibria in markets where aggregate demand is determined on the basis of discrete choice models for individual decision-makers (cf. Anas[1982], Anas and Cho[1986], Eriksson[1986], Smith[1988], Rouwendal[1988b]). Although the results derived in these papers are of a more general nature, in some of these articles application of them to the housing market seems to have been a major motivation for studying such price equilibria. It is well-known, however, that in many countries the housing market is not a 'free' market where prices can have any value required to equilibrate demand and supply. Many governments have taken measures to regulate the market and limit price flexibility. It would therefore be of interest to study the possibilities of regulating the market by means of some rationing mechanism. In the present paper some steps in this direction will be made by considering a particular form of rationing in situations where prices are either completely fixed or can vary within some exogenously determined limits only.

2 Preliminaries

We consider a population consisting of b actors. Each actor belongs to one of M possible classes. This population is distributed over N+1 possible states, indexed 0,1,...,N. State 0 should be identified with non-participation.

In each period every actor is confronted with N+1 alternatives, N of which should be identified with types of the good traded in the market, from which he can choose one, while the remaining one refers to non-participation. When there is rationing he cannot be sure whether the alternative chosen can be realized by him. If realization is not possible it is assumed that such an actor will continue his present state. For the housing market this implies that households that desire to move to another type of dwelling, will not always be able to realize this desire and will then continue living in their present dwelling. It is assumed that the latter possibility is always existent[1].

The various choice alternatives are identified by a vector $\underset{\sim}{x}$ of characteristics. The price associated with each of the alternatives is one of them. The characteristics associated with the various alternatives influence the utilities that will be attached to them by the actors and are for this reason an important determinant of choice behaviour.

When choice for a particular alternative would imply the realization of that alternative with certainty, the characteristics $\underline{x}_n$ would be the only determinants of choice behaviour. It will be assumed here, however, that a choice for a particular alternative does not automatically imply realization of that alternative. When there is excess demabe increased rationing takes place. The ratio of total demand of alternative n and the available supply determines the perunage ψ_n of total demand that can be satisfied.

It will be assumed until section 6 of this paper that rationing takes place in such a way that all actors choosing for type n of the good have a probability ψ_n of realizing their choice. It may be objected that this is not a particularly realistic form of rationing. For instance, in housing markets queueing seems to be of more importance. Furthermore the government usually gives a priority treatment to those actors that are especially in need of another dwelling and this characteristic of actual rationing schemes is excluded by the equal treatment of all households by giving them the same realization probability ψ_n.

In answer to these objections it may be remarked that a realization probability ψ_n implies an expected waiting time of $1/\psi_n$ periods and that the two forms of rationing are, for this reason, not as different as they seem to be at first sight. Furthermore it is possible to make the realization probabilities ψ_n different for different groups of actors as will be pointed out in section 6. For the moment we will therefore use the alocation scheme proposed above because it offers a convenient starting point for analysis.

The probability $\pi_{mn \to n'}$, that an actor (household) of type m who is currently in state (a dwelling of type) n will choose to move to state n' will be assumed to be determined by all vectors $\underline{x}_{n''}$, and the choice probabilities $\psi_{n''}$:

$$\pi_{mn\to n'} = \pi_{mn\to n'}(\underline{x}_1, \ldots, \underline{x}_N, \underline{\psi}^n) \quad , \qquad (1)$$
$$m=1,\ldots,M \ , \ n,n'=0,\ldots,N \ ,$$

where $\underline{\psi}^n$ denotes the vector of realization probabilities $\underline{\psi}$ $(=[\psi_0,\ldots\psi_N]')$ with $\psi_{n'}$ replaced by 1 in order to deal with the fact that every actor has the possibility to continue his present state.

These choice probability functions may be viewed as the outcome of utility maximization, where the utility that is attached to the choice for a particular alternative n may, e.g., be equal to the expected utility $\psi_n . v_{mn\to n'} + (1-\psi_n).v_{mn\to n}$, where $v_{mn\to n'}$ is the utility attached to alternative n' by an actor of type m who is currently in state n [2].

The characteristics that are associated with the various states will be assumed to be fixed, with one exception, viz. the prices p_n. The fixed arguments may be suppressed and we will therefore write instead of (1) :

$$\pi_{mn\to n'} = \pi_{mn\to n'}(\underline{p} \ , \ \underline{\psi}^n) \quad , \qquad (2)$$
$$m=1,\ldots,M \ , \ n,n'=0,\ldots,N \ .$$

The choice probability functions will always be assumed to be continuous in the prices and realization probabilities for all nonnegative prices and realization probabilities. They are also assumed to be non-increasing in the own price, non-decreasing in the own realization probability, non-decreasing in the other prices and non-increasing in the other realization probabilities.

3 Individual Choice Behaviour and Market Demand.

The total number of actors choosing for alternative n , D_n^* can be determined as :

$$D_n^* = \sum_{m=1}^{M} \sum_{n'=0}^{N} b_{mn'} . \pi_{mn'\to n}(\underline{p} \ , \ \underline{\psi}^{n'}) \qquad (3)$$
$$n=0,1,\ldots,N$$

where $b_{mn'}$ denotes the number of actors belonging to class m who are currently in state n'. Since not all actors will be able to realize their choice when ψ_n is smaller than 1 , we will refer to D_n^* as the revealed (as opposed to realized) demand for alternative n.

Realized demand, to be denoted as $\hat{D}_n$, is equal to the sum of ψ_n times the number of actors willing to move to state n from another state and the number of actors choosing to continue their sojourn in state n :

$$\hat{D}_n = \sum_{m=1}^{M} \sum_{\substack{n'=0 \\ n'\neq n}}^{N} \psi_n . b_{mn'} . \pi_{mn' \to n}(\underset{\sim}{p} , \underset{\sim}{\psi}^{n'}) + + b_{mn} . \pi_{mn \to n}(\underset{\sim}{p} , \underset{\sim}{\psi}^{n}) \quad , \quad m=1,\ldots,M \ , \ n,n'=0,1,\ldots,N \ . \tag{4}$$

$\hat{D}_n$ is thus equal to the number of actors who are willing to move to state n (or stay there) and are able to do so. There are also actors who are originally in state n and who want to move to another state, but remain in state n since they are not able to realize that desire. These disappointed searchers will be referred to as $\bar{D}_n$. Their number can be determined as being equal to :

$$\bar{D}_n = \sum_{m=1}^{M} \sum_{\substack{n'=0 \\ n'\neq n}}^{N} [1-\psi_{n'}] . b_{mn} . \pi_{mn \to n'}(\underset{\sim}{p} , \underset{\sim}{\psi}) \quad , \quad m=1,\ldots,M \ , \ n=0,1,\ldots,N \ . \tag{5}$$

The effective demand for state n is the sum of realized demand for that state and the number of disappointed searchers who are in that state and will be denoted as D_n :

$$D_n = \sum_{m=1}^{M} \sum_{n'=0}^{N} \psi_n . b_{mn'} . \pi_{mn' \to n}(\underset{\sim}{p} , \underset{\sim}{\psi}^{n'}) + + [1-\psi_{n'}] . b_{mn} . \pi_{mn \to n'}(\underset{\sim}{p} , \underset{\sim}{\psi}^{n'}) \quad , \quad n=0,\ldots,N \ . \tag{6}$$

This effective demand D_n has to be equilibrated with the available supply. It will be assumed throughout the paper that supply consists of fixed, positive amounts S_n. In the next section we will examine the existence of such an equilibrium when prices are completely fixed.

It would be nice if the aggragated effective demands D_n had the same properties as the choice probability functions, i.e., if they would be non-increasing in the own price and the other realization probabilities and non-decreasing in the other prices and the own realization probability. The following proposition can be proven :

Proposition 3.1 The effective demands D_n , $n=0,\dots,N$, are non-increasing in the own price and non-decreasing in the own realization probability.

Proof. We will rewrite (4) as follows :

$$D_n = \sum_{m=1}^{M} b_{mn}\cdot\pi_{mn\to n}(\underline{p}\ ,\ \underline{\psi}^n) + \sum_{m=1}^{M}\sum_{\substack{n'=0\\ n'\neq n}}^{N} \psi_{n'}\cdot b_{mn'}\cdot\pi_{mn'\to n}(\underline{p}\ ,\ \underline{\psi}^{n'}) + \sum_{m=1}^{M}\sum_{\substack{n'=0\\ n'\neq n}}^{N} [1-\psi_{n'}]\cdot b_{mn}\cdot\pi_{mn\to n'}(\underline{p}\ ,\ \underline{\psi}^n)\ , \quad n=0,\dots,N\ . \tag{7}$$

Now consider the consequences of a small increase in p_n. The first and second expressions on the right-hand-side (rhs) of (7) will decrease or remain the same. The third expression will increase or remain the same. But this increase will never exceed the decrease in the first expression on the rhs of (7) since the choice probabilities $\pi_{mn\to n'}$, $n'=0,\dots,N$ have to add up to 1 and the sum of the changes in these choice probabilities will always add up to zero. This shows that D_n will be non-increasing in the own price p_n.

Now consider a small increase in the realization probability ψ_n. The first and the third expression on the rhs of (7) will not change, the second one will increase or remain the same. D_n is therefore non-decreasing in the own realization probability ψ_n. Q.E.D.

D_n will not always be non-decreasing in the other prices $p_{n''}$. To see this consider the consequences of a small increase in $p_{n''}$, $n''\neq n$. The change in the first two expressions on the rhs of (7) will be nonnegative. The sign of the change in the third expression is ambiguous. The choice probabilities $\pi_{mn\to n'}$ will increase or remain the same when $n'\neq n''$, but $\pi_{mn\to n''}$ may decrease. However, there is one important case in which the sign of the total change in D_n is still determined.

Proposition 3.2 When ψ_n is equal to 1, D_n , $n=0,\ldots,N$, will be non-decreasing in the other prices $p_{n''}$, $n''\neq n$.

Proof. When ψ_n equals 1, the effect of the decrease in $\pi_{mn\to n''}$ on D_n will be nil. This means that we are left with only nonnegative changes and that the total effect on D_n will therefore be nonnegative. Q.E.D.

The effects of a small increase in the realization probability $\psi_{n''}$, $n''\neq n$, are also in general ambiguous in sign. The first and second expression on the rhs of (7) will decrease or remain the same. The change in the third expression is indeterminate because $\pi_{mn\to n''}$ will increase i.e., more people will be inclined to choose alternative n''. This increase will have a positive effect on the number of disappointed searchers who were intending to move to state n'', and may even compensate for the reverse effect on the number of disappointed searchers caused by the increase in $\psi_{n''}$. To see this we will write down the change in third expression on the rhs of (7) for $n'=n''$:

$$\sum_{m=1}^{M} b_{mn}.([1-\psi_{n''}].\Delta\pi_{mn\to n''} -\Delta\psi_{n''}.\pi_{mn\to n''} - \Delta\psi_{n''}.\Delta\pi_{mn\to n''}) \quad ,$$

where $\Delta\pi_{mn\to n''}$ is the change in $\pi_{mn\to n''}$ that occurs as a consequence of the change $\Delta\psi_{n''}$ in $\psi_{n''}$. There seems to be no general way to guarantee that this expression (possibly in combination with other parts of the rhs of (7)) is nonpositive.

We have to conclude that the properties of the choice probability functions $\pi_{mn\to n'}$ with respect to changes in prices and realization probabilities do only partially carry over to the effective demand functions.

4 Rationed Equilibrium

In the present section we will examine the question whether it is possible to determine the realization probabilities ψ_n in such a way that the effective demands will never exceed the available supply S_n and be equal to it whenever the corresponding realization probability is smaller than 1. Such a situation will be referred to as a rationed equilibrium[3]. Formally we define :

Definition 4.1. A rationed equilibrium is a set of realization probabilities ψ_n^* , $n=0,1,\ldots,N$, $0 \leq \psi_n^* \leq 1$ such that $D_n(\underline{p},\underline{\psi}^*) \leq S_n$ for all $n=1,\ldots,N$ for which $\psi_n^* > 0$, $D_n(\underline{p},\underline{\psi}^*) = S_n$ whenever $0 < \psi_n^* < 1$ and $D_n(\underline{p},\underline{\psi}^*) \geq S_n$ whenever $\psi_n^*=0$, for given prices $p_0,\ldots,p_N$.

The effective demand functions $D_n(\underline{p},\underline{\psi})$ have been defined in (7) above. Supply will be assumed to consist of fixed, positive amounts.

Before the existence of a rationed equilibrium will be proven we make some introductory remarks. A trivial equilibrium occurs when all realization probabilities are set equal to 0. Since it is assumed that all actors in the market can continue their present situation the supply S_n has to be equal to at least $\Sigma_m b_{mn}$. It should be noted, however, that this trivial equilibrium satisfies the definition of a fixed price equilibrium only when $S_n = \Sigma_m b_{mn}$ for all $n=1,\ldots,N$.

Since this trivial equilibrium is of little interest we would like to know wether there also exist other ones. One may conjecture that an affirmative answer can be given to this question on the basis of the following reasoning. Consider an arbitrary pair of states n and n'. It is possible that there are actors who want to move from n to n' and also that there are others willing to move in the reverse direction. The number of moves that can be realized is the minimum of both numbers. By considering combinations of more than two states an even higher number of moves can be realized.

A problem that is inherent in this approach is that the number of actors willing to move to a certain state is itself determined partly by the values of the realization probabilities. For this reason the above reasoning does not seem to be of much help in demonstrating the existence of a non-trivial equilibrium, although it strongly suggests such.

Another approach will therefore be adopted here. This approach makes use of Brouwer's fixed point theorem which states that a continuous mapping of a nonempty, closed and convex set into itself has a fixed point, i.e., a point that is mapped into itself (see e.g. Arrow and Hahn[1970] for a proof of the theorem).

Proposition 4.1 For every set of nonnegative prices $p_0,\dots,p_N$ there exists a rationed equilibrium in the market where demand D_n is given by (7) and supply consists of fixed, positive amounts S_n, $n=0,\dots,N$.

Proof. We define Q as the set of vectors $\underline{\psi}$, $0 \le \underline{\psi} \le 1$. The set Q is nonempty, closed and convex. Consider the following function $\underline{F}$:

$$F_n(\psi_n) = \psi_n + \max(0\ ,\ \min[\ -(D_n(\underline{p},\underline{\psi})-S_n)/b)\ ,\ 1-\psi_n\ ]) + \\ -\max(0\ ,\ \min[\ (D_n(\underline{p},\underline{\psi})-S_n)/b)\ ,\ \psi_n\ ])\quad ,\quad n=0,\dots,N\ . \tag{8}$$

This function is continuous and maps the set Q into itself. We can therefore be sure that there exists a fixed point $\underline{\psi}^*$. It can be inferred from (8) that ψ_n^* equal 0 or 1 , or that $D_n(\underline{p},\underline{\psi}^*) = S_n$. When ψ_n^* equals 1 we can be sure that $D_n(\underline{p},\underline{\psi}^*) \le S_n$. When ψ_n equals 0 we can be sure that $D_n(\underline{p},\underline{\psi}^*) \ge S_n$. We can therefore conclude that $\underline{\psi}^*$ is a rationed equilibrium. Q.E.D.

It may be remarked that the realization probability ψ_n^* can only be zero when $S_n=\sum_m b_{mn}$ and the realization probabilities $\underline{\psi}^*$ are such that no single actor who is currently in state n wants to move to another state and is able to do so. Although there is nothing in our assumptions that excludes this state of affairs, it seems to be a very special case only.

5 Mixed Equilibrium

Now that we have proven the existence of a fixed price equilibrium, we turn to the question whether it will be possible to equilibrate the market by means of some mixed regime, in which prices are not completely fixed, but are restricted to vary within a limited range only. Such mixed regimes may be of higher relevance for the analysis of housing markets in Western European countries (where often the rented part of the market is highly regulated, while the owner-occupied part is relatively free) than the fixed price regime analyzed in the preceding section.

It will be assumed that for all alternatives 1,...,N there exists a minimum price $\underline{p}_n$ (possibly equal to 0) and a maximum price $\bar{p}_n$ (possibly equal to ∞). Allocation takes in first instance place only by means of prices. Only when a price has reached its upper bound and there is still excess demand, rationing may occur. These considerations give rise to the following definition of a mixed equilibrium.

Definition 5.1. A mixed equilibrium is a set of prices $(p_1^*,\dots,p_N^*)$, such that $\underline{p}_n \le p_n^* \le \bar{p}_n$ for all $n=1,\dots,N$, and a set of realization probabilities $(\psi_0^*,\dots,\psi_N^*)$, $0 \le \psi_n^* \le 1$ for all $n=0,\dots,N$ such that :

a) $\psi_n^*=1$ for all $n=0,\dots,N$ with $\underline{p}_n \le p_n^* < \bar{p}_n$,

b) $D_n(\underline{p}^*,\underline{\psi}^*) \le S_n$ when $\psi_n > 0$ for all $n=0,\dots,N$,

c) $D_n(\underline{p}^*,\underline{\psi}^*)=S_n$ for all alternatives $n=0,\dots,N$ for which $p_n^* > \underline{p}_n$, $\psi_n>0$.

The existence of a mixed equilibrium will now be proven on the basis of the following two assumptions :

Assumption 5.1

$$\lim_{p_n\to\infty} \pi_{mn\to n'}(\underline{p},\underline{\psi}^n) = 0 \quad , \qquad (9)$$

$$m=1,\dots,M \ , \ n,n'=0,\dots,N \ .$$

Assumption 5.2 When $\bar{p}_n = \infty$, $S_n > b_n$.

The first assumption states that nobody is willing to move to state n when the price associated with such a move increases without an upper bound. The second assumption states that there will be excess supply for all those states for which there does not exist a maximum price. These assumptions are rather weak.

Proposition 5.1 When assumptions 5.1 and 5.2 are satisfied, there exists a mixed equilibrium $(\underline{p}^*, \underline{\psi}^*)$ in the market where demand D_n is described by (6) and supply consists of fixed, positive amounts S_n , $n=0,\ldots,N$.

Proof. We will prove this proposition with the aid of auxiliary variables r_n , $n=0,\ldots,N$. The variables p_n and ψ_n will be defined as functions of these variables in the following way :

$$p_n = \begin{cases} \underline{p}_n & \text{when } -\ln(r_n) \le p_n \\ -\ln(r_n) & \text{when } \underline{p}_n \le -\ln(r_n) \le \bar{p}_n \\ \bar{p}_n & \text{when } -\ln(r_n) \ge \underline{p}_n \end{cases} , \qquad n=0,\ldots,N . \tag{10}$$

$$\psi_n = \begin{cases} 1 & \text{when } r_n < \exp\text{-}\bar{p}_n) \\ r_n/\exp(-\bar{p}_n) & \text{when } r_n \ge \exp(\bar{p}_n) \end{cases} \qquad n=0,\ldots,N . \tag{11}$$

These functions are continuous in r_n. Te set R of vectors $\underline{r}$ contains all vectors for which $\exp(-\underline{p}_n) \le r_n \le 0$. The set R is non-empty, closed and convex. We now define the vector-valued function $\underline{F}$ in the following way :

$$\begin{aligned} F_n = r_n &+ \max(0 , \min[-(D_n(\underline{p}(\underline{r}),\underline{\psi}(\underline{r}))-S_n)/b , \bar{r}_n-r_n]) + \\ &- \max(0 , \min[(D_n(\underline{p}(\underline{r}),\underline{\psi}(\underline{r}))-S_n)/b , r_n] , \qquad n=0,\ldots,N , \end{aligned} \tag{12}$$

where $\bar{r}_n$ is defined as $\exp(-\underline{p}_n)$. This function is continuous in the variables r_n and maps the set R into itself. We can thus be sure that there exists a fixed point r^*. We show that this point can be identified with a mixed equilibrium $(p^*(r^*),\psi^*(r^*))$. First observe that the definitions of the functions $\underline{p}(r)$ and $\underline{\psi}(\underline{r})$ imply that ψ_n can be smaller than 1 only when $p_n=\bar{p}_n$. The definition of the function $\underline{F}$ implies that at the fixed point we have $r_n^*=\bar{r}_n$, $D_n(\underline{p}(r^*),\underline{\psi}(r^*))=S_n$ or $r_n^*=0$. When $r_n^*=\bar{r}_n$, it follows from (12) that $D_n(\underline{p}(\underline{r}^*),\underline{\psi}(\underline{r}^*)) \leq S_n$. In the same way it follows that $D_n(\underline{p}(\underline{r}^*),\underline{\psi}(\underline{r}^*)) \geq S_n$ when $r_n^*=0$. Finally, we can exclude the possibility of some prices being ∞ in equilibrium by our assumptions 5.1 and 5.2. Q.E.D.

It should be observed that the (fixed-price) rationed equilibrium of section 4 is a special case of the mixed equilibrium, that occurs when the lower bound of the price p_n , $\underline{p}_n$, equals the upper bound, $\bar{p}_n$, for all $n=0,\dots,N$.

6 Class and origin-specific realization probabilities

In the present section we will investigate the question whether it is possible to vary the realization probabilities over the various demanders. In concrete situations there may be good reasons for introducing such variation. For instance, on the housing market it is customary to regard some demanders as being more in need of a certain type of dwelling than others and to give them a priority treatment. Also the government may want to favour some types of moves on this market, because the dwelllings that become vacant as a consequence of these moves are needed for other households. There are therefore good reasons to investigate the possibilities of making the realization probabilities dependent on the class to which the actor belongs and on his present state. For this purpose we introduce the notation $\psi_{mn\to n'}$ for the probability that an actor of type m who is currently in state n will be able to move to state n' if he desires to do so. We will define these class and origin-specific realization probabilities as non-increasing functions of general realization probabilities ψ_n , which may be regarded as 'average' realization probabilities, in the following general way :

$$\psi_{mn\to n'} = \psi_{mn\to n'}(\psi_{n'}) \quad , \qquad (13)$$
$$m=1,\ldots,M \ , \ n,n'=0,\ldots,N \ , \ n'\neq n$$

with : $\psi_{mn\to n'}(1)=1$ and $\psi_{mn\to n'}(0)=0$, $m=1,\ldots,M$, $n,n'=0,\ldots,N$, $n'\neq n$. For $n'=n$ we will always have $\psi_{mn\to n}=1$. We will require the sum

$$\sum_{m=1}^{M} \sum_{\substack{n=0 \\ n\neq n'}}^{N} \psi_{mn\to n'}(\psi_{n'})$$

to be strictly decreasing in ψ_n . If one wishes to do so, this sum may even be required to be equal to $M.(N+1).\psi_{n'}$. In the latter case $\psi_{n'}$ is indeed the arithmetic average of the relevant specific averages.

A set of class and origin-specific realization probabilities that fulfills these requirements will be called admissable.

By reducing the realization probabilities for some classes of actors and for some current states earlier than others one may introduce a preferential treatment for some groups of demanders. In this way it can e.g., be guaranteed that demanders which are considered as being especially in need for a particular type of dwelling get a high realization probability even when the average realization probability is low. By making the realization probabilities origin-dependent the government may try to influence the distribution of the housing stock over the population. It may, for instance, stimulate moves of small households from large dwellings to smaller ones or moves of low-income households from expensive dwellings to cheaper ones.

The existence of a rationed or mixed equilibrium with realization probabilities that vary over the different groups of demanders can be proven in exactly the same way as was done for the equilibria with realization probabilities that are only specific for the state of destination. Aggregate demand will be defined analogous to (7) as :

$$D_n(\underline{p},\Psi(\psi)) = \sum_{m=1}^{M} \sum_{n'=0}^{N} \psi_{mn'\to n}(\psi_n).b_{mn'}.\pi_{mn'\to n}(\underline{p},\underline{\psi}_{mn'}(\underline{\psi})) + \qquad (14)$$
$$+ \sum_{m=1}^{M} \sum_{n'=0}^{N} [1-\psi_{mn\to n'}(\psi_{n'})].b_{mn}.\pi_{mn'\to n}(\underline{p},\underline{\psi}_{mn}(\underline{\psi})) \quad , \quad n=0,\ldots,N$$

where Ψ is a 'matrix' with three entries ; its elements are the class and origin-specific realization probabilities.

A rationed equilibrium with variable realization probabilities may now be defined as follows :

Definition 6.1 A mixed equilibrium with class and origin-specific realization probabilities is a set of realization probabilities $\{\psi^*_{mn\to n'}$, $0 \le \psi_n \le 1$, $m=1,\dots,M$, $n,n'=0,\dots,N\}$ and a set of prices $\{p_n$, $\underline{p}_n < p_n < \bar{p}_n$, $n=0,\dots,N$ $\}$ such that :

a) $\psi^*_{mn\to n'}=1$ for all m and n whenever $\underline{p}_{n'} \le p^*_{n'} < p_{n'}$, $n=0,\dots,N$,

b) $D_n \le S_n$ whenever $\psi^*_{mn'\to n} > 0$ for at least one m and n' ,

c) $D_n = S_n$ whenever $\psi^*_{mn'\to n} < 1$ for at least one m and n' or $p_n > \underline{p}_n$.

We will prove a proposition which is closely analogous to proposition 5.1 and restate assumption 5.1 for this purpose as :

Assumption 5.1'

$$\lim_{p_n \to \infty} \pi_{mn\to n'}(\underline{p},\underline{\psi}_{mn}) = 0 \quad , \qquad (15)$$

$$m=1,\dots,M \ , \ n,n'=0,\dots,N \ .$$

Proposition 6.1 When assumptions 5.1' and 5.2 are satisfied there exisits a mixed equilibrium for every set of admissable class and origin-specific realization probability functions.

Proof. The proof is completely analogous to that of proposition 5.1. The vector-valued function $\underline{F}$ that maps the set R into itself will now be defined as :

$$F_n = r_n + \max\{0 \ , \ \min[-(D_n(\underline{p}(\underline{r}),\Psi(\underline{\psi}(\underline{r})))-S_n)/b \ , \ \bar{r}_n - r_n]\} + - \max\{0 \ , \ \min[(D_n(\underline{p}(\underline{r}),\Psi(\underline{\psi}(\underline{r})))-S_n)/b \ , \ r_n]\} \qquad (16)$$

This continuous function maps the set R into itself and has a fixed point $\underline{r}^*$. This fixed point corresponds with a mixed equilibrium $(\underline{p}^*(\underline{r}^*),\Psi^*(\underline{\psi}^*(\underline{r}^*)))$. To see this, first observe that ψ_n can be smaller than 1 only when $p_n=\bar{p}_n$, and that this implies that only when this condition is fulfilled the realization probabilities $\psi_{mn'\to n}$ corresponding with that particular n can be smaller than 1.

Furthermore the definition of F_n implies D_n can only exceed S_n when r_n equals 0, which implies that $\psi_{mn'\to n}=0$ for all m and n'. Finally, it follows also from the definition of F_n that at the fixed point demand D_n will be exactly equal to supply S_n when $\underline{r}_n<r^*_n<\bar{r}_n$, i.e. whenever $p_n\geq \underline{p}_n$ or $\psi_{mn'\to n}>0$ for at least one m and n'(n'≠n). Q.E.D.

The mixed equilibrium which was proven to exist in proposition 5.1 is a special case of the equilibrium of proposition 6.1 that occurs when $\psi_{mn'\to n}(\psi_{n'})=\psi_{n'}$ for all m,n' and n.

8 Discussion

Proposition 6.1 is the most general result that will be proven in this paper. It incorporates propositions 5.1 and 4.1 as special cases. The contents of the proposition can be translated in non-technical terms as follows. When prices are not perfectly flexible, but are only able to vary within a limited range, rationing may be necessary in order to attune demand to the available supply. By choosing a particular set of origin and class specific realization probabilities the government may influence this equilibrium in such a way that some groups of demanders get a preferential treatment.

One may wonder whether the mixed equilibrium will be unique, i.e., whether there exists only one such equilibrium for a given set of upper and lower bounds of the prices and a given set of class and origin specific realization probability functions. An unambiguous answer to this question cannot be given, however, since we are not aware of a set of sufficient conditions for the uniqueness of equilibrium in the type of model under consideration which is satisfied by the model under consideration. In economics one usually applies the Gale-Nikaido[1966] theorem on global univalence of vector-valued functions. This requires differentiability of the aggregate demand functions and semi-definiteness of the matrix of partial derivatives of this function. It is easy to show[4] that the latter condition is fulfilled when the aggregate demand function is increasing in the own price, non-decreasing in the other prices, increasing in the own realization probability and non-decreasing iin the other realization probabilities. It has been shown in section 2, however, that the last property will not automatically be possessed by the aggregate demand function. Furthermore, it would be difficult to find a satisfsactory sufficient condition for this characteristic[5]. We must therefore conclude that uniqueness of the rationed equilibrium can in general not be proven[6].

Throughout this paper the discussion has been cast in static terms, but it should be noticed that the choice probabilities that have been used are in fact transition probabilities. All actors are initially in a certain state and have to decide whether or not they want to move to another one. The model may therefore be considered as describing the short-run equilibrium of a dynamic process.

This implies that there may exist some dynamic phenomena in our model which should be taken into account. The most important of these dynamic phenomena is queueing. To see the potential effects of queueing, assume that a particular choice alternative, say n, is rationed. This means that some actors who have been searching for a dwelling of type n, have not been able to find such a dwelling. In the next period these actors have to make a new choice. One may expect that these people are more likely to search again for alternative n than other actors, who just start searching. If this is indeed the the case it means that the choice probabilities are co-determined by the choice that was made in the former period.

In general, this effect can be taken into account by distinguishing the actors not only on the basis of the class to which they belong and their present housing situation, but also on the basis of the choices they made in the (recent) past. Only when the choice probabilities are the same for groups that made different decisions in the past but are otherwise identical can these dynamic effects be left out of consideration.

In order to specify the complete dynamic model one needs to say something about the transition of demanders from one class to another and about the entry and exit of demanders. The specification of such a complete dynamic model lies, however, beyond the scope of the present paper.

Although there has been a great deal of interest in fixed price equilibria in general equilibrium economics (see e.g. Dreze[1974] and Benassy[1975]), there has been no comparable interest in the study of such equilibria for specific markets. In particular for the housing market we are aware of only two exceptions, viz. Wiesmeth[1985], which uses a framework that is significantly different from the one adopted here, and Anas and Cho[1988]. In the latter paper a model is constructed that is somewhat similar to the one presented here. It is pointed out that the sufficient conditions for uniqueness of equilibrium stated in Anas and Cho[1986][7] may fail to hold (p.221) and also that there may not exist a short run equilibrium in the model (p.220-221). The latter conclusion seems to differ from the one reached above (proposition 6.1) which implies that, under reasonable conditions, there will always exist a short-run equilibrium in the model[9].

The model, that has been developed in the foregoing can in priciple be used to examine the consequenes of changes in the allocation mechanism on the housing market. In Rouwendal[1988c] a discrete choice model has been estimated in which the consequences of disequilibrium and queueing for the observed choice frequencies are incorporated.probabilities for different groups of demanders are not taken into account, however, so that the model corresponds with the situation dealt with in section 5. Extension to the situation of s6, in which the realization probabilities are origin and class specific is conceptually straightforward, but may be dificult to implement empirically. In all probability one needs information about the allocation rules used by the various institutions on the housing market under consideration. Using this approach it may be possible to gain understanding in the functioning of regulated housing markets and on the effects of (changes in) government measures taken with respect to such markets on various groups of households.

Notes

1 In some cases alternative assumptions may be desirable. E.g. when one has to deal with forced movements it may be useful to direct the actors that have not been successful to a different type of dwelling than the one they currently occupy.

2 The derivation of such a model is not an easy task, however. See Rouwendal[1988b] for a discussion of the problem and the formulation of an operational model for discrete choice behaviour in situations of uncertainty.

3 In the literature the term fixed-price equilibrium is also used. The term equilibrium is of course somewhat misleading.

4 One should formulate the aggregate demands as functions of the auxiliary variables $\underline{r}$. A sufficient condition for uniqueness of the fixed point $\underline{r}^*$ is that the Jacobean matrix of the system of demand equations has a dominant diagonal. This will be the case when the demand functions possess the properties mentioned in the text.

5 One may require $\partial\pi_{mn\rightarrow n'}/\partial q_{n'}$ to be smaller than $\pi_{mn\rightarrow n'}$ in order to guarantee that the aggragate demand functions are non-increasing in the other realization probabilities. However, this coauses new problems since $\pi_{mn\rightarrow n'}$ will be zero for $q_{n'}=0$ and then the condition mentioned above would imply that $\pi_{mn\rightarrow n'}$ is always zero, unless one allows for a discontinuity in the choice probability function. Discontinuities would imply a violation of the conditions for the fixed-point theorem.

6 This does of course not automatically imply that simultaneous equilibria will be likely to occur. From the discussion in section 3 it will be clear that the possibility that aggregate demand will be decreasing, although not easy to rule out, is not very likely to occur.

7 Essentiaslly equal to the Gale-Nikaido[1966] theorem.

8 It is not completrely clear what causes the difference in the conclusions, but the fact that Anas and Cho always require equality of supply and demand will probably have somthing to do with it.

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