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EXISTENCE AND UNIQUENESS OF STOCHASTIC PRICE
EQUILIBRIUM IN HETEROGENEOUS MARKETS

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Existence and Uniqueness of Stochastic Price Equilibria in Heterogeneous Markets.

Summary.

In this paper we study price equilibria in a market where the behaviour of individual demanders is determined by means of discrete choice models. The good that is traded in the market is heterogeneous. Demanders may also differ in their preferences. The number of demanders may exceed the the number of units of the good that is available in the market.

In the first part of the paper we give a general description of the model and deal with continuity and differentiability of the (expected) market demand functions. We then study the behaviour of aggregate demand when prices rise without an upper bound. It is also shown that the choice alternatives are always gross substitutes in a weak sense of the term and that strong gross substitutability can be guaranteed by two additional assumptions. The existence of a price equilibrium is then proven under weak gross substitutability and continuity of the demand functions and uniqueness under strong gross substitutability and differentiability of the demand functions. In subsequent sections it is shown that a neccesary and sufficient condition for uniqueness is the non-existence of a market segment with total demand for the goods in that segment is insensitive for changes in prices. The paper is concluded with a comparison of the results that have been reached by others and are reported in the literature and a discussion of the interpretation of stochastic price equilibria.

1 Introduction.

Throughout the paper we consider a market where N different types of the same good (housing may be a good example) are available in fixed (positive) amounts S_n , $n=1,\ldots,N$. These fixed amounts have to be distributed over a fixed number, b , of individual decision units (households). M different types of decision units will be distinguisghed ($M \leq b$). These individual decision makers are initially distributed over the various states in a particular way and should be re-allocated. It is assumed that individual decision-making units do always have the possibility to withdraw from the market (e.g. by means of migration) ; this alternative will be denoted by means of an index 0. Also , there may be actors who have just entered the market ; they are dealt with as being initially in state 0. For simplicity it will also be assumed that each individual decision-making unit can buy (rent) only one unit of the available supply.

The demanders are assumed to be utility maximizers. In each period they are confronted with N+1 choice alternatives , viz. moving to one of the N possible states or withdraw from the market. The utility associated with chosing any of these alternatives can be represented by means of a so-called conditional indirect utility function , i.e. a function that gives the highest value of utiltiy that can be reached by the decision maker concerned , given that he has to choose alternative n , $n=0,1,\ldots,N$. This indirect utility function has as arguments a vector $\underline{x}_n$ of explanatory variables and will be different for different types of households and for different initial states. The indirect utility of moving to choice alternative n for a household of type m that is currently in state n' will be denoted as $\bar{U}_{mn'\rightarrow n}$ and is assumed to be the sum of a deterministic part $U_{mn'\rightarrow n}$, that is a function of $\underline{x}_n$, and a stochastic part $\varepsilon_{mn'\rightarrow n}$:

$$\bar{U}_{mn'\rightarrow n}(\underline{x}_n) = U_{mn'\rightarrow n}(\underline{x}_n) + \varepsilon_{mn'\rightarrow n} \qquad (1)$$

$$m=1,\ldots,M \; ; \; n',n=0,1,\ldots,N$$

The inclusion of a stochastic component in the indirect utility functions makes it impossible to determine the choices of the utility maximizing decision unit with certainty. It is only meaningful to speak about the probability $\pi_{mn'\rightarrow n}$ that alternative n will be chosen by a decision maker belonging to class m and currently occupying a unit of type n' of the good traded in the market. The choice probabilities are defined as :

$$\pi_{mn'\to n} = \text{Prob}(\bar{V}_{mn'\to n} > \bar{V}_{mn'\to n''} \ , \ n'' \neq n \ , \ n''=1,\ldots,N) \quad (2)$$
$$n,n'=0,1,\ldots,N$$

and can be determined as follows :

$$\pi_{mn'\to n} = \int_{-\infty}^{\infty} \int_{-\infty}^{w_{mn'\to 0}} \cdots \int_{-\infty}^{w_{mn'\to N}} h_{mn'}(\underline{\varepsilon}_{mn'}) \, d\varepsilon_N \ldots d\varepsilon_0 . d\varepsilon_n \quad (3)$$
$$m=1,\ldots,M \ . \ n',n=0,\ldots,N.$$

where $w_{mn'\to n''} = U_{mn'\to n} - U_{mn'\to n''} + \varepsilon_{mn'\to n}$ $(n''\neq n)$ and $h_{mn'}(\underline{\varepsilon}_{mn'})$ is the simultaneous probability density function of the vector $\underline{\varepsilon}_{mn'} = [\varepsilon_{mn'\to 0},\ldots,\varepsilon_{mn'\to N}]'$. This function will always be assumed to have a zero mean and a finite variance-covariance matrix.

The choice probabilities $\pi_{mn'\to n}$ will in general be functions of all the indirect utilities $U_{mn'\to n''}$ 1) and therefore of all variables $\underline{x}_0,\underline{x}_1,\ldots,\underline{x}_N$:

$$\pi_{mn'\to n} = \pi_{mn'\to n}(\underline{x}_0,\underline{x}_1,\ldots,\underline{x}_N) \quad (4)$$
$$m=1,\ldots,M \ , \ n',n=0,1,\ldots,N$$

Total expected demand for choice alternative n , D_n ,can , on the basis of (4) be determined as :

$$D_n = \sum_{m=1}^{M} \sum_{n'=0}^{N} b_{mn'} . \pi_{mn'\to n}(\underline{x}_0,\underline{x}_1,\ldots,\underline{x}_N) \quad (5)$$
$$n=0,1,\ldots,N$$

where $b_{mn'}$ denotes the number of decision-making units of type m that are currently in state n'. This total expected demand is therefore a function of all variables that influence the utilities attached to the various alternatives by the individual demanders , but also of the compostion of the population of these demanders and of their initial distribution over the various states. Since all choice probabilities add up to 1 we have :

$$\sum_{n=0}^{N} D_n = b \quad (6)$$

in what follows we will treat the demand functions D_n as if they represented a deterministic demand , although they in fact give only the expected value of a demand that is stochastic. Some justification for this procedure is given by the fact that under general assumptions the ratio of actual stochastic demand and the total number of demanders tends to approach its expected value D_n/b when the total number of demanders b is large (see Lehoczky[1980]).

The limited availability of units of the N types causes the following restrictions on the allocation :

$$D_n \leq S_n \qquad n=1,\dots,N \tag{7}$$

A first requirement for a market equilibrium would be the satisfaction of the constraints (7). A somewhat more demanding definition of equilibrium will be used in section 4.

For proofs of existence and uniqueness of equilibria it is useful to know under what conditions the demand functions D_n are continuous or differentiable in the variables $\underline{x}_0,\underline{x}_1,\dots,\underline{x}_N$. Since these functions are sums of choice probability functions $\pi_{mn'\to n}$, this invokes the question under which conditions these functions are continuous or differentiable in their arguments. The following propositions are proven in the appendix.

<u>Proposition 1.1</u>. The choice probability functions $\pi_{mn'\to n}$, $m=1,\dots,M$, $n',n=1,\dots,N$, are continuous in the variables $\underline{x}_0,\underline{x}_1,\dots,\underline{x}_N$ if and only if the utility functions $v_{mn'\to n}$, $m=1,\dots,M$, $n'n=1,\dots,N$, are continuous in their arguments $\underline{x}_n$ and the cumulative density functions $H_{mn'}(\underline{\varepsilon}_{mn'})$, $m=1,\dots,M$, $n'=0,1,\dots,N$ are continuously differentiable almost everywhere.

<u>Proposition 1.2</u>. The choice probability functions $\pi_{mn'\to n}$, $m=1,\dots,M$, $n',n=1,\dots,N$,are differentiable in the variables $\underline{x}_0,\underline{x}_1,\dots,\underline{x}_N$ if and only if the utility functions $v_{mn'\to n}$, $m=1,\dots,M$, $n',n=1,\dots,N$, are differentiable in their arguments $\underline{x}_n$ and the cumulative density functions $H_{mn'}(\underline{\varepsilon}_{mn'})$, $m=1,\dots,M$, $n'=0,1,\dots,N$ are twice differentiable almost everywhere.

In discrete choice theory it is usually assumed that the probability density functions are normal or (generalized) extreme value. In these cases the conditions with respect to the cumulative density functions mentioned in the propositions are always fulfilled. Also the specifications of the indirect utility functions that are used in empirical work ususally fulfill the continuity and differentiability requirements[2).

2 Some Preliminary Results.

Throughout the rest of the paper it will be assumed that there are prices associated with all types of the good that are available in the market i.e. with choice alternatives 1,...,N. The price p_n is one of the arguments $\underline{x}_n$ of the indirect utilities $U_{mn'\rightarrow n}$, and all other arguments of that vector are assumed to be constant , and can therefore be suppressed. This implies that the utility attached to alternative 0 is constant. Furthermore it will be assumed that the indirect utilties $U_{mn'\rightarrow n}$ are nonincreasing in p_n with :

$$\lim_{p_n \rightarrow \infty} U_{mn'\rightarrow n}(p_n) = -\infty \tag{8}$$

$m=1,\ldots,M$, $n'=0,1,\ldots,N$, $n=1,\ldots,N$

All these assumptions seem to be quite natural[3).
The following preliminary result is easily proven :

Proposition 2.1.

$$\lim_{p_n \rightarrow \infty} D_n = 0 \tag{9}$$

$n=1,\ldots,N$

Proof. The proposition is proven when it can be shown that $\lim_{p_n \rightarrow \infty} \pi_{mn'\rightarrow n} = 0$ for all $m=1,\ldots,M$, $n'=0,1,\ldots,N$ and $n=1,\ldots,N$. Now look at equation (3). When $p_n \rightarrow \infty$, $U_{mn'\rightarrow n} \rightarrow -\infty$ and $w_{mn'\rightarrow n''} \rightarrow -\infty$ for all $n'' \neq 0,n$. The probability density function $h_{mn'}(\underline{\varepsilon}_{mn'})$ was assumed to have zero mean and a finite variance-covariance matrix. This implies that indeed $\lim_{p_n \rightarrow \infty} \pi_{mn'\rightarrow n} = 0$ for all the required m's , n''s and n's.
Q.E.D.

Proposition 2.1 says that the demand for a particular alternative will become arbitrarily close to zero if the price associated with it becomes arbitrary large. It will also be useful to know what happens to the demand D_n when all prices $p_1,\ldots,p_N$ are increasing without an upper bound. The answer is given by the following proposition :

Proposition 2.2.

$$\lim_{p \to \infty} D_n = 0 \qquad (10)$$

$n=1,\ldots,N$

Proof. It can again be observed that the proposition is true when the validity of its analogon for the individual choice propositions can be shown. Consider again equation (3). When all prices rise whithout an upper bound we can be sure that $w_{mn' \to 0}$ becomes more and more negative , without lower bound. Given our assumptions about the probability density function $h_{mn'}(\varepsilon_{mn'})$, this suffices to draw the conclusion that $\lim_{p \to \infty} \pi_{mn' \to n} = 0$ for all $m=1,\ldots,M$, $n'=0,1,\ldots,N$ and $n=1,\ldots,N$. Q.E.D.

This proposition essentially says that when the price of staying in the market , no matter what alternative will be chosen , becomes infinitely large , ultimately everybody will decide to withdraw from the market. E.g. when rents grow higher and higher in one particular region , ultimately everybody prefers living at another place where rents are lower.

3 Gross Substitutability.

In conventional demand theory goods are called gross substitutes when an increase in the price of good n causes demand D_n to fall and demand for all other goods $D_{n'}$, $n' \neq n$ to rise (see e.g. Varian[1978]). The assumption of gross substitutability is used in general equilibrium theory to prove the uniqueness of price equilibria , but is generally regarded to be very restrictive. It is therefore somewhat surprising that in the present context (where the demand functions are based on discrete choice models) the demand functions do always satisfy a slightly weakened version of gross substitutability and that two weak assumptions suffice to guarantee complete gross substitutability. This will be shown in the next two propositions.

Proposition 3.1. The demand functions D_n , $n=1,\ldots,N$, are non-decreasing in the prices $p_{n'}$, $n' \neq n$, $n'=1,\ldots,N'$ and non-increasing in the own price p_n. Moreover , D_0 is non-decreasing in all prices p_n , $n=1,\ldots,N$.

Proof. A sufficient condition for this proposition to be true is that its analogon holds for all choice probability functions $\pi_{mn'\to n}$. Consider equation (3). When the price p_n rises the variables $w_{mn'\to n''}$ all decrease or remain the same. This implies that the same will be true for the choice probability $\pi_{mn'\to n}$. When another price $p_{n''}$ rises the variable $w_{mn'\to n''}$ increases or remains the same , while all other variables $w_{mn'\to n''}$ remain the same. Therefore the choice probability $\pi_{mn'\to n}$ increases or remains the same. It may therefore be concluded that all choice probabilities $\pi_{mn'\to n}$, $m=1,\ldots,M$, $n'=0,1,\ldots,N$, $n=1,\ldots,N$ are non-increasing in the own price p_n and non-decreasing in all other prices. It can in the same way be proven that the choice probabilities $\pi_{mn'\to 0}$ are non-decreasing in all prices $p_1,\ldots,p_N$. The analogon of the proposition for the choice probabilities $\pi_{mn'\to n}$ thus holds. Q.E.D.

We will refer to the property of proposition 2.2 as weak gross substitutability. It is easy to see (from equation (3)) that the stronger definition mentioned above (to be referred to as strong gross substitutability) requires all indirect utility functions to be strictly decreasing in p_n. But it also requires a positive probability of choosing each alternative at all possible prices[4]. This can be guaranteed by assuming that the probability density function is positive almost everywhere[5].

Proposition 3.2. The demand functions D_n , $n=1,\ldots,N$, are increasing in the prices $p_{n'}$, $n'\neq n$, $n'=1,\ldots,N'$, and decreasing in the own price p_n when for at least one m and n' for which $b_{mn'}$ is positive the functions $U_{mn'\to n''}$ are decreasing in the prices $p_{n''}$ and when the simultaneous probability density function $h_{mn'}(\underline{\varepsilon}_{mn'})$ is positive almost everytwhere. Furthermore , D_0 is increasing in all prices $p_1,\ldots,p_N$.

Proof. We will prove this proposition in a somewhat more formal way than was used to show the vality of the preceding ones. Consider equation(3) and assume that a price $p_{n''}$, $n''=n$, increases. The resulting change in the choice probability $\pi_{mn'\to n}$, for the m and n' that fulfill the conditions mentioned in the proposition can be determined as :

$$\Delta\pi_{mn'\to n} = \int_{-\infty}^{\infty}\int_{-\infty}^{w_{mn'\to N}} \dots \int_{w_{mn'\to n''}+\Delta}^{w_{mn'\to n''}} \dots \int_{-\infty}^{w_{mn'\to 1}} h_{mn'}(\underline{\varepsilon}_{mn'})\, d\varepsilon_1 \dots$$

$$\dots d\varepsilon_{n''} \dots d\varepsilon_{N} \cdot d\varepsilon_n \qquad (11)$$

where Δ is the negative change in the value of $w_{mn'\to n''}$ that occurs as a result of a change in $p_{n''}$. The expression on the right-hand-side of (11) is clearly nonnegative. Since $h_{mn'}(\underline{\varepsilon}_{mn'})$ is assumed to be positive almost everywhere we can be sure that it is positive.

Now assume that p_n increases. We then find for the change in the choice probability $\pi_{mn'\to n}$ for the m and n' that fulfill the conditions mentioned in the proposition :

$$\Delta\pi_{mn'\to n} = -\int_{-\infty}^{\infty}\int_{w_{mn'\to N}}^{w_{mn'\to N}+\Delta} \dots \int_{w_1}^{w_1+\Delta} h_{mn'}(\underline{\varepsilon}_{mn'})\, d\varepsilon_1 \dots$$

$$\dots d\varepsilon_N \cdot d\varepsilon_n \qquad (12)$$

where Δ is the positive change that occurs in all $U_{mn'\to n}$ because of the increase in p_n. We can be sure that the exoression on the right-hand-side of (12) is negative since the probability density function is positive almost everywhere.

It can be demonstrated in an analogous way that for the m and n' that fullfill the conditions of the proposition $\pi_{mn'\to 0}$ increases when any of the prices $p_1,\dots,p_N$ increases.

Now look at the definition of aggragate demand , given in equation (5) , where instead of the vectors $\underline{x}_n$ one should read prices p_n. We know from proposition 2.3 that (for n > 1) D_n is non-increasing in its own price p_n and non-decreasing in all other prices since all choice probabilities $\pi_{mn'\to n}$ have this property. We can therefore be sure that D_n will be decreasing in its own price and increasing in all other prices as soon as this is the case for the choice probabilities $\pi_{mn'\to n}$ for one particular m and n'. This is what we have just proven.

In the same way D_0 will be increasing in all prices $p_1,\dots,p_N$ as soon as the choice probabilities $\pi_{mn'\to 0}$ are so for one particular m and n'. This was also proven.

It may therefore be concluded that the proposition is valid. Q.E.D.

When the demand functions D_n are differentiable we have :

$$\sum_{n'=0}^{N} \partial D_{n'}/\partial p_n = 0 \qquad (13)$$

$$n=1,\ldots,N$$

since the choice probabilities $\pi_{mn\to n'}$ have to add up to 1 for all m and n. Because of the weak gross substitutability this implies that the own price effect $\partial D_n/\partial p_n$ will be negative as soon as one of the cross effects $\partial D_{n'}/\partial p_n$, $n'\neq n$ is positive.

Strong gross substitutability of aggregate demand functions based on logit models for individual choice behaviour was used by Anas[1982] and by Anas and Cho[1986]. It seems not to have been realized thus far however that weak gross substitutability is an almost natural characteristic of aggregate demand functions based on discrete choice models.

The weak gross substitutability eases the proof of the existence of a price equilibrium , as will become clear in the next section.

4 Price Equilibrium.

A price equilibrium will now be defined as follows :

Definition. A price equilibrium is a set of nonnegative prices $\{p_1^*,\ldots,p_N^*\}$, such that $D_n \leq S_n$ for all $n=1,\ldots,N$ and $D_n=S_n$ when $p_n^* > 0$.

The remarkable fact , that will be proven below , is that a price equilibrium exists in the market described above under the rather weak assumptions listed at the beginning of section 2 with the only additional requirement that the aggregate demand functions are continuous.

Proposition 4.1. When the demand functions D_n , $n=0,\ldots,N$, are continuous , there exists a price equilibrium in the market described above.

Proof. We use a simple price adjustment mechanism. Suppose that all prices are initially equal to 0. Let $\dot{p}_n$ $(=dp_n/dt)$ equal δ $(\delta>0)$ when D_n exceeds S_n and equal to 0 otherwise. Because of the weak gross substitutability (proposition 3.1) we can be sure that for those goods for which demand initially exceeded supply it will always remain greater than or equal to supply as the price adjustment process runs. It is therefore clear that a price equilibrium has been reached when the process terminates. From the properties of the demand functions that were established in section 2 (proposition 2.2) we can be sure that the process stops since S_n is positive for all n=1,...,N and demand for all alternatives 1 to N' becomes arbitrary small when its price rises without an upper bound. Therefore a price equilibrium exists. Q.E.D.

An attractive feature of the proof given above is that it suggests a way to compute the price equilibrium : start with all prices equal to zero and increase the prices of those choice alternatives for which there is excess demand[6). This simple approach is possible because of the weak gross substitutability of the demand functions.

5 Uniqueness of Price Equilibrium.

It is useful to know whether a price equilibrium is unique , i.e. whether there is only one such equilibrium. When one believes that markets function in such a way that prices equilibrate supply and demand , uniqueness implies that the market equilibrium is fully determinate in the sense that there can only be one set of prices that correspond to equilibrium.

In the present section a sufficient condition for the uniqueness of a price equilibrium will be derived on the basis of the global univalence theorem of Gale and Nikaido[1965]. This theorem states that a function is (globally) one-to-one if the matrix of its first-order partial derivatives is (positive or negative) quasi-definite. The use of this mathematical tool makes it necessary to assume differentiability of the demand functions D_n. We will also assume strong gross substitutability in the aggregate , i.e. we require $\partial D_n/\partial p_{n'}$ to be negative for n'=n and positive otherwise.

Proposition 5.1 When the demand functions D_n , n=0,...,N ,are differentiable and the choice alternatives are strong gross substitutes , there exists a unique price equilibrium in the market described above.

Proof. The existence follows from proposition 4.1. To prove uniqueness we define C^* to be the set of indices with positive equilibrium prices p_n^*. Moreover let D_p^* be the matrix with elements $\partial D_n / \partial p_{n'}$ for $n,n' \in C^*$. The matrix D_p^* has a dominant diagonal (see McKenzie[1959]) i.e. :

$$|\partial D_n / \partial p_n| > |\Sigma_{n' \neq n} \partial D_{n'} / \partial p_n| \qquad n,n' \in C^* \qquad (14)$$

The validity of this inequality is guaranteed by the strong gross substitutability (see equation(14))[7]. Diagonal dominance implies that the matrix D_p^* is quasi-definite (see again McKenzie[1959] or note 13 of chapter 2). Quasi-definiteness implies global univalence by the Gale-Nikaido theorem (see Gale and Nikaido[1965] or Nikaido[1968] , chapter 7). And this means that if there exists a set of prices $\{p_n^* \mid n \in S^*\}$ for wich D_n equals S_n , it is unique. This ensures uniqueness of the set of positive equilibrium prices , given the fact that the other prices are equal to zero. When all prices are positive in equilibrium , this also completes the proof. When some prices are zero however we still have to show the uniqueness of the complete set of equilibrium prices $\{p_1^*,...,p_{N'}^*\}$. To show this consider the possibility that there exists another price equilibrium $\{\hat{p}_1^*,...,\hat{p}_{N'}^*\}$. In this alternative equilibrium at least one of the prices $\hat{p}_n^*$, $n \notin S^*$ has to be positive. So at least one price $\hat{p}_n^*$ has to be larger than p_n^*. Now let all prices $\hat{p}_n^*$ which exceed p_n^* decrease to p_n^*. Because of the strong gross substitutability the total demand for the alternatives whose prices remain unchanged during this operation and for those that have no price associated with them decreases , while the total demand for the goods whose prices have decreased increases. Then let the prices that remained unchanged thus far (if any) increase to p_n^*. It follows from the gross substitutability that during this second process the total demand for the alternatives whose price initially decreased , increases again. This means that for the set of new prices total demand for these alternatives is higher than in equilibrium. But the new prices are the equilibrium prices $\{p_1^*,...,p_N^*\}$. We have therefore reached a contradiction and must conclude that the price equilibrium is completely unique. Q.E.D.

It may be noted here that in case all prices are positive at equilibrium the assumption of strong gross substitutibility can be relaxed to one of negativeness of the 'own' partial derivatives $\partial D_n/\partial p_n$. Uniqueness of the price equilibrium can still be demonstrated in this case[8].

Proposition 5.1 gives a sufficient condition for uniqueness. It would be useful to have sufficient conditions for non-uniqueness to exist as well. This issue will be discussed in the next ection.

6 Non-Uniqueness of Price Equilibrium.

One obvious possibility for non-unique price equilibria to exist arises when the demand is not sensitive for changes in one price , say $p_{n'}$, in a particular range , given the values of all other prices , i.e. when $\partial D_n/\partial p_{n'}=0$ for all $n=0,\ldots,N$ and some $n'\in\{1,\ldots,N'\}$.

Proposition 6.1. Non-unique price equilibria may occur when the demand functions D_n , $n=0,1,\ldots,N$ are differentiable if there exists an interval $P_{n'}=[p^1_{n'}, p^2_{n'}]$, $(p^1_{n'}\neq p^2_{n'})$ for price $p_{n'}$ and values $\bar{p}_n$ for all other prices , such that $\partial D_{n'}/\partial p_{n'}=0$ whenever $p_{n'}\in P_{n'}$ and $p_n=\bar{p}_n$ for all other , $n=1,\ldots,N'$, $n\neq n'$.

Proof. Note first that the fact that $\partial D_{n'}/\partial p_{n'}=0$ implies that $\partial D_n/\partial p_n=0$ for all $n'=0,\ldots,N$ (this follows from the weak gross substitutability and equation (13)). Choose $p_{n'}\in P_{n'}$ and let $p_n=\bar{p}_n$ for all other $n=1,\ldots,N'$. Determine the associated values for the total demands D_n $(n=0,\ldots,N)$. Set S_n equal to D_n for all $n=1,\ldots,N$. Now let $p_{n'}$ change by a small amount $\Delta p_{n'}$, in such a way that $p_{n'}+\Delta p_{n'}$ remains within $P_{n'}$. Then the values of the demands D_n $(n=0,\ldots,N)$ do not change. We have therefore shown the existence of a non-unique price equilibrium. Q.E.D.

A second , less obvious , possibility for non-uniqueness to occur arises when a certain market segment functions independently of the rest of the market (at least for certain values of the prices) , i.e. whenever $\Sigma_{n \in C}\ \partial D_n/\partial p_{n'} = 0$ for all $n' \in C$, where C denotes the set of states which belong to the market segment. In this case changes of prices associated with the states belonging to the market prices do only influence allocation of the demand within that segment.

It is useful to introduce some additional notation at this point. The prices p_n , $n \in C$ can be incorporated in a vector that may be denoted by $\underline{p}_C$. The number of elements of this vector is equal to the number of elements of the subset C. A set of such vectors $\underline{p}_C$ will be denoted as P_C.

Proposition 6.2. Non-unique price equilibria may occur when the demand functions D_n , $n=0,1,\ldots,N$ are differentiable if there exists a subset C of alternatives for which $\Sigma_{n \in C} \partial D_n/\partial p_{n'} = 0$ for all $n' \in C$ and $\partial D_n/\partial p_{n''} = 0$ for all $n'' \notin C$ on a set P_C of prices $\underline{p}_C$ which has a nonempty interior and for some fixed values $\bar{p}_n$ for $n \notin C$

Proof. Set all prices p_n, equal to $\bar{p}_n$, for those $n \notin C$. Now consider the submarket consisting of the alternatives $n \in C$. On this submarket total demand $\Sigma_{n \in C} D_n$ has a fixed value D , independent of the prices p_n , $n \in C$, as long as these remain within the set P_C. Choose values for these prices p_n , $n \in C$, from the interior of P_C. Determine the associated demands D_n. First assume that all D_n's are positive. Then choose $S_n = D_n$ for all $n \in C$ and a price equilibrium will be established. Now increase one positive price $p_{n''}$, $n'' \in C$, by a small finite amount $\Delta p_{n''}$, in such a way that the new vector of prices remains within the set P_C. When the demand D_n does not change the non-uniqueness of the equilibrium prices has been established. When it decreases , demand for another type of dwelling n' ($n' \in C$) must have increased. Now start the algorithm used in the proof of proposition 4.1 with the new prices as starting point. Then a new vector of equilibrium prices will be established with all new prices greater than or equal to those at the starting point and some greater. When this new vector falls outside the set P_C the procedure can be repeated with a smaller value of Δp_n until a new vector of equilibrium prices is reached that is an element of P_C [9].

Next , assume that some demands D_n , $n \in C$, are zero for the level of prices chosen. Define C' to be the set of states for which demand is positive. Let $S_n = D_n$ for $n \in C'$ and choose arbitrary positive values S_n for $n \in C$, $n \notin C'$. Set the prices p_n, $n \notin C'$, equal to zero. Then we have found a price equilibrium. Now let one of the positive prices p_n, $n \in C'$, increase by a small amount Δp_n and use the same procedure as in the case when all demands D_n, $n \in C$ were positive and take care that at the new equilibrium prices the demands for the alternatives whose prices were set to zero do not exceed their supply volumes.

Finally, let one of the prices p_n for which demand was initially equal to zero increase by a small amount. If the demand for this alternative remains equal to zero we have again found non-uniqueness. Since this demand can only decrease or remain the same as a consequence of the rise in the price associated with it this has to be the case.

This demonstrates the non-uniqueness of the price equilibrium.
Q.E.D.

The situation considered in the last proposition is that of a segmented market. When the supply on this market segment is equal to the total demand on this segment and this demand is constant for all possible price vectors such a market segment, when considered on its own, is called a balanced market (see Anas[1982], Eriksson[1986] and Smith[1988]).

As is clear from the last two propositions, the violation of the assumption of strong gross substitutability on an open set of price vectors may result in non-uniqueness. Clearly something more than weak gross substitutability is needed for uniqueness, although strong gross substitutability may be too much.

On the basis of the results reached above one may conjecture that a necessary and sufficient condition for a unique price equilibrium when the demand functions are differentiable is the non-existence of a subset C of all states and a set set P_C of prices p_n, $n \in C$, with a non-empty interior, with $\Sigma_{n \in C} \partial D_n / \partial p_{n'} = 0$ for all $n' \in C$. This covers the case of proposition 6.1 as well as that of 6.2.

It should be noted that the conditions stated in these propositions show the possibility of non-uniqueness, but that this does not imply that non-uniqueness will occur in practice. In fact this may sometimes be less probable given the values of the supply volumes S_n in a given situation.

7 A Necessary and Sufficient Condition for Uniqueness.

In the present subsection we will prove the correctness of the conjecture mentioned at the end of the previous section , viz. that the existence of a set of states that function as an independent market segment for some set of prices is a necessary and sufficient condition for the possibility of non-uniqueness and therefore that the nonexistence of such a set is necessary and sufficient for uniqueness.

Proposition 7.1. When the demand functions are differentiable , there exists a unique price equilibrium in the model described above if and only if there does not exist a nonempty subset C of states such that $\Sigma_{n\in C}\partial D_n/\partial p_{n'}=0$ for all $n'\in C$ on a set P_C of prices p_n , $n\in C$, which has a non-empty interior , when all other prices have some fixed values $\bar{p}_n$, $n\notin C$.

Proof. Existence of a price equilibrium follows from proposition 4.1. We therefore concentrate our attention on the uniqueness.

The proposition says that non-uniqueness can occur if and only if there exists a set C of states and a set P_C of associated prices as described above , together with fixed values of the prices for other states , such that $\Sigma_{n\in C}\ \partial D_n/\partial p_{n'}=0$ for all $n'\in C$. We will prove this equivalent version of the proposition.

(i) Sufficiency. If there exists a set C with the required properties a non-unique equilibrium is possible by proposition 6.2.

(ii) Necessity. Suppose there are two vectors of equilibrium prices $\underline{p}^*$ and $\hat{\underline{p}}^*$. We assume , without loss of generality , that $\underline{p}^*$ has at least one of its elements greater than the corresponding element of $\hat{p}^*$. We now define three sets of states. A state n is an element of C_1 when p_n^* exceeds $\hat{p}_n^*$, an element of C_2 when both prices are equal to each other , and an element of C_3 when $\hat{p}_n^*$ exceeds p_n^*. Fix all prices at their equilibrium level $\underline{p}^*$. Now decrease the prices for the states $n\in C_1$ in the direction of $\hat{p}_n^*$. As a result of this operation total demand $D_{C_1}=\Sigma_{n\in C_1}D_n$ will increase or remain the same. Suppose it increases. Then let the price p_n , $n\in C_1$ drop further to the other equilibrium levels $\hat{p}_n^*$ and let the prices p_n , $n\in C_3$ increase to the other equilibrium levels $\hat{p}_n^*$. As a result of these subsequent changes D_{C_1} will not decrease , but may increase further. We can therefore be sure that D_{C_1} will be higher at the equilibrium prices $\hat{\underline{p}}^*$ than it was at the equilibrium prices $\underline{p}^*$. This gives rise to a contradiction however , since D_{C_1} was , at the equilibrium $\underline{p}^*$ equal to $\Sigma_{n\in C_1} S_n$. ($D_n(\underline{p}^*)<S_n$ is excluded for all $n\in C_1$ by the fact that for those n $p_n^* > \hat{p}_n^* \geq 0$.) We must therefore conclude that D_{C_1} remains equal to $\Sigma_{n\in C_1} D_n(p_n^*)$ for all prices p_n , $n\in C_1$, such that $p_n^* \geq p_n \geq \hat{p}_n^*$. Call this set of prices P_{C_1}. We have thus found that C_1 functions as an independent market segment (i.e. $\Sigma_{n\in C_1} \partial D_n/\partial p_{n'} = 0$ for all $n' \in C_1$) when the prices associated with its states are in the set P_{C_1} and the other prices are fixed at the values p_n^*. It is clear moreover that the set P_{C_1} has a non-empty interior.

We thus have to conclude that the existence of a subset C with the properties mentioned in the proposition is necessary and sufficient for the possibility of non-uniqueness. It follows that the non-existence of such a set is necessary and sufficient for uniqueness. Q.E.D.

8 Comparison with other Results.

Anas[1982] has probably been the first one who studied the existence of price equilibria in markets where there are M classes of actors who all have to choose among a finite number of alternatives and take their decisions on the basis of discrete choice theory (in his case more specifically the multinomial logit model). He also introduced the methodology of concentrating the attention on the deterministic version of the model , which is valid only when the number of market participants is large. The equilibrium of this system was named a stochastic market equilibrium by Anas , because of the relation of the model with random utility theory. It should be clear however that the equilibrium whose existence is proven is completely deterministic and that the relation of this equilibrium with the stochastic model needs further discussion (see the next section). The term stochastic market equilibrium has become more or less common in the literature to denote market clearing situations in models where demand is determined on the basis of discrete choice models and supply is fixed.

Anas[1982] considered the situation in which the total number of dwellings is exactly equal to that of the actors participating in the market and where there is no exit or entry of participants. A market for which these conditions are fulfilled is called a balanced market. As a consequence of the assumption of balancedness the price equilibrium is not unique. The value of one price (say p_1) can be fixed in advance.

The assumption of a balanced market is not a very realistic one. Consider e.g. the housing market. When the prices of dwellings become very high , the formation of new households may be postponed ; when they are very low a larger number of new households may be formed. Also migration may be expected to be influenced to some extent by the prices of dwellings in the region under consideration. This will especially be the case when commuting to and from other regions is possible.

Notwithstanding this lack of realism , the balancedness assumption is maintained in two other contributions to the literature on stochastic market equilibria , viz. Eriksson[1986] and Smith[1988]. Eriksson[1986] studies the class of generalized extreme value models but does not consider price equilibria , but equilibria in terms of the values of the systematic utilities. In his model all individuals attach the same systematic utility value to all alternatives. Differences between individuals are differences in the simultaneous probability density functions of the random terms. The assumption of equal values for the systematic utilities for all individual actors should be considered as unrealistic. Eriksson reaches existence results as well as a necessary and sufficient condition for uniqueness of the equilibrium values of the systematic utilities up to additions of the same scalar value. This condition states , loosely speaking , that the balanced market should not have a market segment C , as in proposition 6.2 , that functions independently for some set of prices.

Smith[1988] considers the class of balanced markets in general and the case where individual choices are modeled by discrete choice models as one possibility. For this special case he restricts his attention to specifications of the systematic utility functions that are scale- or translation-invariant , which means that multiplication of all prices with the same positive scalar resp. addition of the same scalar to all prices does not influence the values of the differences $U_{mn'\to n} - U_{mn'\to n'}$,. In his model the systematic parts of the utilities attached to each alternative may be functions of all prices[10).

Anas and Cho[1986] dropped the assumption of balancedness , and proved existence and uniqueness of equilibrium in a model where exit is possible and where the probabilistic choice functions are given by the logit model. Their proof of uniqueness uses the Gale-Nikaido theorem. Proposition 5.1 above is closely analogous to their uniqueness result.

The necessary and sufficient condition for uniqueness stated in proposition 7.1 has not been not mentioned in the literature for models as general as the ones studied in the present paper. It is however closely related to the necessary and sufficient condition Eriksson[1986] developed for a much more restricted type of market.

9 Interpretation of the price equilibrium.

There are (at least) two possible interpretations of the price equilibria discussed in the previous section. They can be considered as the outcome of a market process. They can also be interpreted as the optimal values (in some sense) for the government to set the prices.

The former interpretation of price equilibria is common in the economic literature , although not without difficulties. The question of who changes the prices when they do not equilibrate the market , turns out to be difficult to answer in a way that is consistent with price-taking behaviour of participants. Notwithstanding that , it may be said that the main reason for economists to study the existence and other properties of price equilibria is the idea that markets do function in reality in such a way that demand may approximately equal supply.

Market clearing in the stochastic version of the model requires in principle different prices for different periods (given the same initial distribution of the actors over the various states) , as a result of differences in the realizations of the random terms. When the same original situation of the market could be realized more than once , the market clearing prices will nevertheless most likely be somewhat different , although less so when the number of market participants is large.

The second interpretation of price equilibria is less common in economics , but is mentioned repeatedly in the literature on stochastic price equilibria. Anas[1982] tries to motivate the concept by showing that at the stochastic price equilibrium the probability of market clearing is maximal [11]. Eriksson[1986] shows that at the stochastic price equilibrium the expected value of the unsatisfied demand is minimal. Clearly the idea behind these statements is that there are good reasons for a governmental agency that controls the housing market to set the prices at their stochastic equilibrium values.

Nevertheless the possibility exists that when this rule is adopted the market - in the words of Eriksson[1986] - fails to clear miserably [12]. This would force the government to use some second-round-allocation procedures , which are considered briefly in Anas[1982]. Moreover , when the random terms in the utilities attached to the various alternatives are serially correlated this price-setting rule may give rise to persistent excess-demands and/or excess supplies in some segments of the market.

The difference between the two interpretations may also be illustrated by again considering the possibility of observing the same situation of the market more than once. Now the same prices would be set , but the demands for the various alternatives would differ from period to period.

It should be clear however that , in case there is no serial correlation in the random terms the differences between the two interpretations disappear when the number of market participants becomes large.

Notes.

1)E.g. when all $\varepsilon_{mn'\to n''}$ are assumed to be independent and identically extreme value distributed the logit model , $\pi_{mn'\to n} = \exp(U_{mn'\to n})/\Sigma_{n''}\exp(U_{mn'\to n''})$ results.
2)See the many applications of the logit model , e.g. in Domencich and McFadden[1974] and Anas[1982].
3)Sometimes it may be useful however to have more than one price as an argument of the utility function. E.g. when the model refers to a housing market where dwellings are owner-occupied the difference between the price of the old dwelling , $p_{n'}$, and that of the new one , p_n , may be taken as an argument of the utility function $U_{mn'\to n}$.
4)Consider e.g. the case in which the utility of choice alternative n has to exceed a treshold value before it will ever be chosen. If the utiltiy of this choice alternative is below that treshold value its probability of being chosen is insensitive to (small) changes in prices which contradicts strong gross substitutability.
5)Almost everywhere is a technical term meaning : 'everywhere , possibly except on a set of measure zero'.
6)The algorithm starts with all prices equal to zero and increases the prices of the choice alternatives for which there is excess demand. One should be careful however not to increase prices so much that the excess demand turns into excess supply.
7)It should be noted that $0 \notin C^*$ and that $\partial D_0/\partial p_n > 0$.
8)This follows from the fact that every matrix with a quasi-dominant diagonal and its diagonal terms non-zero has a dominant diagonal. See McKenzie[1959].
9)This is possible by the continuity of the aggregate demand functions.
10)This covers the case mentioned in note 3 , while incorporation of the price difference $p_n - p_{n'}$ as an argument of the utiltiy function guarantees translation invariance.
11)He shows this to be the case when there is only one class of actors. When there are more classes he claims to have established that this is no longer the case. However Eriksson[1986] points out that this claim is false. This leaves the issue open for the case of more than one class of actors.
12)Eriksson[1986] p. 553.

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Appendix. Continuity and Differentiability of the Aggregate Demand Equations.

In this appendix propositions 1.1 and 1.2 will be proven. We will do this by showing that the propositions hold for the choice probability functions $\pi_{mn'\to n}$. Since the aggregate demand functions are sums of these choice probability functions the proof for the aggregate demand functions follows at once.

The cumulative density function $H_{mn'}(\underline{\varepsilon}_{mn'})$ mentioned in the propositions 1.1. and 1.2 is defined as follows :

$$H_{mn'}(\underline{\varepsilon}_{mn'}) = \int_{-\infty}^{\varepsilon_{mn'\to N}} \cdots\cdots \int_{-\infty}^{\varepsilon_{mn'\to 1}} h_{mn'}(\underline{\varepsilon}_{mn'})\, d\varepsilon_{mn'\to 1} \cdots \;\cdots d\varepsilon_{mn'\to N} \qquad (A1)$$

$$m=1,\ldots,M \;;\; n',n=0,1,\ldots,N$$

The choice probabilities $\pi_{mn'\to n}$ can be written as :

$$\pi_{mn'\to n} = \int_{-\infty}^{\infty} \frac{\partial}{\partial \varepsilon_n} H_{mn'}(w_{mn'\to 1},\ldots,w_{mn'\to n-1},\varepsilon_n,w_{mn'\to n+1},\ldots,w_{mn'\to N})\,d\varepsilon_n \qquad (A2)$$

$$m=1,\ldots,M \;,\; n,n'=0,1,\ldots,N$$

where the variables $w_{mn'\to n''}$ are defined in the text. Equation (A2) is equivalent to (3). We will denote the transpose of the vector $[w_{mn'\to 1},\ldots,w_{mn'\to n-1},\varepsilon_n,w_{mn'\to n+1},\ldots,w_{mn'\to N}]$ as $\underline{w}_{mn'}$.

When the indirect utility functions $U_{mn'\to n}$ are continuous in their arguments $\underline{x}_n$, the variables $w_{mn'\to n''}$ will be continuous in $\underline{x}_{n''}$ and $\underline{x}_n$. The change that occurs in $\pi_{mn'\to n}$ as a consequence of a change in one of the explanatory variables $x_{n'',i}$ will be denoted as $\Delta\pi_{mn'\to n}$ and can be determined as being equal to :

$$\Delta\pi_{mn'\to n} = \int_{-\infty}^{\infty} \Big[\frac{\partial}{\partial \varepsilon_{mn'\to n}} H_{mn'}(\underline{w}_{mn'}+\Delta\underline{w}_{mn'}) - - \frac{\partial}{\partial \varepsilon_{mn'\to n}} H_{mn'}(\underline{w}_{mn'}) \Big] d\varepsilon_{mn'\to n} \qquad (A2)$$

$$m=1,\ldots,M \;,\; n',n=0,1,\ldots,N$$

where $\Delta\underline{w}_{mn'}$ denotes the change in $\underline{w}_{mn'}$ that occurs as a consequence of the change in $x_{n'',i}$. It is now easy to see that the continuity of the variables $w_{mn'\to n''}$ and of the partial derivative $\partial H_{mn'}/\partial\varepsilon_n$

guarantees continuity of $\pi_{mn'\to n}$ in all variables $\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_N$. This proves that continuity of the indirect utilities and continuous differentiability of the density function is sufficient for continuity of the choice probabilities. It is easy to see that these conditions are also necessary by considering what would happen when either the variables $w_{mn'\to n''}$ or the partial derivative $\partial H_{mn'}/\partial \varepsilon_n$ were not differentiable. This proves proposition 1.1.

In order to prove proposition 1.2 we first observe that it is sufficient to prove differentiability of the choice probabilities in the indirect utilities $u_{mn'\to n''}$ since we have , by the chain rule :

$$\partial \pi_{mn'\to n}/\partial x_{n''i} = (\partial \pi_{mn'\to n}/\partial u_{mn'\to n''}).(\partial u_{mn'\to n''}/\partial x_{n''i}) \qquad (A3)$$
$$m=1,\ldots,M \ , \ n,n'=0,1,\ldots,N$$

and the indirect utilities are , by assumption , differentiable in their arguments.

Now consider the consequences of a small change in one of the indirect utilties $U_{mn'\to n''}$, $n'' \neq n$. The vector $\Delta \underline{w}_{mn'}$ then has as its arguments only zero's , except for the n''-th position , where the element equals $-\Delta U_{mn'\to n''}$.

The ratio of the change in $\pi_{mn'\to n}$ that occurs as a consequence of the change in $U_{mn'\to n}$ can be determined as being equal to :

$$\Delta \pi_{mn'\to n}/\Delta U_{mn'\to n''} = \int_{-\infty}^{\infty} \{ [\frac{\partial}{\partial \varepsilon_{mn'\to n}} H_{mn'}(\underline{w}_{mn'} + \Delta \underline{w}_{mn'}) - $$
$$- \frac{\partial}{\partial \varepsilon_{mn'\to n}} H_{mn'}(\underline{w}_{mn'})] / \Delta U_{mn'\to n''} \} \, d\varepsilon_{mn'\to n} \qquad (A4)$$
$$m=1,\ldots,M \ , \ n,n'=0,1,\ldots,N$$

with $\Delta \underline{w}_{mn'}$ as discussed above. Taking the limit for $\Delta U_{mn'\to n} \to 0$, the right hand side of (A4) becomes equal to :

$$- \int_{-\infty}^{\infty} \frac{\partial^2}{\partial \varepsilon_{mn'\to n} \partial \varepsilon_{mn'\to n''}} . H_{mn'}(\underline{w}_{mn'}) . d\varepsilon_{mn'\to n} \qquad (A5)$$

When this integral exists it is equal to $\partial \pi_n/\partial U_{mn'\to n''}$. We know , by assumption , that the second order partial derivative in A5 exists everywhere and is continuous. Since it is the second-order derivative of a well-behaved we can also be sure that the integral (A5) exists.

Since $\partial \pi_{mn'\to n}/\partial U_{mn'\to n}$ is equal to $-\Sigma_{n''\neq n} \partial \pi_{mn'\to n}/\partial U_{mn'\to n''}$ we can be sure that this partial derivative will also exist. We have therefore shown that differentiability of the indirect utility functions and twice differentiability of the density function guarantees differentiability of the aggregate demand functions.

By considering what would happen when one (or both) of these conditions are not fulfilled it is easy to see that they are also necessary. This completes the proof of proposition 1.2.

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