SERIE RESEARCH MEMORANDA

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HETEROSKEDASTIC AND AUTOCORRELATED
COVARIANCE MATRICES

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A M S T E R D A M

A Note on Consistent Estimation of Heteroskedastic and Autocorrelated Covariance Matrices

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Recently Newey and West (1987) propose estimators for the variance covariance matrix

$$S_T = T^{-1}E[\Sigma_{t=1}^{T}\Sigma_{s=1}^{T}h_t(\theta*)h_s'(\theta*)]$$

where

$$E[h_t(\theta*)] = 0.$$

If $\hat{\theta}$ solves

$$T^{-1}\Sigma_{t=1}^{T}h_t(\hat{\theta}) = 0$$

and $\hat{\Omega}_j$ is defined by

$$\hat{\Omega}_j = T^{-1}\Sigma_{t=j+1}^{T}h_t(\hat{\theta})h_{t-j}'(\hat{\theta})$$

then they show in their theorem 2 that the weighted estimator

$$\hat{S}_T = \hat{\Omega}_0 + \Sigma_{j=1}^{m(T)}w(j,m(T))(\hat{\Omega}_j + \hat{\Omega}_j')$$

converges in probability to S_T as long as m(T) growths to infinity more slowly than $T^{1/4}$. In this note we show that under the conditions of that theorem the growth rate of m(T) can be increased to $o(T^{1/2})$.

Theorem: If the conditions of theorem 2 of Newey and West (1987) are satisfied, except that m(T) is now chosen such that

$$\lim_{T\to\infty} T^{-1/2}m(T) = 0,$$

then

$$\hat{S}_T - S_T \xrightarrow{p} 0.$$

Proof: In the proof we will use the notation of Newey and West (1987). Throughout the proof $M_1, M_2, \ldots$ are finite constants. The theorem will only be proved for α mixing, as for φ mixing the proof is similar. Our proof is nearly identical to that of Newey and West (1987), except that we will sharpen their inequalities (10) and (11). These two inequalities are needed for the second term of their inequality (9). If we define

$$Z_{tj} = h_t h_{t-j} - E[h_t h_{t-j}]$$

then this second term can be written as

$$T^{-1}\Sigma_{t=1}^{T}(h_t^2 - E[h_t^2]) + 2\ T^{-1}\Sigma_{j=1}^{m}w(j,m)\ \Sigma_{t=j+1}^{T}(h_t h_{t-j} - E[h_t h_{t-j}])$$

$$= T^{-1}\Sigma_{t=1}^{T} Z_{t0} + 2\ T^{-1}\Sigma_{j=1}^{m} w(j,m)\ \Sigma_{t=j+1}^{T} Z_{tj}.$$

From the Liapounov inequality follows that

$$(1)\qquad E\left|T^{-1}\Sigma_{t=1}^{T} Z_{t0} + 2\ T^{-1}\Sigma_{j=1}^{m} w(j,m)\ \Sigma_{t=j+1}^{T} Z_{tj}\right|$$

$$\le T^{-1}E\left|\Sigma_{t=1}^{T} Z_{t0}\right| + 2\ T^{-1}\Sigma_{j=1}^{m}\left|w(j,m)\right|\ E\left|\Sigma_{t=j+1}^{T} Z_{tj}\right|$$

$$\le M_1 T^{-1}\Sigma_{j=0}^{m} E\left|\Sigma_{t=j+1}^{T} Z_{tj}\right| \le M_1 T^{-1}\Sigma_{j=0}^{m}[E(\Sigma_{t=j+1}^{T} Z_{tj})^2]^{1/2}.$$

For fixed j application of the triangle inequality yields

$$(2)\qquad E(\Sigma_{t=j+1}^{T} Z_{tj})^2 \le \Sigma_{t=j+1}^{T} E[Z_{tj}^2] + 2\ \Sigma_{s=1}^{j}\Sigma_{t=s+1+j}^{T}\left|E[Z_{tj}Z_{t-s,j}]\right|$$

$$+ 2\ \Sigma_{s=j+1}^{T-j-1}\ \Sigma_{t=s+1+j}^{T}\left|E[Z_{tj}Z_{t-s,j}]\right|.$$

Now consider the three sums on the right hand side of (2). First observe that

$$(3)\qquad \Sigma_{t=j+1}^{T} E[Z_{tj}^2] \le M_2 T.$$

If we define

$$x_t = h_t h_{t-j}$$

then according to lemma 6.18 of White (1984)

$$(4)\qquad \alpha_x(k) \le \begin{cases} 1 & k\le j \\ \alpha(k-j) & k\ge j. \end{cases}$$

For the terms of (2) with $s\ge j+1$ we deduce from (4) and corrollary 6.16 of White (1984) that there exists a constant M_3 such that

$$\left|E[(h_t h_{t-j} - E[h_t h_{t-j}])(h_{t-s}h_{t-s-j} - E[h_{t-s}h_{t-s-j}])]\right|$$

$$\le M_3[\alpha_x(s)]^\eta \le M_3[\alpha(s-j)]^\eta$$

where

$$\eta = (r+\delta-1)/2(r+\delta),$$

hence

$$(5)\qquad \Sigma_{s=j+1}^{T-j-1}\ \Sigma_{t=s+1+j}^{T}\left|E[Z_{tj}Z_{t-s,j}]\right|$$

$$\le \Sigma_{s=j+1}^{T-j-1}\ \Sigma_{t=s+1+j}^{T} M_3[\alpha(s-j)]^\eta \le \Sigma_{s=j+1}^{T-j-1} M_3 T[\alpha(s-j)]^\eta \le M_4 T.$$

For the remaining terms of (2) with $1\le s\le j$ rearranging and application of the triangle inequality yields

$$\left|E[(h_t h_{t-j} - E[h_t h_{t-j}])(h_{t-s}h_{t-s-j} - E[h_{t-s}h_{t-s-j}])]\right|$$

$$= \left|E[(h_t h_{t-s} - E[h_t h_{t-s}])(h_{t-j} h_{t-s-j} - E[h_{t-j} h_{t-s-j}])] + E[h_t h_{t-s}]\, E[h_{t-j} h_{t-s-j}] - E[h_t h_{t-j}]\, E[h_{t-s} h_{t-s-j}]\right|$$

$$\leq \left|E[(h_t h_{t-s} - E[h_t h_{t-s}])(h_{t-j} h_{t-s-j} - E[h_{t-j} h_{t-s-j}])]\right| + \left|E[h_t h_{t-s}]\right| \left|E[h_{t-j} h_{t-s-j}]\right| + \left|E[h_t h_{t-j}]\right| \left|E[h_{t-s} h_{t-s-j}]\right|.$$

Similar as above we conclude from (4) and corollary 6.16 of White (1984) that

$$\left|E[(h_t h_{t-s} - E[h_t h_{t-s}])(h_{t-j} h_{t-s-j} - E[h_{t-j} h_{t-s-j}])]\right| \leq M_3 [\alpha_x(j)]^\eta \leq M_3 [\alpha(j-s)]^\eta.$$

Since for all t

$$E[h_t] = 0$$

it follows along the same lines of reasoning that

$$\left|E[h_t h_{t-s}]\right| \left|E[h_{t-j} h_{t-s-j}]\right| \leq (M_5 [\alpha(s)]^{\eta/2})^2 = M_6 [\alpha(s)]^\eta,$$

and

$$\left|E[h_t h_{t-j}]\right| \left|E[h_{t-s} h_{t-s-j}]\right| \leq (M_5 [\alpha(j)]^{\eta/2})^2 = M_6 [\alpha(j)]^\eta.$$

Because for $j \geq s$

$$\alpha(j) \leq \alpha(s)$$

it follows that

(6) $$\Sigma_{s=1}^{j} \Sigma_{t=s+1+j}^{T} \left|E[Z_{tj} Z_{t-s,j}]\right| \leq \Sigma_{s=1}^{j} \Sigma_{t=s+1+j}^{T} (M_3 [\alpha(j-s)]^\eta + M_6 [\alpha(s)]^\eta + M_6 [\alpha(j)]^\eta)$$
$$\leq \Sigma_{s=1}^{j} M_7 (T-s-j) \{[\alpha(j-s)]^\eta + [\alpha(s)]^\eta + [\alpha(j)]^\eta\} \leq 3\ M_7 T\ \Sigma_{s=0}^{j} [\alpha(s)]^\eta = M_8 T.$$

Substitution of (3),(5) and (6) into (2) yields that there exists a constant M_9 such that for fixed j

(7) $$E\{\Sigma_{t=j+1}^{T} Z_{tj}^2\}^2 \leq M_9 T$$

which sharpes inequality (10) of Newey and West (1987). Finally from substitution of (7) into (1) we conclude that

$$E\left|T^{-1} \Sigma_{t=1}^{T} Z_{t0} + 2\ T^{-1} \Sigma_{j=1}^{m} w(j,m)\ \Sigma_{t=j+1}^{T} Z_{tj}\right| \leq M_1 T^{-1} \Sigma_{j=0}^{m} [M_9 T]^{1/2} = M_{10} (m+1) T^{-1/2}$$

which converges to zero if $m = o(T^{1/2})$. The proof is completed by observing that the remaining three terms of inequality (9) of Newey and West (1987) still converges to zero if m growths to infinity more slowly than $T^{1/2}$.

For testing unit roots Phillips (1987) needed a statistic which is similar to S_T. Along the same lines as above it can be shown that results of theorem 4.2 of Phillips (1987) still hold true if also in that case the growth rate is increased from $o(T^{1/4})$ to $o(T^{1/2})$.

References

Newey,W.K. and K.D.West, (1987), "A Simple, Positive Semi-Definite, Heteroskedasticity and Autocorrelated Consistent Covariance Matrix", Econometrica, 55, 703-708.

Phillips, P.C.B., (1987), "Time Series Regression with a Unit Root", Econometrica, 55, 277-301.

White,H. (1984), Asymptotic Theory for Econometricians, New York, Academic Press.