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PRODUCT FORMS FOR QUEUEING NETWORKS
WITH LIMITED CLUSTERS

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Product Forms for Queueing Networks with Limited Clusters

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Queueing networks are considered with limited station clusters. A general characterization is given, in terms of local solutions of state dependent traffic equations, from which the existence and structure of a product form can be concluded. This characterization leads to new product form examples such as with a non-reversible routing both within and in between limited clusters.

Cluster, blocking, product form, invariance condition, non-reversible routing.

1. Introduction

Over the last decades queueing network analysis has gained popularity in telecommunication, computer performance evaluation and flexible manufacturing. A vast majority of the associated literature has been concerned with the validity of explicit product type expressions, related to Jackson's celebrated product form, when realistic phenomena such as blocking, synchronous serving, resource sharing or prioritizations are involved. Particularizing to systems with blocking, product forms have so been obtained under the strong condition of a reversible routing (cf. [1], [3], [4], [7], [8], [9], [10], [11], [14], [15], [16]), and the assumption of capacity limitations or storage constraints imposed upon only individual stations. Though some extensions to non-reversible routing situations have been obtained (cf. [4], [5]), from a practical point of view the reversible routing can be regarded as almost necessary for the case of limited individual stations.

In practice, however, one often encounters capacity constraints upon clusters of stations rather than upon individual stations. In telecommunication, for example, messages from different sources may have to contend for available trunks of a limited trunk group. In computer networks, pooling of store and forward buffers frequently arises. In manufacturing, a number of workstations along an assembly line may have to share a common storage pool.

For such systems one cannot adopt the above mentioned results as special station interdependences need to be incorporated. Nor can aggregation results (cf. [13]) be applied as detailed states are to be distinguished to justify a Markovian analysis. This paper, therefore, will be concerned with networks of queues with constraints upon station clusters such as clusters with limited capacities. The main results are:

1. A concrete condition, in terms of local solutions of traffic equations, from which the existence and structure of a product form expression can be concluded.
2. A number of novel product form examples with limited clusters. Particularly, examples with a non-reversible routing within and in between clusters are included.

To highlight the essential features, the presentation will first be restricted to closed exponential systems with one type of customer. Extensions to open, non-exponential and multi-class networks are briefly discussed afterwards.

The model is described in section 2 while the conditions and product form result are given in section 3. A scala of examples is presented in section 4 and the above extensions are briefly discussed in section 5. An evaluation concludes the paper.

2. Model

Consider a closed network with N service stations and M jobs. A job at station i requires a random amount of service with mean μ_i^{-1}, i=1,...,N. Throughout we will use the notation

$$\bar{n} = (n_1, n_2, \ldots, n_N) \quad \text{with } n_1 + n_2 + \ldots + n_N = M$$
$$\bar{m} = (m_1, m_2, \ldots, m_N) \quad \text{with } m_1 + m_2 + \ldots + m_N = M-1$$

and we will consider states of the form:

$$\bar{n} = \bar{m} + e_j = (m_1, m_2, \ldots, m_{j-1}, m_j+1, m_{j+1}, \ldots, m_N)$$

to indicate the state in which $\bar{n}$ constitutes the population vector $\bar{m}$ with one additional job at station j, j=1,...,N. For a vector $\bar{n} = (n_1, \ldots, n_N)$, let $\bar{n}+e_i-e_j$ denote the same vector with one job moved from station i to j. Also $m+e_j$ is the vector $\bar{m}$ with one job more at station j and $\bar{n}-e_j$ is the vector $\bar{n}$ with one job less at station j.

Servicing. Service requirements are assumed to be independent and exponential with parameter μ_i at station i, i=1,...,N. For arbitrary strictly positive functions $\chi(\bar{n})$ and $\emptyset(\bar{n})$ the service speed of station j when the system is in state $\bar{n}=\bar{m}+e_j$ is given by

$$f_j(\bar{m}+e_j) = \chi(\bar{m})/\emptyset(\bar{m}+e_j) \tag{2.1}$$

Routing. Assume that the N stations are partitioned in disjoint clusters, say clusters C(1),...,C(P) numbered 1,...,P. We introduce the notation:

$$\bar{s} = (s_1,\dots,s_P) \quad \text{with } s_1 + \dots + s_P = M$$
$$\bar{t} = (t_1,\dots,t_P) \quad \text{with } t_1 + \dots + t_P = M-1$$

and consider cluster configurations of the form:

$$\bar{s} = \bar{t} + e_p = (t_1,\dots,t_{p-1},t_p+1,t_{p+1},\dots,t_P)$$

to indicate that $s_j=t_j$ jobs are currently within cluster $j\neq p$ and $s_p=t_p+1$ within cluster p. The vectors $\bar{s}+e_q-e_p$, $\bar{t}+e_p$ and $\bar{s}-e_p$ are defined similarly as for $\bar{s}-\bar{n}$ and $\bar{t}-\bar{m}$. Let $B_{pq}(\bar{t})$ for all possible p,q and vectors $\bar{t}$ be a given value between 0 and 1 (0 and 1 included) with $B_{pp}(.)=1$ and denote by c(i) the number of the cluster which contains station i.

The routing protocol can now be described as follows. Upon completion of a service at station i a job requests to route to station j with probability p_{ij}. When the other jobs constitute cluster vector $\bar{t}$, that is t_ℓ from the other jobs are present in cluster ℓ for $\ell=1,\dots,P$, this request is accepted with probability

$$B_{c(i)c(j)}(\bar{t}).$$

When the routing request is rejected, the job has to remain at station i where it has to undergo a new service. Let us give two examples of the blocking function B. Herein as well as in the sequel the symbol 1(A) denotes the indicator of an event A, i.e. 1(A)=1 if event A is satisfied and 1(A)=0 otherwise.

Example 2.1. (Independent departure/arrival blocking). Frequently a separate independent blocking mechanism for leaving and for entering a cluster is involved. This is reflected by

$$B_{pq}(\bar{t}) = D_p(\bar{t})A_q(\bar{t}), \tag{2.2}$$

where $D_p(.)$ represents a departure blocking or delay factor for leaving cluster p, while $A_q(.)$ represents an arrival blocking for entering cluster q. Particularly, these departure and arrival blockings may depend upon only the individual cluster size, in which case for certain functions $d_p(.)$ and $a_q(.)$:

$$D_p(\bar{t}) = d_p(t_p) \tag{2.3}$$
$$A_q(\bar{t}) = a_q(t_q).$$

As a special example, a capacity limit of no more than N_q jobs at cluster q is modeled by

$$B_{pq}(\bar{t}) = 1(t_q < N_q). \tag{2.4}$$

Example 2.2. (Resource sharing between clusters). Beyond capacity limitations within clusters, such as due to storage constraints, different clusters themselves may have to share some restricted resource such as a storage pool, central processor or transmission system. For example, consider a three cluster system with capacity limitations N_i for cluster $i=1,2,3$ and an addition commonly shared pool of size K for clusters 2 and 3. Then this is modeled by

$$B_{ij}(\bar{t}) = 1(\bar{t}+e_j \in C)$$

with

$$C = \{\bar{s} \mid s_i \le N_i,\ i=1,2,3,\ s_2+s_3 \le K\}. \tag{2.5}$$

3. Product form characterization

Assume that there exists a unique stationary distributoin $\pi(.)$ for the set S of admissible states. To verify that $\pi(\bar{n})$ has a particular form it suffices to verify the global balance (or forward Kolmogorov) equations. These in turn are guaranteed by the more detailed "station balance equations" stating that for any station j separately and any admissible state:

"the rate out of that state due to a departure at station j =
the rate into that state due to an arrival at station j." (3.1)

To formalize (3.1), consider a fixed station j and a fixed state $\bar{n}$ say with cluster vector $\bar{s}$. Let $\bar{m}=\bar{n}-e_j$ and $\bar{t}=\bar{s}-e_p$ with $p=c(j)$. Then (3.1) requires that

$$\pi(\bar{m}+e_j)\mu_j[\chi(\bar{m})/\varnothing(\bar{m}+e_j)] \ \Sigma_i \ p_{ji}B_{c(j)c(i)}(\bar{t}) =$$

$$\Sigma_i \ \pi(\bar{m}+e_i)\mu_i[\chi(\bar{m})/\varnothing(\bar{m}+e_i)] \ p_{ij}B_{c(i)c(j)}(\bar{t}). \qquad (3.2)$$

Here it is noted that a blocked routing request is not included as it would actually contribute equally to both sides. Conversely, by checking (3.2) for all possible $\bar{m}$ and j such that $\bar{n}=\bar{m}+e_j$ is admissible global balance and thus stationarity is proven.

To this end, first consider the local version of (3.1) for a fixed $\bar{m}$. Without restriction of generality (see remark 3.7 below), assume that this local equation has a unique probability solution $\{y(j|\bar{m})|\bar{m}+e_j\in S\}$. That is, for any j=1,...,N we have:

$$y(j|\bar{m}) \ \Sigma_i \ p_{ji}B_{c(j)c(i)}(\bar{t}) = \Sigma_i \ y(i|\bar{m})p_{ij}B_{c(i)c(j)}(\bar{t}). \qquad (3.3)$$

Consider the local solutions $\{y(\ell|\bar{m})\}$ at $S(\bar{m})=\{\bar{m}+e_q|\bar{m}+e_q\in S\}$ for all possible $\bar{m}$. Then we can define an artificial Markov chain with transition rates $\bar{q}(\bar{m}+e_i\to\bar{m}+e_j)$ for a change from state $\bar{m}+e_i\in S(\bar{m})$ into $\bar{m}+e_j\in S(\bar{m})$ such that

$$\frac{\bar{q}(\bar{m}+e_j \ \to \ \bar{m}+e_i)}{\bar{q}(\bar{m}+e_i \ \to \ \bar{m}+e_j)} = \frac{y(i|\bar{m})}{y(j|\bar{m})} \qquad (3.4)$$

while transition rates not of this form are equal to 0. First observe that this Markov chain is restricted to the same state space S. Also note that its transition rates $\bar{q}$ are unique up to a constant factor at $S(\bar{m})$ for any possible $\bar{m}$. Consider such a Markov chain with $\bar{q}$ fixed and denote its stationary distribution by $\Psi(.)$. Theorem 3.1 below will relate this artifial $\bar{q}$-model with the possibility of a product form expression for the original model. First, we need a definition in correspondence with the literature (cf. [8]).

Definition. The $\bar{q}$-model is reversible if for all $\bar{m}$ and $\bar{m}+e_i,\bar{m}+e_j \in S(\bar{m})$:

$$\frac{\Psi(\bar{m}+e_i)}{\Psi(\bar{m}+e_j)} = \frac{y(i|\bar{m})}{y(j|\bar{m})} , \qquad (3.5)$$

Theorem 3.1. (Characterization of product form). The stationary distribution $\pi(.)$ satisfies (3.2) for all $\bar{m}+e_j \in S$ if and only if the $\bar{q}$-model is reversible. Further, for all $\bar{n} \in S$ and with c a normalizing constant it then holds that

$$\pi(\bar{n}) = c\ \Psi(\bar{n})\ \emptyset(\bar{n}) \sum_{i=1}^{N} (1/\mu_i)^{n_i} \tag{3.6}$$

Proof. Assume that the $\bar{q}$-model is reversible and consider the distribution $\pi(.)$ defined by (3.6). Then from (3.6) and (3.5):

$$\frac{\pi(\bar{m}+e_i)}{\pi(\bar{m}+e_j)} = \frac{y(i|\bar{m})\ \emptyset(\bar{m}+e_i)\ \mu_j}{y(j|\bar{m})\ \emptyset(\bar{m}+e_j)\ \mu_i} . \tag{3.7}$$

By substituting (3.7) in (3.2), after dividing the left and right hand side of (3.2) by $\pi(\bar{m}+e_j)$ and cancelling $\chi(\bar{m})$, equation (3.2) reduces to (3.3), which is guaranteed by definition of $\{y(\ell|\bar{m})\}$. The distribution defined by (3.6) is hereby proven to be a stationary distribution which satisfies (3.2). Conversely, assume that the stationary distribution $\pi(.)$ satisfies (3.2) for any fixed $\bar{m}$. Then $\{y(.|\bar{m})\}$ chosen by

$$\frac{y(i|\bar{m})}{y(j|\bar{m})} = \frac{\pi(\bar{m}+e_i)\ \emptyset(\bar{n}+e_i)^{-1}\mu_i}{\pi(\bar{m}+e_j)\ \emptyset(\bar{n}+e_j)^{-1}\mu_j} \tag{3.8}$$

will satisfy (3.3). Recalling that $y(.|\bar{m})$ is uniquely determined up to a constant factor, reversibility of the $\bar{q}$-model, that is (3.5), will then be guaranteed by choosing $\Psi(.)$ such that

$$\frac{\Psi(\bar{m}+e_i)}{\Psi(\bar{m}+e_j)} = \frac{\pi(\bar{m}+e_i)\ \emptyset(\bar{n}+e_i)^{-1}\mu_i}{\pi(\bar{m}+e_j)\ \emptyset(\bar{n}+e_j)^{-1}\mu_j} \tag{3.9}$$

and normalizing $\Psi(.)$. Furthermore, from (3.9) the form (3.6) is easily concluded. □

Theorem 3.2. (Routing invariance condition). There exists a stationary distribution $\pi(.)$ of the form (3.2) which satisfies the station balance equations (3.1) if and only if for some function $\Psi(.)$, some reference state $\bar{n}_0$ and all $\bar{n} \in S$:

$$\prod_{k=0}^{z-1} \left[\frac{y(i_{k+1}|\bar{m}_k)}{y(i_k\ \ |\bar{m}_k)} \right] = \Psi(\bar{n}) \tag{3.10}$$

for all possible trajectories of the form

$$\bar{n}_0 = \bar{m}_0+i_0 \rightarrow \bar{m}_0+j_1 = \bar{m}_1+i_1 \rightarrow \bar{m}_1+j_1 = \ldots$$
$$= \bar{m}_k+i_k \rightarrow \bar{m}_k+j_k = \bar{m}_{k+1}+i_{k+1} \rightarrow \bar{m}_{k+1}+j_{k+1} = \ldots = \bar{m}_z+j_z = \bar{n}$$

for which all denominators in (3.10) are positive and where z is arbitrary.

Proof. An immediate consequence of the reversibility condition (3.4) and the well-known Kolmogorov criterion for reversibility (cf. [8], p.21). □

Conclusion 3.3. In order to investigate the existence of a stationary distribution of the form (3.5), it suffices to:
(i) compute local solutions $y(.|.)$ of the equations (3.3) and
(ii) check (3.5) or (3.10) for some function $\Psi(.)$.

Remark 3.4. (Non-reversible routing). It is emphasized that condition (3.10) allows the original Markov chain to be non-reversible. For example, we may have non-reversible routing probabilities p_{ij} as will be illustrated in the next section.

Remark 3.5. (Limited station case). Even for the more standard case in which capacity limitations are imposed upon individual stations (in the present setting that is with each station seen as a separate cluster), theorems 3.1 and 3.2 have not been explicitly reported in the literature. In that case, however, the results relate to the so-called K_1-criterion in [5] for job-local-balance to hold.

Remark 3.6. (Checking (3.10) directly). As will be illustrated in the next section, the local solutions $y(.|.)$ are often easily calculated. These local solutions may directly suggest a required form of $\Psi(.)$ in order to satisfy (3.5). As a consequence, rather than checking (3.10) for all possible trajectories, instead we can simply check whether expression (3.6) with this suggested form of $\Psi(.)$ indeed satisfies (3.2). The validity of all examples in the next section are so verified directly.

Remark 3.7. (Irreducibility of local chains). The assumption of a unique solution, up to normalization, of the local balance equation (3.3) for any fixed $\bar{m}$ is equivalent to requiring that the local Markov chain at $S(\bar{m})$ is irreducible. However, from (3.4), (3.5) and the proof of theorem 3.1 it is readily concluded that only ratios of the local solutions $y(.|.)$ need to be unique for states $\bar{m}+e_i$ and $\bar{m}+e_j$ that communicate at $S(\bar{m})$. The results, therefore, remain valid if this local irreducibility assumption is relaxed to local chains which decompose in irreducible (or ergodic) sets. In other words that is, the local chain at $S(\bar{m})$ for the original q-model may not contain transient states. Such a relaxation can be useful for modeling blocked situations (cf. [5]).

Remark 3.8. (Service speed functions). Note the specific form (2.1) for the service speed of station j with a different function χ and $\emptyset$. To the best of the authors knowledge, this form has not been explicitly reported earlier in the literature. In contrast, in the literature (cf. [2], [8], [15], [16]) one finds $\chi=\emptyset$. This extension allows more flexibility in network dependent service speeds, such as with special service delay or acceleration factors. However, as this paper is primarily concerned with cluster limitations, this extension will not be elaborated upon.

4. Applications

In this section we provide some product form examples that are based upon theorems 3.1 and 3.2. With partial exception of example 3.1, these examples are non-standard and appear to be new. Most notably, examples with non-reversible routing are given. The verification of the product forms by (3.2) or (3.5) (see Remark 3.6) is left to the reader. The notational conventions from sections 2 and 3 are adopted.

Example 4.1. (Reversible routing; no internal routing). After a job completes a service at a station in some cluster p it always requires a next service at a station in some other cluster, say cluster $q\neq p$, with probability R_{pq}. Upon acceptance by a cluster q a job is assigned station j with probability $b_q(j)$. Hence, for all i,j and with $p=c(i)$ and $q=c(j)$ we have:

$$p_{ij} = R_{pq}b_q(j) \tag{4.1}$$

The function $B_{pq}(\bar{t})$ is assumed to be of the form (2.2.) and (2.3) where specifically the specification (2.4) can be kept in mind. Then, by assuming that R_{pq} is reversible, i.e. for some $\{\gamma_p\}$ and all p,q:

$$\gamma_p R_{pq} = \gamma_q R_{qp} \tag{4.2}$$

one immediately verifies (3.3) and (3.5) for all $\bar{m}+e_j \in S$ and $\bar{n}\in S$ with

$$y(j|\bar{m}) = \gamma_q b_q(j), \quad q=c(j) \tag{4.3}$$

and

$$\Psi(\bar{n}) = \left\{ \prod_{p=1}^{P} \prod_{k=1}^{s_p} [q_p(k-1)/d_p(k)] \right\} \cdot \left\{ \prod_{j=1}^{N} [\gamma_{c(j)} b_{c(j)}]^{n_j} \right\}, \tag{4.4}$$

provided $d_p(k) > 0$ for all $k \le s_p$. Particularly, under (2.4):

$$\Psi(\bar{n}) = 1(s_q \le N_q,\ q=1,\dots,P)\left\{ \prod_{j=1}^{N} [\gamma_{c(j)} b_{c(j)}]^{n_j} \right\}, \qquad \bar{n}\in S. \tag{4.5}$$

The product form (3.6) thus holds with the above form of $\Psi(.)$ substituted. For the particular case where P=2, this product form result has been widely identified in the literature (cf. [1], [3], [7] and [10]) with typical applications as a material handling system in manufacturing (cf. [16]), a machine-interference model or an interconnected metropolitan area network (cf. [12]).

Examples 4.2. Station independent cluster and non-reversible internal routing. Now assume that inside a cluster jobs can route from one station to another in a non-reversible manner while between clusters a station independent routing is applicable. More precisely, consider arbitrary probabilities p_{ij} for $j\in C(i)$ and assume that for all $j\notin C(i)$:

$$\begin{aligned} p_{ij} &= g_p(i)\ R_{pq} b_q(j) \qquad &&, p=c(i),\ q=c(j), \\ g_p(i) &= [1 - \Sigma_{j\in C(i)} p_{ij}] > 0 \qquad &&, i=1,\dots,N. \end{aligned} \tag{4.6}$$

In words that is, given that a job leaves its cluster it routes to another cluster with fixed probability R_{pq}. Given that it is accepted at cluster q it is assigned station j with probability $b_q(j)$. Then with $\bar{t}$ the cluster vector corresponding to $\bar{m}$ and

$$y(i|\bar{m}) = [b_p(i)/g_p(i)]\ x(p|\bar{t}) \qquad , p=c(i), \tag{4.7}$$

(3.3) reduces to the pure cluster equation

$$x(p|\bar{t})\ \Sigma_{q\neq p}\ R_{pq}B_{pq}(\bar{t}) = \Sigma_{q\neq p}\ x(q|\bar{t})\ R_{qp}B_{qp}(\bar{t})\ . \tag{4.8}$$

The reversibility condition (3.5) or (3.10) is then guaranteed by the existence of a function $\Theta(.)$ satisfying for all p,q and $\bar{t}$:

$$\frac{\Theta(\bar{t}+e_p)}{\Theta(\bar{t}+e_q)} = \frac{x(p|\bar{t})}{x(q|\bar{t})} \tag{4.9}$$

and the substitution , with $\bar{s}$ the cluster vector corresponding to $\bar{n}$,

$$\Psi(\bar{n}) = \Theta(\bar{s})\ \prod_{i=1}^{N}[b_{c(i)}(i)/g_{c(i)}(i)]^{n_i}. \tag{4.10}$$

(i) (Reversible cluster routing). For example, again by assuming that $B_{pq}(\bar{t})$ is of the independent departure/arrival form (2.2) and (2.3), while R_{pq} satisfies the reversibility condition (4.1), we directly verify (4.5) with

$$\Theta(\bar{s}) = \prod_{p=1}^{P}\ (\gamma_p)^{s_p}\ \prod_{k=1}^{s_p}[a_p(k-1)/d_p(k)] \tag{4.11}$$

assuming $d_p(k)>0$ for all $k\leq s_p$. Example 4.1 is hereby extended to internal non-reversible routings within each cluster. Particularly, for the natural form (2.4) again, (4.5) is generalized by setting

$$d_p(.) = 1 \quad\text{and}\quad a_p(k) = 1(k < N_p). \tag{4.12}$$

(ii) (Non-reversible cluster routing). Another example under the assumption of (4.6) but now with non-reversible routing between clusters is the following. Consider a three cluster model with cyclic routing $R_{12} = R_{23} = R_{31} = 1$ and capacity limitation N_3 for cluster 3. Let

$$B_{12}(\bar{t}) = B_{23}(\bar{t}) = 1(t_3 < N_3). \tag{4.13}$$

That is, upon saturation of cluster 3 a job is not allowed to leave any of the other clusters. Then (4.9) applies with $\Theta(\bar{s})=1$ for all states with $s_3\leq N_3$.

This example is related to example 4.4(i) below. In contrast, however, due to the special routing assumption (4.6), note that here jobs can continue to route within clusters 1 and 2 even though cluster 3 is congested.

(iii) (Resource sharing of clusters; see example 2.2). As a particular application of the general framework (4.6) and (4.8), reconsider the resource sharing example 2.2 with three clusters and assume that the transition probabilities are given by (4.6) with $R_{21}=R_{31}=1$, $R_{12}=p$ and $R_{13}=1-p$ for some $0<p<1$. Then (4.7) through (4.10) hold with

$$\theta(\bar{s}) = 1(s_q \le N_q,\ q=1,2,3;\ s_2+s_3 \le K) \prod_{q=1,2,3} [\gamma_q]^{s_q} \tag{4.14}$$

with $\gamma_1=1$, $\gamma_2=p$ and $\gamma_3=(1-p)$. The product form expression (3.6) with (4.6) substituted is thus valid, restricted to S given by (2.5).

Example 4.3. (Weak reversible routing). As a relaxation of the standard reversible routing case, let λ_j, $j=1,\ldots,N$ be the unique solution, up to normalization, of the standard traffic equations

$$\lambda_j = \Sigma_i \lambda_i p_{ij}, \tag{4.15}$$

and assume that for any i and $q\neq c(i)$

$$\lambda_i \Sigma_{j\in C(q)} p_{ij} = \Sigma_{j\in C(q)} \lambda_j p_{ji}. \tag{4.16}$$

Then, for example with $B_{p,q}(\bar{t})$ of the form (2.2), (2.3) and (2.4), the reversibility condition (3.5) or (3.10) is immediately verified with

$$\begin{aligned} &y(i|\bar{m}) = \lambda_i \qquad (\forall\ i,\bar{m}) \quad \text{and} \\ &\Psi(\bar{n}) = 1(t_p \le N_p,\ p=1,..,P) \prod_{j=1}^{N} [\lambda_j]^{n_j}\ . \end{aligned} \tag{4.17}$$

Examples 4.4. (Non-reversible routing)

In the examples below no general conditions are imposed upon the routing probabilities $p_{ij}(\bar{t})$ such as of no internal routing (as in example 4.1), a uniform station assignment within a cluster (as in examples 4.2) or a weak reversible routing (as in example 4.3).

(i) (Preventive blocking). Consider an arbitrary network with routing probabilities p_{ij} and capacity limitation N_p for cluster p=1,...,P. Upon saturation of a cluster p, however, a job is not allowed to leave any station outside this cluster p. The corresponding parametrization is

$$B_{pq}(\bar{t}) = 1(t_c < N_c \quad \text{for all } c \neq p) \tag{4.18}$$

With $\{\lambda_j\}$ determined by the traffic equations (4.15) and

$$S = \{\bar{n} | s_p \leq N_p,\ p=1,\ldots,P,\ s_p+s_q < N_p+N_q \text{ for all } p \neq q\} \tag{4.19}$$

one verifies (3.5) or (3.10) with

$$\begin{aligned} \Psi(\bar{n}) &= \prod_{j=1}^{N} [\lambda_j]^{n_j} \\ y(i|\bar{m}) &= \lambda_i \end{aligned} \tag{4.20}$$

The product form expression (3.6) restricted to S thus applies. Here it is noted, in contrast with examples 2.1, 2.2 and 2.3, that no conditions at all are imposed upon the routing probabilities p_{ij}. The above protocol can be practical, such as in manufacturing, as it prevents more than one cluster to become saturated at the same time so that deadlocks or accumulating congestions are avoided.

(ii) (Minimal workload blocking; cyclic cluster routing). Consider a network in which the routing between clusters is cyclic. Each cluster, however, requires a minimal workload or occupancy, say of at least M_p jobs in cluster p=1,...,P. Hence,

$$B_{p,p+1}(\bar{t}) = 1(t_p \geq M_p),$$

with p+1=1 for p=P. Note that the blocking is of the form (2.2), (2.3) and (2.4) with

$$a_q(.)=1 \text{ but } d_p(\bar{t})=1(t_p \geq M_p). \tag{4.21}$$

The condition (3.5) or (3.10) and thus expression (3.6) is readily verified with (4.10) and (4.11) substituted with $a_p(.)=d_p(.)=1$ restricted to S given by

$$S = \{\bar{n} \mid s_p \geq M_p,\ p=1,\ldots,P\}. \tag{4.22}$$

A minimal workload is thus secured for each cluster. This can be practical for workload balancing along assembly lines in manufacturing systems.

5. Extensions

5.1. Open systems. Similar results can be derived for open networks. To this end, one can either use standard limiting arguments or one can include an extra station to represent the exterior of the network. The main difference will arise in relation (3.3) in which an extra term at either side is to be included. By introducing corresponding local solutions $y(0|\bar{m})$ the results in the remainder can then be adapted easily. For example, the open analogue of example 4.1 with P=2 is a standard Jackson network with a single total capacity constraint.

5.2. Multi-job classes. Clearly, when multi-jobclasses are allowed while jobs of one class route independently from other jobclasses, the above results directly transfer per jobclass. However, in practical situations with multiple jobclasses, interferences of different jobclasses is most common. In order to extend the present results to these situations, (e.g. similarly to [2] or [5]), a more refined notion of station-class balance is to be applied. Similar results can then be derived in analogy with the job-local-balance analysis in [5]. The local traffic equations such as (4.8), relations however, will become more complicated.

5.3 Non-exponential servicing. It is well-known that product form expressions, as based upon station or local balance notions in the exponential case, remain valid without exponentiality assumptions provided "locally balanced", such as so-called "symmetric" service disciplines, are in order (e.g. [1], [2], [6], [8], [14], [15], [16]). Based upon the station balance property (3.1), this latter statement applies just as well in the present setting. For example, with each station behaving as an infinite server queue with the capacity limitations taken into account, the product form expression (3.6) can be shown to be insensitive for service distributional forms. More precisely, it applies with $\emptyset(\bar{n})=[n_1!n_2!\ldots n_N!]^{-1}$.

Evaluation

Closed queueing networks are studied in which groups of stations rather than individual stations have capacity limitations, for instance due to a common storage pool or commonly shared resource such as frequently arising in communication, computer or manufacturing applications. A general condition is derived, based upon a notion of station balance, which guarantees a product form expression. This condition is given purely in terms of local solutions of state dependent routing equations. As explicit expressions for these local solutions are often obtainable, a practical tool for investigating the existence and computation of a product form is thus provided. A number of novel product form examples with blocking is so derived. Particularly, examples with non-reversible routing both within clusters and in between clusters are concluded, thus relaxing the standard restrictive reversibility conditions for a product form to hold in queueing networks with blocking. Extensions to open, non-exponential and multi jobclasses are possible.

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